LECTURE
SURFACES OF CONSTANT MEAN CURVATURE SS 10

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Inhaltsverzeichnis

References

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Introduction

The class was given for 4th and 5th year students of mathematics. The prerequisites included only a short course in the differential geometry of surfaces. Some students in the audience also knew about manifolds and Riemannian geometry, but the class only assumed knowledge on submanifolds of Euclidean space.

The theorems of Alexandrov and Hopf were the main topic of the class. They are supplemented with more basic material on the first and second variation, and the example of the Delaunay surfaces of revolution. At the end I presented more advanced material on non-compact constant mean curvature surfaces with finite topology, namely the first part of the paper by Korevaar, Kusner, and Solomon. Together with Kapouleas' existence results, this paper has been very influential in the recent develpment of the theory.

An attractive feature of the theory is that it makes use of a lot of mathematics worth knowing: The Alexandrov Theorem builds on the main tool to study elliptic PDE's, the maximum principle. The Hopf theorem makes use of classical global surface theory, the Poincaré-Hopf index theorem, together with some complex analysis; moreover it provides an opportunity to present the integrability equations for hypersurfaces. Since Stokes' theorem and homology theory was not required, I could only outline the use of the mapping degree.

Certainly, there is room for improvement: The global notion of a surface should be included in the introductory chapter. A more Riemannian approach could serve to derive the first variation, together with the principle of force balancing. This would suffice to solve the ODE for the Delaunay surfaces, perhaps even to characterize them as roulettes of conic sections. Also, the stability arguments should be presented explicitly. Finally, the Hopf theorem should be described in terms of quadratic differentials.

There is also much further material worth presenting. I have not touched on any regularity issues, in particular analyticity of the solutions to the mean curvature equation. Periodic surfaces are missing. I have not derived the linearization of the mean curvature operator, a topic crucial for all analytic existence proofs. The construction of the Wente tori would be nice to include. Also, there is a lot more to say about the case of minimal surfaces.

I used several sources to prepare the class. Besides the books by Hopf and Spivak, I also used handwritten notes by K. Steffen. Particular thanks go to St. Fröhlich for his notes on the second variation. Parts in small print were not presented in the lectures.

I thank Miroslav Vrzina and Dominik Kremer for suggesting corrections to me.

Darmstadt, July 2010 Karsten Große-Brauckmann

`2. Lecture, Thursday 15.4.10`

1. The equations

1.1. **Review of mean curvature for parametrized hypersurfaces.** For this section, suppose $f\colon U \to \mathbb{R}^{n+1}$ is a hypersurface with Gauss map $\nu\colon U \to \mathbb{S}^n$. Let us recall from differential geometry:

Definition. (i) The *shape operator [Weingartenabbildung]* of (f, ν) is the mapping

$$S\colon U \times \mathbb{R}^n \to \mathbb{R}^n, \qquad (p, X) \mapsto S_p(X) := -(df_p)^{-1}\big(d\nu_p(X)\big).$$

(ii) The *second fundamental form* of (f, ν) is

$$b\colon U \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad (p, X, Y) \mapsto b_p(X, Y) := \big\langle \nu(p), d^2f_p(X, Y)\big\rangle. \tag{1}$$

Since $0 = \langle \nu, df(Y)\rangle$ we have

$$0 = \partial_X\big\langle \nu, df(Y)\big\rangle = \big\langle d\nu(X), df(Y)\big\rangle + \big\langle \nu, df^2(X, Y)\big\rangle.$$

This gives the following relationship of the bilinear form b_p and the shape operator S_p:

$$b(X, Y) = \big\langle \nu, d^2f(X, Y)\big\rangle = -\big\langle d\nu(X), df(Y)\big\rangle = g(SX, Y) \qquad \text{for all } X, Y \in \mathbb{R}^n, \tag{2}$$

and so by the Schwarz lemma b is bilinear or S self-adjoint. Moreover, identity (2) explains our sign choice for S.

The shape operator S is self-adjoint, and so has a basis of eigenvectors:

Theorem 1. *For each $p \in U$ there exists a g-orthonormal basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ of eigenvectors for S_p, called* principal curvature directions *[Hauptkrümmungsrichtungen], with eigenvalues $\kappa_1, \ldots, \kappa_n$, called* principal curvatures *[Hauptkrümmungen]. The eigenvalues are independent of the parameterization chosen.*

To see the last claim, note that any two representations S and $\tilde{S}$ are similar:

$$\tilde{S}_{\tilde{p}} := -(d\tilde{f}_{\tilde{p}})^{-1} \cdot d\tilde{\nu}_{\tilde{p}} = -(d\varphi_{\tilde{p}})^{-1} \cdot (df_p)^{-1} \cdot d\nu_p \cdot d\varphi_{\tilde{p}} = (d\varphi_{\tilde{p}})^{-1} \cdot S_p \cdot d\varphi_{\tilde{p}} \tag{3}$$

Remark. Another characterization of a principal curvature direction is a critical direction for the normal curvature $X \mapsto g(SX, X)$ on the g-unit sphere $\|X\|_p^2 = 1$. In particular, for dimension $n = 2$, minimal and maximal normal curvature must agree with $\kappa_{1,2}$.

Definition. The *mean curvature [mittlere Krümmung] H* is given by

$$H(p) := \frac{1}{n}\operatorname{trace} S_p = \frac{1}{n}\big(\kappa_1(p) + \ldots + \kappa_n(p)\big).$$

A surface with $H \equiv 0$ is called *minimal.*

Often, it is advantageous to consider the *mean curvature vector* $\mathbf{H}(p) := H(p)\nu(p)$, which is independent of the Gauss map chosen (why?).

Example. The sphere $\mathbb{S}^n_R$ of radius $R > 0$ with inner normal $\nu = -\frac{1}{R}f$ has $d\nu = -\frac{1}{R}df$ and thus $S = \frac{1}{R}\,\mathrm{id}$. All directions are principal, and $\kappa_i = \frac{1}{R}$, which also gives the average $H = \frac{1}{R}$.

