Fachbereich Mathematik
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9. Problems for CMC Surfaces

Problem 31 – Example for integrability conditions:
Suppose the fundamental forms of an immersion $f\colon U \to \mathbb{R}^3$ satisfy

$$g_{ij} = \delta_{ij} \quad \text{and} \quad b_{11} = b_{12} = b_{21} = 0, \quad b_{22}(x,y) = b(x,y)$$

a) Derive a necessary condition from the Gauss- and Codazzi equations.
b) Given the condition, the fundamental theorem for surfaces guarantees the existence of a surface with fundamental forms g and b. What is the Gauss curvature of this surface?
c) Give two examples of such surfaces (or characterize all such immersions $f\colon \mathbb{R}^2 \to \mathbb{R}^3$).

Problem 32 – Gauss curvature and Christoffel symbols:
Consider an immersion $f\colon U^2 \to \mathbb{R}^3$ whose Gauss curvature does not vanish identically.
a) Can all Christoffel symbols vanish identically?
b) Can the first fundamental form be constant?

Problem 33 – Gauss curvature of the hyperbolic plane:
Suppose the upper halfplane $U = \{(x,y) : y > 0\}$ parameterizes an immersion into $\mathbb{R}^3$ with the conformal first fundamental form $g = \frac{1}{y^2}\delta$.
a) What are the eight Christoffel symbols Γ_{ij}^k?
Hint: We calculated Christoffel symbols for conformal parameterizations in class.
b) Suppose a subdomain of U immerses to $\mathbb{R}^3$ with first fundamental form g. What is its Gauss curvature? By a theorem of Hilbert the entire hyperbolic plane (U,g) does not immerse into $\mathbb{R}^3$.

Problem 34 – Euler characteristic and genus:

a) Position a torus of revolution in space suitably, so that you can the Poincaré-Hopf theorem for the torus by choosing the gradient vector field of the height function to compute the index.
b) Now do the same for a nice model of an oriented surface $\Sigma_g \subset \mathbb{R}^3$ of genus $g \in \mathbb{N}_0$, homeomorphic to a sphere with g handles attached. Show that the Euler characteristic is
$$\chi(\Sigma) = 2 - 2g.$$