Fachbereich Mathematik
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10. Problems for CMC Surfaces

Problem 35 – Properness:

a) Prove that the composition of proper maps is proper.

b) What does it mean for a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ to be proper?

c) Show that for a compact domain A, a continuous map $f\colon A \to \mathbb{R}^n$ is always proper.

d) Give an example for a homeomorphism $\varphi\colon B^n \to \mathbb{R}^n$; it is useful to construct a homeomorphism $\psi\colon (0,1) \to (0,\infty)$ first. Prove that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is proper, then $f \circ \varphi\colon B^n \to \mathbb{R}^n$ is proper. That is, it makes no difference if we define a properly embedded plane as the image of D or $\mathbb{R}^2$.

e) More advanced: Prove that the following two characterizations of the properness of a map $F\colon S \subset \mathbb{R}^m \to \mathbb{R}^n$ for an arbitray domain S are equivalent.
• Compact sets $K \subset \mathbb{R}^n$ have compact preimage $F^{-1}(K) \subset S$,
• For all paths γ which leave any compact subset the image path $c = F \circ \gamma$ also leaves any compact subset.

Problem 36 – Stereographic projection:

Let $f_\pm$ be projection of $\mathbb{S}^n$ from the north or south pole onto the equatorial plane. Under the assumption that this map is differentiable, check geometrically (no computation!) that the transition map $\tau(x) := (f_+^{-1} \circ f_-)(x) = x/|x|$

a) is conformal.

b) Is τ orientation preserving?

Problem 37 – Totally umbilic surfaces are spheres:

We want to prove that a surface Σ for which each point is umbilic is a subset of the sphere $\mathbb{S}^2$ or the plane. (We do not assume constant mean curvature.)

a) Consider a parameterization $(f(x,y), \nu(x,y))$ and differentiate the equation for a principal curvature direction to derive the equation $\nu + \kappa f \equiv C$ where $\kappa(x,y)$ and C are constant.

b) Why is Σ contained in a plane when $\kappa \equiv 0$? Otherwise, take the equation from part a) and show that f has constant distance $1/|\kappa|$ to some point.

Problem 38 – Laplace-Beltrami and mean curvature of a graph:

a) Write down the Laplace-Beltrami operator for polar coordinates $f\colon (0,\infty) \times \mathbb{R}$, $f(r,\varphi) = (r\cos\varphi, r\sin\varphi)$.

b) Let $u \in C^\infty(\Omega^n, \mathbb{R})$ and consider the graph $M = \{(x, u(x)) : x \in \Omega\}$. Check the equation $\Delta_M(x, u(x)) = -nH(x, u(x))$, where Δ_M is the Laplace-Beltrami operator for the graph.