Let us state a simple but important fact:

Proposition 2. *If f has mean curvature function H, then the dilated surface af for $a > 0$ has mean curvature function $\frac{1}{a}H$.*

Proof. The parameterization of the dilated surface $\tilde{f} = af$ satisfies $d\tilde{f} = a\,df$ and $\tilde{\nu} = \nu$. Hence $\tilde{S} = -(d\tilde{f})^{-1}d\tilde{\nu} = \frac{1}{a}(-df)^{-1}d\nu = \frac{1}{a}S$. □

Consequently, if we are interested in surfaces with constant mean curvature H, then either $H \equiv 0$ or we can achieve $H \equiv 1$ by scaling and possibly a change of orientation. Note, however, that the existence of dilations is a special feature of the ambient space $\mathbb{R}^n$.

To be able to calculate H, let us derive representations with respect to the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. We set

$$S(e_j) = \sum_{i=1}^n S^i_j e_i \qquad \text{and} \qquad b_{ij} := b(e_i, e_j) = \Big\langle \nu, \frac{\partial^2 f}{\partial x_i \partial x_j} \Big\rangle = -\Big\langle \frac{\partial \nu}{\partial x_i}, \frac{\partial f}{\partial x_j} \Big\rangle. \tag{4}$$

Then

$$b_{kj} \overset{(2)}{=} g\big(e_k, Se_j\big) = g\big(e_k, \textstyle\sum_i S^i_j e_i\big) = \displaystyle\sum_i g(e_k, e_i) S^i_j = \sum_i g_{ki} S^i_j,$$

or in matrix notation $b = gS$. So $S = g^{-1}b$ has the not necessarily symmetric matrix

$$S^i_j = \sum_k g^{ik} b_{kj}.$$

We conclude the following formulas which we will need lateron: The *Weingarten formula*

$$-\partial_j \nu = df(Se_j) = df\Big(\sum_i S^i_j e_i\Big) = \sum_{i,k} g^{ik} b_{kj} \partial_i f, \qquad j = 1, \ldots, n \tag{5}$$

and formulas for mean curvature

$$H = \frac{1}{n}\operatorname{trace} S = \frac{1}{n}\operatorname{trace}(g^{-1}b) = \sum_{i,j} \frac{1}{n} g^{ij} b_{ij}, \tag{6}$$

and Gauss curvature

$$K = \det S = \det(g^{-1}b) = \det g^{-1} \det b = \frac{\det b}{\det g}. \tag{7}$$

Remark. For a Riemannian manifold (N^{n+1}, ∇), the mean curvature of a submanifold $M^n \subset N^{n+1}$ with normal vector field ν can be similarly defined in terms of the shape operator $SX := -\nabla_X \nu$ and second fundamental form $b(X,Y) := g(SX,Y)$. Again, we can set $H := \frac{1}{n} \operatorname{trace} S$.

1.2. **Mean curvature equation for parameterized surfaces.** In dimension 2, the inverse of the metric g has the simple representation $g^{-1} = \frac{1}{\det g} \left(\begin{smallmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{smallmatrix}\right)$, and so (6) gives:

Theorem 3. *The mean curvature of a surface $f\colon U^2 \to \mathbb{R}^3$ satisfies*

$$\begin{aligned} H &= \frac{1}{2\det g}(g_{22}b_{11} - 2g_{12}b_{12} + g_{11}b_{22}) \\ &= \frac{|f_y|^2\langle f_{xx}, \nu\rangle - 2\langle f_x, f_y\rangle\langle f_{xy}, \nu\rangle + |f_x|^2\langle f_{yy}, \nu\rangle}{2\big(|f_x|^2|f_y|^2 - \langle f_x, f_y\rangle^2\big)}. \end{aligned} \tag{8}$$

Let us know introduce an assumption which simplifies the equation:

Definition. A surface $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ has a *conformal [konform]* parametrization if its first fundamental form is some multiple of the standard metric, that is, there is $\lambda\colon U \to (0,\infty)$ such that

$$g_{ij}(p) = \lambda(p)\,\delta_{ij} \qquad \text{for all } p \in U \text{ and } 1 \le i,j \le n.$$

For example, a holomorphic map is conformal and orientation preserving. Conformal parameterizations are useful in dimension two. A two-dimensional surface f is conformal if and only if $g = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ for $\lambda(x,y) > 0$, or

$$|f_x| \equiv |f_y| > 0 \qquad \text{and} \qquad \langle f_x, f_y\rangle \equiv 0.$$

According to a deep result, locally any two-dimensional surface has a conformal parameterization, see [Sy2, XII, §8, Satz 2].

Under the assumption that a parameterization is conformal the equation for mean curvature simplifies considerably:

Theorem 4. *Suppose $f\colon U^2 \to \mathbb{R}^3$ is a two-dimensional surface in conformal parameterization. Moreover assume the Gauss map is chosen positively oriented, i.e.,* $\det(f_x, f_y, \nu) > 0$. *Then f satisfies the* parametric mean curvature equation

$$\Delta f = 2H f_x \times f_y \qquad \textit{for all } p \in U; \tag{9}$$

here $\Delta f = f_{xx} + f_{yy}$ is the standard Laplacian.

Recall that the *outer* or *vector product* of any two vectors $v, w \in \mathbb{R}^3$ is given by

$$v \times w = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}.$$

The cross product is perpendicular to v and w, and $v, w, v \times w$ are positively oriented. Geometrically, $|v \times w|$ is the area content of the parallelgram spanned by v, w. In $\mathbb{R}^3$ the *Lagrange identity*

$$|v|^2 |w|^2 = \langle v, w\rangle^2 + |v \times w|^2 \qquad \text{for all } v, w \in \mathbb{R}^3, \tag{10}$$

is a quantitative version of the Cauchy-Schwarz inequality (see problems). It shows that for perpendicular v, w and oriented normal ν we must have $v \times w = |v||w|\,\nu$.

