Fachbereich Mathematik
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6. Problems for CMC Surfaces

Problem 19 – Dirichlet problem over a non-convex domain:

The following example indicates that the solvability of the Dirichlet problem for arbitrary boundary data requires convexity of the domain.

a) Let C be a catenoid whose axis of revolution is the z axis. Find a closed simple (Jordan) curve $\Gamma \subset C$ with the following properties:
1. The projection $\pi(\Gamma)$ into the xy-plane is injective.
2. $\pi(\Gamma)$ is the boundary of a domain U in the xy-plane which is not convex.
3. Γ bounds an open bounded subset $M \subset C$ such that M is not graph over U.

b) Prove that M is the unique minimal surface bounded by Γ; perhaps you need to modify Γ suitably. Hence Γ cannot bound a minimal surface which is a graph over U.
Hint: Which theorem of the lecture can only prove this claim?

c) Can you prescribe other non-constant boundary values over ∂U, such that there is a unique minimal graph over U?

Problem 20 – Weierstrass data:

The Enneper-Weierstrass representation formula is

$$f(z) = \operatorname{Re} \int_{z_0}^{z} h(w) \left(\frac{1}{2}\Big(\frac{1}{g(w)} - g(w) \Big), \frac{i}{2}\Big(\frac{1}{g(w)} + g(w) \Big), 1 \right) dw.$$

a) Prove that on $U = \mathbb{C}$ the Weierstrass data $g(z) = z$, $h(z) = 2z$ give the Enneper surface f.

b) Prove that on $U = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $g(z) = z$, $h(z) = \frac{1}{z}$ the function f parameterizes a catenoid.
Hint: After integration, use conformal polar coordinates $z = \exp(r + i\varphi)$, that is, calculate $f\big(\exp(r + i\varphi)\big)$.

Problem 21 – Isoperimetric sets in 2-tori:

In class we showed that isoperimetric sets are bounded by constant mean curvature surfaces, or bounded by constant curvature curves in the case of 2-dimensional domains. For the present problem we can also assume that the solution domains are connected.

a) Determine explicitely isoperimetric sets in a square 2-torus, say: with unit area. To do so, plot the function $L(A)$, giving the length of the boundary of a set with area A for various candidates. Note that in the torus there is no difference between inside and outside.

b) Discuss the same problem for a general 2-torus.

c) Do you have conjectures about the analogous problem for 3-tori? Plot $A(V)$ for some obvious candidates.

Problem 22 – Sets invariant under reflections:

Prove that that a bounded set $A \subset \mathbb{R}^{n+1}$ invariant under reflections in every direction must be empty if the symmetry planes do not all contain a common point.