Fachbereich Mathematik
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4. Problems for CMC Surfaces

Problem 12 – Force balancing for the catenoid:

Let M be a minimal surface, and $K \subset M$ be a compact subset with piecewise smooth boundary (and non-empty interior); the boundary ∂K then is the union of smooth loops. Rob Kusner discussed in his lecture the principle of *force balancing [Kräftegleichgewicht]*: the total *force*

$$F(\partial K) := \int_{\partial K} \eta \, ds = 0.$$

Here η is the *exterior conormal*, that is, η is a unit tangent vector of M which is normal to ∂K. We now apply this principle specifically to minimal surfaces of revolution.

a) Specify $K = K(t)$ in a way to prove that for a minimal surface of revolution $f(t, \varphi) = \big(r(t)\cos\varphi, r(t)\sin\varphi, h(t)\big)$ the function $t \mapsto \int_{C(t)} \eta \, ds$ over the circle $C(t) = \{f(t, \varphi) : 0 \le \varphi < 2\pi\}$ is constant.

b) Check with the explicit representation of the catenoid $r(t) = (a \cosh t/a)$ that the force is independent of the circle chosen.

c) Conversely, derive the ODE and perhaps the representation of the catenoid (or plane) from the principle of force balancing.

The same principle for nonzero constant mean curvature can be used to give a qualitative discussion of the Delaunay surfaces. It also applies to higher dimensions or other ambient spaces.

Problem 13 – Force balancing for Scherk's doubly periodic surface:

Consider Scherk's doubly periodic surface over one square, that is, the graph $u(x, y) = \log(\cos y / \cos x)$ over $(x, y) \in (-\pi/2, \pi/2)^2$.

a) What is the vertical component of the force over the intersection of a horizontal plane P_z at height z with the surface (choose η with $\eta^3 \ge 0$)? Do not engage in a calculation! What can you say about the horizontal component of the force for the intersection curve with a vertical axis-parallel plane?

b) Scherk's surface is a graph over the square $(-\pi/2, \pi/2)^2$ with boundary values alternating between $+\infty$ and $-\infty$. Similar surfaces exist over more general quadrilaterals *[Vierecke]*. Derive a necessary condition the quadrilaterals must satisfy from force balancing. (The condition is sufficient by work of Jenkins and Serrin from the 60's.)

Problem 14 – Maximum of harmonic functions on unbounded domains:

Exhibit an unbounded domain $U \subset \mathbb{R}^n$ with non-empty boundary and a harmonic function $u \colon U \to \mathbb{R}$ such that u does not take a maximum on the boundary.

Problem 15 – Versions of the maximum principle:

Let $U \subset \mathbb{R}^n$ be bounded.

a) Prove for a harmonic function $u \in C^2(U, \mathbb{R}) \cap C^0(\overline{U}, \mathbb{R})$:

$$\sup_U |u(x)| = \sup_{\partial U} |u(x)|$$

b) A mapping $u \in C^2(U, \mathbb{R}^n) \cap C^0(\overline{U}, \mathbb{R}^n)$ is called *harmonic* if each component is a harmonic function. Prove that harmonic mappings satisfy the above maximum principle.