A function $u \colon \mathbb{R}^n \to \mathbb{R}$ is called *harmonic [harmonisch]* if $\Delta u \equiv 0$. Thus a conformally parametrized surface f is minimal if and only if $\Delta f \equiv 0$, that is, if and only if each of the three component functions is harmonic.

Examples. 1. The helicoid of pitch 1 has a conformal parameterization

$$f \colon \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(x, y) = (\sinh x \cos y, \sinh x \sin y, y).$$

By the theorem, the three coordinate functions are harmonic (check!). Find a conformal parameterization for the helicoid of pitch $h > 0$!

2. Let us represent an arbitrary plane $P \subset \mathbb{R}^3$. Choose two vectors v, w spanning P, such that $|v| = |w| = 1$, $\langle v, w\rangle = 0$. Then $f(x, y) := xv + yw$ is a conformal parameterization. The fact that the plane is minimal ($H \equiv 0$) is equivalent to the fact that the three coordinate functions $f^k(x, y) = xv^k + yw^k$, are harmonic.

Proof. Conformality means $|f_x| = |f_y| =: \lambda$ and $\langle f_x, f_y\rangle = 0$. Using this in (8) gives

$$H = \frac{1}{2\lambda^2}\langle f_{xx} + f_{yy},\, \nu\rangle = \frac{1}{2\lambda^2}\langle \Delta f, \nu\rangle. \tag{11}$$

To derive (9) we claim that the vectors $\Delta f = f_{xx} + f_{yy}$ and ν are parallel. To see this we differentiate the conformality conditions:

$$\begin{aligned} \frac{\partial}{\partial x}|f_x|^2 = \frac{\partial}{\partial x}|f_y|^2 \quad &\Rightarrow \quad \langle f_{xx}, f_x\rangle = \langle f_{xy}, f_y\rangle \\ \frac{\partial}{\partial y}\langle f_x, f_y\rangle = 0 \quad &\Rightarrow \quad \langle f_{yy}, f_x\rangle = -\langle f_{xy}, f_y\rangle \end{aligned}$$

Adding these identities gives $\langle f_{xx} + f_{yy},\, f_x\rangle = 0$, and similarly $\langle f_{xx} + f_{yy},\, f_y\rangle = 0$. Since (f_x, f_y, ν) is an orthogonal frame, this proves $\Delta f \parallel \nu$.

Decomposing with respect to the basis (f_x, f_y, ν) therefore gives

$$\Delta f = \langle \Delta f, \nu \rangle \nu \overset{(11)}{=} 2H\lambda^2 \nu.$$

Using the positive orientation of (f_x, f_y, ν) we prove (9) by

$$\lambda^2 \nu = |f_x||f_y|\nu = f_x \times f_y. \qquad \square$$

`3. Lecture, Tuesday 20.4.10`

1.3. **Mean curvature equation for graphs.** Let $u\colon U^n \to \mathbb{R}$, and $f(x) = \big(x, u(x)\big)$ be a graph in $\mathbb{R}^{n+1}$. There is a frame consisting of the vectors spanning the tangent space,

$$\partial_i f(x) = \begin{pmatrix} e_i \\ \partial_i u \end{pmatrix}, \qquad \text{for } i = 1, \dots, n,$$

and an upper normal

$$\nu(x) = \frac{1}{\sqrt{1 + |\nabla u(x)|^2}} \begin{pmatrix} -\nabla u(x) \\ 1 \end{pmatrix}. \tag{12}$$

Theorem 5. *Let $f\colon U \to \mathbb{R}^{n+1}$ be a graph, $f(x) = \big(x, u(x)\big)$, and let ν be its upper normal. Then the mean curvature $H = H(x)$ satisfies*

$$H = \frac{1}{n} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right). \tag{13}$$

Proof. To calculate $\operatorname{trace} S = \sum_i S_i^i$ we use the defining equation for S,

$$-\partial_i \nu = -d\nu(e_i) = df(Se_i) \overset{(4)}{=} df\Big(\sum_k S_i^k e_k\Big) = \sum_k S_i^k df(e_k) = S_i^1 \begin{pmatrix} e_1 \\ \partial_1 u \end{pmatrix} + \dots + S_i^n \begin{pmatrix} e_n \\ \partial_n u \end{pmatrix},$$

valid for $i = 1, \dots, n$. The i-th component $(i \le n)$ reads $-\partial_i \nu^i = S_i^i$ and so (12) gives

$$\partial_i \left(\frac{\partial_i u}{\sqrt{1 + |\nabla u|^2}}\right) = S_i^i.$$

Consequently, the trace can be written

$$H = \frac{1}{n} \sum_i S_i^i = \frac{1}{n} \sum_i \partial_i \left(\frac{\partial_i u}{\sqrt{1 + |\nabla u|^2}}\right) = \frac{1}{n} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right). \qquad \square$$

The theorem gives the mean curvature equation in divergence form. Upon differentiation, the standard form of this second order partial differential equation arises. For the two-dimensional case, it is easier to derive this form of the equation directly from (8). Note that

$$g_{ij} = \langle \partial_i f, \partial_j f \rangle = \left\langle \begin{pmatrix} e_i \\ \partial_i u \end{pmatrix}, \begin{pmatrix} e_j \\ \partial_j u \end{pmatrix} \right\rangle = \delta_{ij} + \partial_i u \, \partial_j u \tag{14}$$

and

$$b_{ij} = \langle \nu, \partial_{ij} f \rangle = \left\langle \frac{1}{\sqrt{1+|\nabla u|^2}} \begin{pmatrix} -\nabla u \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \partial_{ij} u \end{pmatrix} \right\rangle = \frac{\partial_{ij} u}{\sqrt{1+|\nabla u|^2}}. \tag{15}$$

