Fachbereich Mathematik
Prof. K. Große-Brauckmann
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UNIVERSITÄT
DARMSTADT

1. Problems for CMC Surfaces

Problem 1 – Lagrange identity:
Let $x, y \in \mathbb{R}^n$ and consider the $(n \times n)$-matrix C with entries

$$c_{ij} := x_i y_j - x_j y_i = \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}.$$

a) For $\|C\|^2 := \sum_{i<j} c_{ij}^2$ half the L^2-norm of C, prove that $\|C\|^2 = |x|^2|y|^2 - \langle x, y\rangle^2$.

b) Conclude the Lagrange identity

$$|v|^2|w|^2 = \langle v, w\rangle^2 + |v \times w|^2 \qquad \text{for all } v, w \in \mathbb{R}^3.$$

c) Use a) to prove that the Cauchy-Schwarz inequality for $\mathbb{R}^n$ is exactly attained with equality when x, y are linearly dependent.

Problem 2 – Parallel surfaces of a cylinder:
Let $C(r)$ be a cylinder in $\mathbb{R}^3$ with radius r.

a) Show that for any pair of points $p, q \in C(r)$ there is an isometry of $\mathbb{R}^3$ which maps p to q (is it unique?). Conclude that the Gauss curvature is constant.

b) Consider the cylinder $C_h(r)$ of radius r with height h (without the bounding disks). Insert the area of $C_h(r+t)$ and $C_h(r)$ into the expansion of area for parallel surfaces and conclude that K must vanish.

Problem 3 – Graphs and minimality:
Let the graph $(x, y, u(x, y))$ represent a minimal surface. Examine which of the following graphs $(x, y, \tilde{u}(x, y))$ are also minimal:

a) $\tilde{u} = u + c$ for $c \in \mathbb{R}$,

b) $\tilde{u} = cu$ for $c \in \mathbb{R}$,

c) $\tilde{u} = cu(cx, cy)$ or $\tilde{u} = cu\left(\frac{x}{c}, \frac{y}{c}\right)$ for $c \neq 0$, where the domain is chosen suitably.

Problem 4 – Minimal Graphs:

a) Differentiate the divergence form of the mean curvature equation for graphs to obtain a second order equation in the standard form

$$nH = \sum_{1\le i,j\le n} a^{ij}(x, u, Du)\, \partial_{ij} u.$$

Compare the result with the formula for $n = 2$ obtained in class.

b) Prove that the equation is elliptic in the following sense: Suppose $u\colon \Omega \to \mathbb{R}$ satisfies $|\nabla u| < K$. Then there exists $\lambda = \lambda(K)$ such that

$$\sum_{1\le i,j\le n} a^{ij}(x, u, Du)\, \xi_i \xi_j > \lambda|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\} \text{ and } x \in \Omega.$$