Model Theory

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Exercises No.1

Exercise 1 [filters]
Consider $I \neq \emptyset$ and $\emptyset \neq \mathcal{B} \subseteq \mathcal{P}(I)$ with the *finite intersection property* (*f.i.p.*). Show the following:

(a) closure under intersection ($\cap$) and supersets ($\supseteq$) preserves f.i.p.

(b) $\mathcal{F}(\mathcal{B})$, defined as the closure under supersets of the closure under (finite) intersections of $\mathcal{B}$, is a filter extending $\mathcal{B}$.

(c) if $s, \overline{s} \notin \mathcal{B}$ then at least one of $\mathcal{B} \cup \{s\}$ or $\mathcal{B} \cup \{\overline{s}\}$ has the f.i.p. (and hence extends to a filter).

(d) if $\mathcal{B} \subseteq \mathcal{P}(I)$ is *maximal* with f.i.p., then $\mathcal{B} = \mathcal{F}(\mathcal{B})$ is an ultrafilter on I.

Exercise 2 [cf. Lemma 1.3]
Let $\mathcal{F}$ be a filter on I.

(a) Show that the relation $\sim_{\mathcal{F}}$ is an equivalence relation on $\prod_i A_i$ (for any family of non-empty sets $(A_i)_{i \in I}$).

(b) Show that $\sim_{\mathcal{F}}$ is a congruence w.r.t. any function $f^{\mathfrak{A}}$ in the direct product $\mathfrak{A} := \prod_i \mathfrak{A}_i$ of a family of structures $(\mathfrak{A}_i = (A_i, f^{\mathfrak{A}_i}))_{i \in I}$.

(c) Show that for a family of structures $(\mathfrak{A}_i = (A_i, R^{\mathfrak{A}_i}))_{i \in I}$ with relation R, and for $\mathbf{a} \sim_{\mathcal{F}} \mathbf{a}'$ (component-wise equivalence): $\|R\mathbf{a}\| \in \mathcal{F} \ \Leftrightarrow\ \|R\mathbf{a}'\| \in \mathcal{F}$.

Exercise 3 [non-standard models]
Describe non-standard models of the first-order theories of arithmetic on $\mathbb{N}$ and $\mathbb{R}$ as ultrapowers. Which elements of the direct powers will represent infinitely large natural (or real) numbers; which will represent infinitesimals (in the real case)?

Exercise 4 [cf. Łos theorem, theorem 1.5]
Which steps in the inductive treatment of the usual connectives and quantifiers of FO fail for filters instead of ultrafilters. Consider some simple (counter)examples to illustrate what can happen in arbitrary reduced products.

Exercise 5 [preview; this will also be done in class]
Find a simple way to prove the compactness theorem for countable sets of formulae (w.l.o.g., sentences) straight from Łos theorem.
What more does it take to get a proof that works in arbitrary cardinality?

Exercise 6 [ultrapowers as elementary extensions]
Show that the natural embedding of $\mathfrak{A}$ into its ultrapower $\mathfrak{A}^I/\mathcal{F}$ is elementary.