Hence for the two-dimensional case of surfaces

$$g = \begin{pmatrix} 1+u_x^2 & u_x u_y \\ u_x u_y & 1+u_y^2 \end{pmatrix}, \qquad b = \frac{1}{\sqrt{1+(\nabla u)^2}} \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}.$$

Using $\det g = 1 + u_x^2 + u_y^2 = 1 + |\nabla u|^2$ we obtain from (8)

$$H = \frac{1}{2\big(1+|\nabla u|^2\big)} \frac{(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy}}{\sqrt{1+|\nabla u|^2}}$$

Note that at a point p with horizontal tangent plane, this equation becomes $2H(p) = \Delta u(p)$. In general we obtain:

Theorem 6. *A surface which is a graph* $\big(x, y, u(x,y)\big)$, *where* $u \in C^2(U^2, \mathbb{R})$ *and* $H : U \to \mathbb{R}$ *is the mean curvature at* $(x, y, u(x,y))$ *satisfies the* mean curvature equation

$$2H\big(1+|\nabla u|^2\big)^{3/2} = (1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} \qquad \textit{for all } (x,y) \in U. \tag{16}$$

This equation is a second order partial differential equation, depending on second and first derivatives of u. The function u itself does not enter: Indeed, $(x, y, u(x,y))$ and $(x, y, u(x,y) + c)$ for $c \in \mathbb{R}$ have the same mean curvature. The equation is nonlinear, meaning that if u and v satisfy (16), then $(u+v)$ need not. Nevertheless we will later prove a maximum principle.

1.4. **Problems.**

Problem 1 – Lagrange identity:

Let $x, y \in \mathbb{R}^n$ and consider the $(n \times n)$-matrix C with entries

$$c_{ij} := x_i y_j - x_j y_i = \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}.$$

a) For $\|C\|^2 := \sum_{i<j} c_{ij}^2$ half the L^2-norm of C, prove that $\|C\|^2 = |x|^2|y|^2 - \langle x, y\rangle^2$.

b) Conclude the Lagrange identity

$$|v|^2|w|^2 = \langle v, w\rangle^2 + |v \times w|^2 \qquad \text{for all } v, w \in \mathbb{R}^3.$$

c) Use a) to prove that the Cauchy-Schwarz inequality for $\mathbb{R}^n$ is attained with equality exactly when x, y are linearly dependent.

2. Variations of area and volume

In 1762 Lagrange introduced what we nowadays call the *calculus of variations [Variationsrechnung]*. He considered the area content of a surface with fixed boundary which is a graph of minimal area and showed that the graph satisfies the minimal surface equation in divergence form (13)

$$\frac{\partial}{\partial x}\frac{u_x}{\sqrt{1+u_x^2+u_y^2}}+\frac{\partial}{\partial y}\frac{u_y}{\sqrt{1+u_x^2+u_y^2}}=0.$$

Only in the 19th century was it recognized that the latter condition is precisely the condition that twice the mean curvature vanishes.

The present section introduces the variational ideas. We find it advantageous to use expansions for area and volume rather than derivatives.

2.1. **Area.** Let us review here material from Analysis IV. Given an immersion $f\colon U\subset\mathbb{R}^n\to\mathbb{R}^{n+k}$, its first fundamental form $g_{ij}=\langle\partial_i f,\partial_j f\rangle$ determines the n-dimensional area content $A(f)=A_U(f)$

$$A_U(f):=\int_U\sqrt{\det g}\,dx.$$

Remarks. 1. For an embedding $A_U(f)$ agrees with the n-dimensional measure of the set $f(U)$. However, for an immersion, $A_U(f)$ counts the area of $f(U)$ with the multiplicity it is attained, and so $A_U(f)$ can be different from the measure of $f(U)$.
2. By means of a partition of unity, we can define the area of a hypersurface or immersed manifold, in case more than one chart is needed.

Let us recall the computation proving that $A_U(f)$ remains invariant under change of parameters: For $\varphi\colon V\to U$ a diffeomorphism and $\tilde f:=f\circ\varphi$ we have according to chain rule and change of variables formula

$$\begin{aligned}A_V(\tilde f)&=\int_V\sqrt{\det(d\tilde f^T d\tilde f)}\,dx=\int_V\sqrt{\det(d\varphi^T df_\varphi^T df_\varphi d\varphi)}\,dx\\&=\int_V\sqrt{\det d\varphi^T\det(df_\varphi^T df_\varphi)\det d\varphi}\,dx=\int_V\sqrt{\det(df_\varphi^T df_\varphi)}\,|\det d\varphi|\,dx\\&=\int_U\sqrt{\det(df^T df)}\,dx=A_U(f);\end{aligned}$$

here, the notation $df_\varphi(x)$ means the differential df is evaluated at the point $\varphi(x)$.

We call $\det g=\det(df^T df)$ the *Gram determinant*. For the case $n=2$ and $f=f(x,y)$ we have $g=\begin{pmatrix}f_x\\f_y\end{pmatrix}(f_x,f_y)=\begin{pmatrix}|f_x|^2&\langle f_x,f_y\rangle\\\langle f_x,f_y\rangle&|f_y|^2\end{pmatrix}$ and so the Gram determinant reduces to

$$\sqrt{\det g}=\sqrt{g_{11}g_{22}-g_{12}^2}=\sqrt{|f_x|^2|f_y|^2-\langle f_x,f_y\rangle^2}\overset{(10)}{=}\big|f_x\times f_y\big|. \tag{17}$$

From (17) we conclude $A(f) = \int |f_x \times f_y| \, dxdy$ which says that the area of a surface can be obtained by integrating the area element $|f_x \times f_y|$.

