Fachbereich Mathematik
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5. Problems for CMC Surfaces

Problem 16 – Test:

a) Why is it no loss of generality to assume that the matrix A in $Lu = A\,d^2u = \sum a^{ij}\partial_{ij}u$ is symmetric?

b) Prove that if u is subharmonic ($\Delta u \geq 0$) on a bounded domain U, and h is a harmonic function ($\Delta h = 0$) with the same boundary values, $u|_{\partial U} = h|_{\partial U}$, then $u(x) \leq h(x)$ for all $x \in U$. Discuss also the equality case $u(p) = h(p)$ for an interior point $p \in U$.

c) A standard linear algebra result is that a linear map $L\colon V \to W$ with $\ker L = 0$ gives $Lx = b$ has at most one solution x. Draw the analogy to the uniqueness theorem for the Poisson equation $Lu = f$. (What are the vector spaces V, W?)

Problem 17 – Uniqueness and symmetry of solutions:

Suppose σ is reflection in the hyperplane $\{x_n = 0\} \subset \mathbb{R}^n$,

$$\sigma\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad \sigma(x_1, \dots, x_n) := (x_1, \dots, x_{n-1}, -x_n).$$

We call a domain $U \subset \mathbb{R}^n$ *mirror symmetric* if $\sigma(U) = U$. For a bounded mirror symmetric domain U, consider a function $u \in C^2(U, \mathbb{R}) \cap C^0(\overline{U}, \mathbb{R})$ whose boundary values are invariant under σ, that is, $u(x) = u\big(\sigma(x)\big)$ for all $x \in \partial U$.
Consider the following cases:

1. u is harmonic,
2. u solves a uniformly elliptic equation $Lu = 0$,

a) Decide for each of the two cases if u respects the symmetry σ, i.e., $u(x_1, \dots, x_n) = u(x_1, \dots, x_{n-1}, -x_n)$ for all $x \in U$.

b) On the other hand, find a solution v of the equation $\Delta v + v = 0$ which has symmetric boundary values, but is not invariant under σ (it suffices to consider $n = 1$).

c) Consider the cases for which the answer under a) is in the affirmative. Prove the same statement more generally for isometries $A \in \mathsf{O}(n)$, for instance for rotations.

Problem 18 – Maximum principle with exceptional points:

Let us first state two facts:
1. $\log|x|\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is harmonic,
2. If $f\colon \Omega^2 \to \mathbb{R}^2$ is conformal then $\Delta(u \circ f) = (\Delta u) \circ f$.
Use these facts to prove the following:

a) Let $D \subset \mathbb{R}^2$ be the unit disk, and set $D^* := \overline{D} \setminus \{(1,0)\}$, $S^* := \mathbb{S}^1 \setminus \{(1,0)\}$. Find a harmonic function $u \in C^2(D, \mathbb{R}) \cap C^0(D^*, \mathbb{R})$ with boundary values $u|_{S^*} = 0$ such that u is not constant.
Hint: Exhibit a nonzero harmonic function with zero boundary values on the upper halfplane.

b) Prove that each bounded harmonic function $u \in C^2(D, \mathbb{R}) \cap C^0(D^*, \mathbb{R})$ is constant.
Hint: Compare with $\epsilon \log|z - 1|$.

c) Generalize: Can you admit more than just one exceptional point? Can you replace the boundedness assumption on u by a growth condition at the exceptional points? What is the n-dimensional generalization?

d) Prove the two facts stated above by calculation.