Model Theory | Martin Otto
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Exercises No.2

Exercise 1 [non-standard models, continuation of Ex. 1.3&6]
Analyse non-standard extensions $\mathfrak{R}^* \succcurlyeq \mathfrak{R}$ and $\mathfrak{N}^* \succcurlyeq \mathfrak{N}$ obtained as ultrapowers $\mathfrak{A}^{\mathbb{N}}/\mathcal{U}$ w.r.t. suitable ultrafilters $\mathcal{U}$ over $\mathbb{N}$ in relation to the base structures $\mathfrak{A} = \mathfrak{R}, \mathfrak{N}$.
For instance,

(i) which sequences $(a_n) \in \mathbb{N}^{\mathbb{N}}$ represesent infinitely large numbers?
(ii) which sequences $(a_n) \in \mathbb{R}^{\mathbb{N}}$ represesent points in the infinitesimal neighbourhood of $a \in \mathbb{R}$?
(iii) what are the possible order-types for $\mathfrak{N}^*$ and $\mathfrak{R}^*$?
(iv) investigate the equivalence relations of being "finitely far apart" in $\mathfrak{N}^*$ and of being "of the same order of magnitude" in $\mathfrak{R}^* \setminus \{0\}$.

Exercise 2 [Lemma 2.5]
Prove the following assertions about the algebraic and elementary diagrams $\mathrm{D}_{\mathrm{alg}}(\mathfrak{A})$ and $\mathrm{D}(\mathfrak{A})$ of a τ-structure $\mathfrak{A}$.

(a) T.f.a.e. for any τ-structure $\mathfrak{B}$:
  (i) there is an isomorphic embedding of $\mathfrak{A}$ into $\mathfrak{B}$
  (ii) $\mathfrak{B} = \mathfrak{B}_A \restriction \tau$ for some $\mathfrak{B}_A \models \mathrm{D}_{\mathrm{alg}}(\mathfrak{A})$.
(b) T.f.a.e. for any τ-structure $\mathfrak{B}$:
  (i) there is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$
  (ii) $\mathfrak{B} = \mathfrak{B}_A \restriction \tau$ for some $\mathfrak{B}_A \models \mathrm{D}(\mathfrak{A})$.

Exercise 3 [preservation of $\forall$-FO under substructures]
Show by syntactic induction on $\varphi(\mathbf{x}) \in \forall\text{-FO}[\tau]$ that universal FO-formulae are preserved under substructures, i.e., that for all $\mathfrak{A} \subseteq \mathfrak{B}$ and all $\mathbf{a}$ in $\mathfrak{A}$:

$$\mathfrak{B} \models \varphi[\mathbf{a}] \quad \Rightarrow \quad \mathfrak{A} \models \varphi[\mathbf{a}].$$

Give examples of common model classes that are closed under substructures and consider whether they admit an axiomatisation in $\forall$-FO$[\tau]$ (note that this may depend on the choice of signatures; why?).

Exercise 4 [$\preccurlyeq$-criterion, Lemma 2.7]
Show that the following are equivalent for any subset $A \subseteq B$ of a τ-structure $\mathfrak{B}$:

(i) $\mathfrak{A} := \mathfrak{B} \restriction A$ is a τ-structure s.t. $\mathfrak{A} \preccurlyeq \mathfrak{B}$.
(ii) for every $\varphi(\mathbf{x}, x) \in \mathrm{FO}[\tau]$ and $\mathbf{a}$ over A: $\left\{ \begin{array}{ll} & \text{ex. } b \in B \text{ s.t. } \mathfrak{B} \models \varphi[\mathbf{a}, b] \\ \Rightarrow & \text{ex. } a \in A \text{ s.t. } \mathfrak{B} \models \varphi[\mathbf{a}, a]. \end{array} \right.$