2.2. **A lemma.**

Lemma 7. *If f is a two-dimensional surface with Gauss map ν we have*

$$\big\langle d\nu(X), d\nu(Y)\big\rangle = 2Hb(X,Y) - Kg(X,Y) \qquad \text{for all } X, Y \in \mathbb{R}^2. \tag{18}$$

The bilinear form on the left is called the *third fundamental form.*

Proof. Let κ_1, κ_2 be the principal curvatures and define the bilinear form

$$T(X,Y) := \big\langle (d\nu + \kappa_1 df)X, (d\nu + \kappa_2 df)Y \big\rangle.$$

Let X_1, X_2 be two linearly independent principal curvature directions for κ_1, κ_2. Then on the one hand

$$T(X_1, Y) = T(Y, X_2) = 0 \qquad \text{for all } Y \in \mathbb{R}^2. \tag{19}$$

On the other hand we claim T is symmetric. Indeed we can express T in terms of symmetric forms as follows:

$$\begin{aligned} T(X,Y) &= \langle d\nu(X), d\nu(Y)\rangle + \kappa_1\kappa_2\langle df(X), df(Y)\rangle + \kappa_1\langle df(X), d\nu(Y)\rangle + \kappa_2\langle df(Y), d\nu(X)\rangle \\ &= \langle d\nu(X), d\nu(Y)\rangle + Kg(X,Y) - 2Hb(X,Y). \end{aligned}$$

Since X_1, X_2 is a basis, an arbitrary vector X can be represented as $X = aX_1 + bX_2$. Therefore,

$$T(X,Y) = aT(X_1,Y) + bT(X_2,Y) \overset{T \text{ symm.}}{=} aT(X_1,Y) + bT(Y,X_2) \overset{(19)}{=} 0 \qquad \text{for all } Y,$$

meaning that T vanishes identically. Plugging this into the previous expression for T implies our claim. □

2.3. **Expansion of the area functional.**

Theorem 8. *Let $f\colon \Omega \to \mathbb{R}^3$ be a surface with Gauss map ν and $u \in C_0^1(\Omega, \mathbb{R})$ be differentiable with compact support $U := \operatorname{supp} u \subset \Omega$. Then the* normal variation

$$f^t := f + tu\nu\colon \Omega \to \mathbb{R}^3 \tag{20}$$

is an immersion for sufficiently small $|t|$, whose area has the following expansion as $t \to 0$,

$$A_U(f^t) = A_U(f) - 2t\int_U uH\,dS + t^2\int_U \frac{1}{2}\|\nabla u\|^2 + u^2K\,dS \; + O(t^3). \tag{21}$$

Here $\|\nabla u\|^2 := \sum_{i,j=1}^2 g^{ij}\partial_i u\,\partial_j u$ is the Riemannian gradient*, and $dS = \sqrt{\det g}\,d\lambda$ is the Riemannian area element of f, and H, K denote the mean or Gauss curvature of f at x.*

Example. Vary the 2-sphere S_r through spheres S_{r+t} of radius $r+t$. With respect to the outer normal ν we have $u \equiv 1$. Expanding the area content by powers of t we obtain

$$A\big(S_{r+t}\big) = 4\pi(r+t)^2 = 4\pi r^2 + t\,8\pi r + t^2 4\pi.$$

Indeed, the coefficient of t gives $2\int_{S_r} H = 2(4\pi r^2)\frac{-1}{r} = -8\pi r$, while the coefficient of t^2 gives $\int_{S_r} K = 4\pi r^2 \frac{1}{r^2} = 4\pi$.

Remarks. 1. A normal variation seems more restrictive than a general variation $f^t := f + tX$, where $X \in C^1_0(U, \mathbb{R}^n)$ is a vector field with compact support. However, for t small and upon reparametrization, any general variation can be written as a normal variation; this doesn't affect the area.

2. For $u\colon \Omega \to \mathbb{R}$ the Riemannian gradient ∇u is a tangent vector (a column), while du is a cotangent vector (a row). By definition, $g(\nabla u, X) = du(X)$ or $\sum_{jk} g_{jk}(\nabla u)^j X^k = \sum_i \partial_i u X^i$, and so $(\nabla u)^j = \sum_l g^{jl}\partial_l u$. Hence

$$\|\nabla u\|^2 = \sum_{ij} g_{ij}(\nabla u)^i(\nabla u)^j = \sum_{ijkl} g_{ij} g^{ik}\partial_k u\, g^{jl}\partial_l u = \sum_{kl} g^{kl}\partial_k u\,\partial_l u.$$

`4. Lecture, Thursday 22.4.10`

Proof. Step 1: We calculate the first fundamental form of $f^t(U)$. We have for each i

$$\partial_i f^t = \partial_i f + tu\,\partial_i \nu + t\,\partial_i u\,\nu. \tag{22}$$

Using that ν and df are orthogonal, and $b_{ij} = -\langle \partial_i f, \partial_j \nu\rangle = b_{ji}$ we obtain

$$\begin{aligned} g^t_{ij} := \big\langle \partial_i f^t, \partial_j f^t \big\rangle &= g_{ij} - 2tub_{ij} + t^2\Big(u^2\langle \partial_i\nu, \partial_j\nu\rangle + \partial_i u\,\partial_j u\Big) \\ &\overset{(18)}{=} g_{ij} - 2tub_{ij} + t^2\Big(u^2(2Hb_{ij} - Kg_{ij}) + \partial_i u\,\partial_j u\Big). \end{aligned}$$

Step 2: We calculate the determinant of g^t, that is, the coefficients of the expansion

$$\det g^t = g^t_{11}g^t_{22} - (g^t_{12})^2 = (I) + t\,(II) + t^2(III) + O(t^3).$$

Clearly,

$$(I) = g_{11}g_{22} - g_{12}^2 = \det g.$$

The first order terms are

$$(II) = -2u\big(b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{21}\big) \overset{(8)}{=} -4uH\det g$$

Finally, the quadratic terms are

$$\begin{aligned}(III) &= u^2\Big(2H(b_{11}g_{22}+b_{22}g_{11}-2b_{12}g_{12}) - K(g_{11}g_{22}+g_{22}g_{11}-2g_{12}^2) + 4(b_{11}b_{22}-b_{12}^2)\Big)\\ &\qquad + g_{22}\,\partial_1 u\,\partial_1 u + g_{11}\,\partial_2 u\,\partial_2 u - 2g_{12}\,\partial_1 u\,\partial_2 u\\ &\overset{(7),(8)}{=} u^2\big(4H^2\det g - 2K\det g + 4K\det g\big) + \det g\sum_{i,j=1} g^{ij}\,\partial_i u\,\partial_j u.\end{aligned}$$

Collecting all terms, we arrive at the desired expansion

$$\det g^t = \det g\Big[1 - 4tuH + t^2\Big(u^2\big(4H^2+2K\big) + \|\nabla u\|^2\Big)\Big] + O(t^3). \tag{23}$$

Since $\det g \neq 0$ and u has compact support this formula shows that indeed $\det g^t \neq 0$ for small $|t|$. This verifies f^t is an immersion.

