Fachbereich Mathematik
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2. Problems for CMC Surfaces

Problem 5 – Alternative derviation of the first variation of a graph:
Consider a hypersurface $f\colon U \to \mathbb{R}^{n+1}$ which is a graph, $f(x) = \big(x, u(x)\big)$. For simplicity, we assume U is bounded and $u \in C^2(\overline{U}, \mathbb{R})$. We set

$$J(u) := \int_U \sqrt{1 + |\nabla u|^2}\, dx + n \int_U Hu\, dx,$$

where for now $H = H(x)$ is an arbitrary continuous function.

a) Prove that

$$\int_U \partial_i \eta\, dx = 0 \qquad \text{for all } \eta \in C_0^1(U, \mathbb{R}) \text{ and each } i = 1, \dots, n.$$

Conclude the law of *partial integration* in several variables,

$$\int_U \varphi \partial_i \eta\, dx = -\int (\partial_i \varphi) \eta\, dx,$$

provided one of the two functions η, φ has compact support. Finally, deduce

$$\int_U \sum_i \varphi_i \partial_i \eta\, dx = -\int \eta\, \mathrm{div} \varphi\, dx.$$

b) Calculate the first variation of J, that is,

$$\delta_\eta J(u) := \frac{d}{dt} J(u + t\eta)\Big|_{t=0}$$

for $\eta \in C_0^1(U, \mathbb{R})$. You obtain the mean curvature equation for a graph in divergence form.
Remark: This is the most elegant derivation of the mean curvature for graphs, which for the case of minimal surfaces, $H \equiv 0$, goes back to Lagrange. However, this derivation leaves open that H agrees with the mean curvature.

Problem 6 – Expansions of volume and J for a cylinder:
As in problem 2, let $C_l(r)$ be a cylinder of height $l > 0$ and radius $r > 0$ (do not include the top and bottom disks).

a) Compute both sides of the volume expansion for the variation $t \mapsto C_l(r + t)$.
b) Compute similarly the expansion for the functional J^H where H is the mean curvature of $C_l(r)$.

Problem 7 – Catenary:

Consider a rope whose endpoints are fixed at two points of $\mathbb{R}^3$, under the influence of gravity, such as an electrical power line between two posts, or a railway catenary. We want to determine its shape, the so-called catenary (*Kettenlinie*).
We consider the vertical $\mathbb{R}^2$ containing the two points, and suppose the points are not related by a vertical translation. Moreover, we suppose the curve can be represented as a graph

$$\{(x, f(x)), x \in [0, b]\}.$$

For $0 \le t \le b$ we consider the portion $(x, f(x))$ of the curve with $0 \le x \le t$. The tangent vectors at its endpoints,

$$T_0 := (1, f'(0)) \qquad \text{and} \qquad -T_t := -(1, f'(t)),$$

correspond to the forces which pull tangentially on the catenary (sketch!). Note that the lengths of $T_0, -T_t$ are chosen in a way that the horizontal components $1, -1$ of the forces balance.

a) Formulate the force balance for for the vertical components of the forces: The gravity force of the rope corresponds to $\rho > 0$ times the length of the portion of the rope considered. It agrees with the sum of the two vertical components of T_a and T_s.

b) Deduce a differential equation of second order.

c) Solve the differential equation by separation of variables (substition with a hyperbolic function!). Was Galileo correct, when he claimed in 1638 that the solution curve is a parabola?

d) Here are some suggestions for further thoughts:
   - Does the catenary of a suspension bridge have the same shape? Assume that all the wiring is weightless, while the weight is concentrated on the bridge deck.
   - At the mathematics museum at Giessen there is an exhibit modelling the Gateway Arch at St. Louis, Missouri, see `http://en.wikipedia.org/wiki/Gateway_Arch`. It can be assembled from building blocks without any glue. Explain its shape and how the faces of the building blocks are chosen.
   - Will a self-supporting dome have the same cross-section? (It is claimed that St. Paul's Cathedral in London is very close to be self-supporting.)