Fachbereich Mathematik
Prof. K. Große-Brauckmann
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8. Problems for CMC Surfaces

Problem 26 – Singularities of the curvature line field:

On the reverse, you find singularities of line fields depicted (from Hopf, Differential geometry in the large, p.109).

a) Assuming these represent a curvature line field of some surface, draw the other curvature line field and convince yourself that it has the same index.

b) Can you imagine an example of a surface with the depicted curvature line field? Consider only the indices admitted by the Loewner conjecture.

Problem 27 – Index of spherical vector fields:

Draw unit vector fields on $\mathbb{S}^2$. It can helps to regard them as the stereographic projection of vector fields on $\mathbb{R}^2$ (see back side!).

a) With singularities at north and south pole (recall from class).

b) With only one singularity – what must be its index?

c) With three singularities. (A strategy is to merge singularities.)

d) Can you find similar vector fields on $\mathbb{S}^n$?

Problem 28 – Index of other spaces:

a) If you happen to know the spaces, calculate the Euler characteristic of a Klein bottle and of $\mathbb{R}P^2$.

b) If you know about stereographic projection: Calculate the Euler characteristic from the 4-dimensional cube (also called hypercube) or 4-dimensional tetrahedron.

c) Is there a vector field on $\mathbb{S}^3$ without zeros?
Hint: Try to find $X(x)$ in terms of the coordinates of x.

Problem 29 – Index of islands:

a) Consider an island with no lakes. Show that # peaks $-$# passes $+$# sinks $= 1$, provided all these numbers are finite. Here, peaks are local maxima of height, sinks local minima, and passes critical points which are neither peaks nor sinks.
Hint: Poincaré-Hopf. Which vector field is appropriate?

b) Find an island with 3 peaks, 1 pass, and no sinks. Correct the formula in part a) (!).

c) Now admit lakes and show that

$$\#\text{ peaks} - \#\text{ passes } + \#\text{ sinks } = 1 - \#\text{ lakes}\,.$$

Problem 30 – Symmetries and umbilics:

Let $f\colon U^2 \to \mathbb{R}^3$ be a two-dimensional surface. For simplicity, suppose $f(p) = 0$ for some $p \in U$ such that $T_p f$ agrees with the xy-plane.

a) Let $R_3\colon \mathbb{R}^3 \to \mathbb{R}^3$ be 120° rotation about the z-axis. Prove: If $f(U)$ is invariant under the rotation R, then p is an umbilic *[Nabelpunkt]*.

b) Does part a) hold for an arbitrary rotation R_k of angle $2\pi/k$ where $k = 2, 3, \ldots$?

c) Let $R_6\colon \mathbb{R}^3 \to \mathbb{R}^3$ be rotation by 60° about the z-axis, and S reflection in the xy-plane. Prove: If $f(U)$ is invariant under $R_{\overline{6}} := S \circ R_6$, then the principal curvatures at p vanish, that is p is a *flat point*.