Step 3: We compute the root of the determinant $\sqrt{\det g^t}$. To find the root of $[\dots]$, we use Taylor's formula

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{1}{8}x^2 + O(x^3) \qquad \text{as } x \to 0.$$

Thus (23) gives

$$\begin{aligned}\sqrt{\det g^t} &= \sqrt{\det g}\Big(1 - 2tuH + t^2u^2\big(2H^2+K\big) + \frac{t^2}{2}\|\nabla u\|^2 - 2t^2u^2H^2 + O(t^3)\Big)\\ &= \sqrt{\det g}\Big(1 - 2tuH + t^2\big(\tfrac{1}{2}\|\nabla u\|^2 + u^2K\big) + O(t^3)\Big)\end{aligned}$$

Step 4: Since $A_U(f^t) = \int_U \sqrt{\det g^t}\,dx$ we obtain the claim by integration. $\square$

Remarks. 1. For two reasons we required the variation to have compact support. First, all integrals become finite. Second, all functions being integrated then take a maximum; thus f^t is an immersion for small $|t|$. In case these two properties hold for Ω there is no need to replace Ω with U.

2. We can avoid using the Taylor expansion of the square root by noting

$$\delta g = \delta(\sqrt{g}\sqrt{g}) = 2\sqrt{g}\,(\delta\sqrt{g}) \quad\Rightarrow\quad \delta\sqrt{g} = \frac{1}{2\sqrt{g}}\,\delta g, \tag{24}$$

Moreover, $\delta^2 g = \delta^2(\sqrt{g}\sqrt{g}) = \delta(2\sqrt{g}\,\delta\sqrt{g}) = 2(\delta\sqrt{g})^2 + 2\sqrt{g}\,\delta^2\sqrt{g}$ and so

$$\delta^2\sqrt{g} = \frac{1}{2\sqrt{g}}\,\delta^2 g - \frac{1}{\sqrt{g}}\,(\delta\sqrt{g})^2 \overset{(24)}{=} \frac{1}{2\sqrt{g}}\,\delta^2 g - \frac{1}{4g^{3/2}}\,(\delta g)^2$$

For various purposes, parallel surfaces with $u \equiv 1$ are interesting. As a corollary to the variation formula (21) we find that their area behaves as follows.

Corollary 9. *Let $f\colon U \to \mathbb{R}^3$ be a surface. Then for sufficiently small $|t|$ the* parallel surface *$f^t := f + t\nu$ is an immersion, and, provided U has compact closure, we have*

$$A_U(f^t) = A_U(f) - 2t\int_U H\,dS + t^2\int_U K\,dS + O(t^3). \tag{25}$$

That means that the mean curvature at a point can be understood as the first order change in area element. This can also be verified in a completely elementary way.

2.4. Stationary surfaces and local minima.

Definition. We call a surface $f\colon \Omega \to \mathbb{R}^{n+1}$ *stationary for area* if

$$\delta_{u\nu} A_U(f) := \frac{d}{dt} A_U(f + tu\nu)\Big|_{t=0} = 0$$

for all normal variations $u \in C_0^1(\Omega, \mathbb{R})$ with $U := \operatorname{supp} u$.

Theorem 10. *A surface $f\colon U \to \mathbb{R}^3$ is stationary for area if and only if $H \equiv 0$.*

In particular, if f has minimal area compared with admissable surfaces $f + tu\nu$ then $H \equiv 0$. This leads to the name *minimal surface* for surfaces with $H \equiv 0$.

Proof. From (21) we find

$$\delta_{u\nu} A_V(f) = -2 \int_V uH \, dS \qquad \text{for all } u \in C_0^1(U, \mathbb{R}).$$

The proof then follows using the next lemma. □

Lemma 11 (Fundamental Lemma of the Calculus of Variations)**.** *Suppose $f \in C^0(U, \mathbb{R})$ and*

$$\int_U uf \, dx = 0 \qquad \textit{for all } u \in C_0^1(U, \mathbb{R}).$$

Then $f \equiv 0$.

Proof. Suppose that $f(x) \neq 0$ for some $x \in U$, say $f(x) > 0$. By continuity of f there is $\varepsilon > 0$ such that $f(y) > 0$ for all $y \in B_\varepsilon(x)$. Then choose a function u with compact support in $B_\varepsilon(x)$, such that $u(x) > 0$. The existence of such functions is not hard to show. Then $\int uf \, dx > 0$, contradicting the assumption. □

Why does the lemma fail for general, discontinuous f?

`5. Lecture, Tuesday 27.4.10` ———

2.5. **Volume.** We want to associate to a surface a reference volume, having the following properties:

• The volume need only be defined up to an additive constant since we will only be interested in measuring changes in volume.

• The volume is declared for immersed surfaces, and represents an algebraic volume to a surface, having both signs.

• We seek a volume formula in terms of a surface integral. (This is also essential for computational purposes!)

The last requirement lets us invoke the divergence theorem.

Example. One dimension lower, the Green integral formula gives for a compact domain $A \subset \mathbb{R}^2$, the vector field $x \mapsto \frac{1}{2}x$, and an oriented boundary parameterization $c\colon I \to \partial A$:

$$|A| = \int_A \operatorname{div} \frac{x}{2} = \frac{1}{2}\int_{\partial A} \langle x, \nu\rangle \, dS \overset{dS=|c'|dt}{=} \frac{1}{2}\int_I \langle c, -Jc'\rangle \, dt = \frac{1}{2}\int_I c \times c' \, dt$$

Here, $\frac{1}{2}c \times c'$ can be considered the signed area of an infinitesimal triangle with vertices $0, c(t), c(t) + \varepsilon c'(t)$; these triangles tesselate A. In case ∂A has self-intersections we can still use the right hand side to define a signed area content of c.

We use a similar idea in general dimension. The divergence theorem gives for U compact $\int_U \operatorname{div} X = \int_{\partial U} \langle X, \nu\rangle \, dS$, where ν is the outer normal. The identical vector field $\xi \mapsto \xi$ has $\operatorname{div} \xi = n$. Hence for $U \subset \mathbb{R}^n$ compact with smooth boundary

$$V(U) = \int_U 1 \, dx = \frac{1}{n}\int_{\partial U} \big\langle \xi, \nu(\xi)\big\rangle \, dS.$$

For an immersed embedded hypersurface $f\colon \overline{U}^n \to \mathbb{R}^{n+1}$ consider the cone

$$C := \{tf(x) : t \in [0,1], x \in U\} \subset \mathbb{R}^{n+1}.$$

Let us assume for a moment that also $(t, x) \mapsto tf(x)$ is an embedding, so that

$$\partial C = \big\{f(x) : x \in U\big\} \cup \big\{tf(x) : t \in [0,1], x \in \partial U\big\} =: S_f \cup S_\Lambda.$$

The boundary ∂C of the cone is smooth except at 0 and along $f(\partial U)$. But these sets have measure 0 in ∂C and so the divergence theorem is still valid for C.

The conical surface S_Λ is foliated by straight segments $t \mapsto t\xi$, and so ξ is tangent to S_Λ at $t\xi$. That means $\nu(\xi) \perp \xi$ and hence

$$\int_{S_\Lambda} \langle \xi, \nu(\xi)\rangle \, dS_\xi = 0.$$

From now on suppose that $n = 2$. Then, possibly after changing the orientation of U, we can represent the normal as

$$\nu \circ f = \frac{\partial_1 f \times \partial_2 f}{\left|\partial_1 f \times \partial_2 f\right|} \qquad \Rightarrow \qquad (\nu \circ f)\sqrt{g} = \partial_1 f \times \partial_2 f. \tag{26}$$

Consequently

$$\int_{S_f} \langle \xi, \nu(\xi)\rangle \, dS = \int_U \langle f, \partial_1 f \times \partial_2 f\rangle \, dxdy,$$

If $f\colon \overline{U}^2 \to \mathbb{R}^3$ is only an immersion and the cone not necessarily embedded, we want to use the same formula to define a volume, which now has a sign:

Definition. Let $f\colon \overline{U}^2 \to \mathbb{R}^3$ be a surface where $\overline{U}$ is compact. Then we define the (*algebraic*) *volume* of f by

$$V_U(f) := \frac{1}{3}\int_U \langle f,\ \partial_1 f \times \partial_2 f\rangle \, dxdy.$$

The functional $V(f) = \frac{1}{n}\int \langle f, *(\partial_1 f \wedge \ldots \wedge \partial_n f)\rangle$ generalizes the algebraic volume to n dimensions. For manifolds we can use a similar formula in each chart.

2.6. **Expansion of the volume functional.**

Theorem 12. *For the normal variation* (20) *the algebraic volume has the expansion*

$$V_U(f^t) = V_U(f) + t\int_U u \, dS - t^2 \int_U Hu^2 \, dS + O(t^3) \qquad \text{for } t \sim 0, \tag{27}$$

where $u \in C^1_0(\Omega, \mathbb{R})$ with $U := \operatorname{supp} u$ and H denotes the mean curvature of f at x.

Here the variation f^t and H must be calculated w.r.t. the same normal.

Example. For the sphere $\mathbb{S}^2_{r+t}$, the left hand side is $\frac{4}{3}\pi(r+t)^3 = \frac{4}{3}\pi r^3 + t4\pi r^2 + t^2 4\pi r + O(t^3)$. Indeed, these terms agree with the right hand side since $\int_U u\, dS = \int_{\mathbb{S}^2_r} 1\, dS = 4\pi r^2$ and $\int_U Hu^2\, dS = 4\pi r^2 \frac{-1}{r}$.

Proof. We insert (22) into the integrand to obtain the expansion

$$\begin{aligned}
\left\langle f^t, \partial_1 f^t \times \partial_2 f^t\right\rangle &= \Big\langle f + tu\nu,\ \big(\partial_1 f + t\,\partial_1(u\nu)\big) \times \big(\partial_2 f + t\,\partial_2(u\nu)\big)\Big\rangle \\
&= \left\langle f, \partial_1 f \times \partial_2 f\right\rangle + t\left\langle u\nu, \partial_1 f \times \partial_2 f\right\rangle + t\underbrace{\left\langle f,\ \partial_1(u\nu) \times \partial_2 f + \partial_1 f \times \partial_2(u\nu)\right\rangle}_{=:(I)} \\
&\quad + t^2 \underbrace{\left\langle f,\ \partial_1(u\nu) \times \partial_2(u\nu)\right\rangle}_{=:(II)} + t^2 \underbrace{\left\langle u\nu,\ \partial_1(u\nu) \times \partial_2 f + \partial_1 f \times \partial_2(u\nu)\right\rangle}_{=:(III)} + O(t^3).
\end{aligned}$$

Let us now calculate (I) to (III).

First, we use that the product $(a,b,c)\mapsto\langle a, b\times c\rangle$ is alternating to obtain

$$\begin{aligned}(I) &= \partial_1 \underbrace{\langle f, u\nu\times\partial_2 f\rangle}_{=:X^1} + \partial_2 \underbrace{\langle f, \partial_1 f\times u\nu\rangle}_{=:X^2} \\ &\quad - \langle \partial_1 f, u\nu\times\partial_2 f\rangle - \langle \partial_2 f, \partial_1 f\times u\nu\rangle - \langle f, u\nu\times\partial_{12} f\rangle - \langle f, \partial_{12} f\times u\nu\rangle \\ &= \operatorname{div} X + 2\big\langle u\nu, \partial_1 f\times\partial_2 f\big\rangle\end{aligned}$$

Due to the compact support of u, the divergence term will not contribute after integration. Moreover, we can rewrite the contributing term as

$$\big\langle u\nu, \partial_1 f\times\partial_2 f\big\rangle \overset{(26)}{=} u\big\langle \nu, \sqrt{g}\,\nu\big\rangle = u\sqrt{g}.$$

We now calculate the first quadratic term, using $\nu\times\nu=0$ at the first equality sign:

$$\begin{aligned}(II) &= \big\langle f,\, u\,\partial_1 u(\nu\times\partial_2\nu)\big\rangle + \big\langle f,\, u\,\partial_2 u(\partial_1\nu\times\nu)\big\rangle + \big\langle f,\, u^2\partial_1\nu\times\partial_2\nu\big\rangle \\ &= \partial_1 \underbrace{\big\langle f,\, \tfrac12 u^2\,\nu\times\partial_2\nu\big\rangle}_{=:Y^1} + \partial_2 \underbrace{\big\langle f,\, \tfrac12 u^2\,\partial_1\nu\times\nu\big\rangle}_{=:Y^2} - \big\langle \partial_1 f,\, \tfrac12 u^2\,\nu\times\partial_2\nu\big\rangle \\ &\qquad - \big\langle \partial_2 f,\, \tfrac12 u^2\,\partial_1\nu\times\nu\big\rangle - \big\langle f,\, \tfrac12 u^2\,\nu\times\partial_{12}\nu\big\rangle - \big\langle f,\, \tfrac12 u^2\,\partial_{12}\nu\times\nu\big\rangle \\ &= \operatorname{div} Y - \frac{u^2}{2}\big\langle \partial_1 f,\, \nu\times\partial_2\nu\big\rangle - \frac{u^2}{2}\big\langle \partial_2 f,\, \partial_1\nu\times\nu\big\rangle \\ &= \operatorname{div} Y + \frac{u^2}{2}\big\langle \nu,\, \partial_1 f\times\partial_2\nu + \partial_1\nu\times\partial_2 f\big\rangle \\ &\overset{(*)}{=} \operatorname{div} Y - u^2 H\sqrt{g}.\end{aligned}$$

At two places of this calculation we used the fact that $(a,b,c)\mapsto\langle a, b\times c\rangle$ is alternating. The identity $(*)$ follows from the Weingarten formula and the expression for mean curvature:

$$\begin{aligned}\partial_1 f\times\partial_2\nu + \partial_1\nu\times\partial_2 f &\overset{(5)}{=} -\partial_1 f\times\sum_j b_{2j}g^{j2}\partial_2 f - \sum_j b_{1j}g^{j1}\partial_1 f\times\partial_2 f \\ &\overset{(6)}{=} -2H\,\partial_1 f\times\partial_2 f \overset{(26)}{=} -2H\sqrt{g}\,\nu.\end{aligned} \tag{28}$$

Let us finally deal with the second quadratic term. Note first that $\nu\times\partial_2 f$ and $\partial_1 f\times\nu$ are perpendicular to ν, meaning that these terms cannot contribute to the scalar product. Thus we remain with

$$(III) = \big\langle u\nu,\, u\,\partial_1\nu\times\partial_2 f + \partial_1 f\times u\,\partial_2\nu\big\rangle \overset{(28)}{=} -2u^2 H\sqrt{g}.$$

Let us now collect all terms:

$$\big\langle f^t, \partial_1 f^t\times\partial_2 f^t\big\rangle = \big\langle f, \partial_1 f\times\partial_2 f\big\rangle + 3tu\sqrt{g} + t\operatorname{div} X - 3t^2u^2H\sqrt{g} + t^2\operatorname{div} Y + O(t^3)$$

The divergence terms have compact support and hence do not contribute to the integral. Thus upon integration we remain with the desired terms:

$$\frac{1}{3}\int_U \big\langle f^t, \partial_1 f^t \times \partial_2 f^t \big\rangle \, dxdy = \frac{1}{3}\int_U \big\langle f, \partial_1 f \times \partial_2 f \big\rangle \, dxdy + t\int_U u \, dS - t^2 \int_U Hu^2 \, dS + O(t^3)$$

□

Remark. Interestingly enough, the term $O(t^3)$ in (27) has a simple precise form, namely

$$\frac{1}{3}t^3 \int_U Ku^3 \, dS.$$

Let us include a proof of this result for completeness. First we write

$$t^3(IV) := t^3\big\langle u\nu, \partial_1(u\nu) \times \partial_2(u\nu)\big\rangle = t^3u^3\big\langle \nu, \partial_1\nu \times \partial_2\nu\big\rangle$$

For the second equality, note that if the outer product contains a factor involving ν, then it is perpendicular to ν, and so the scalar product vanishes. We can calculate this term using again the Weingarten formula:

$$\begin{aligned}
\partial_1\nu\times\partial_2\nu &= \sum_k b_{1k}g^{k1}\partial_1 f \times \sum_m b_{2m}g^{m2}\partial_2 f + \sum_k b_{1k}g^{k2}\partial_2 f \times \sum_m b_{2m}g^{m1}\partial_1 f \\
&= \Big(\big(b_{11}g^{11} + b_{12}g^{21}\big)\big(b_{12}g^{12} + b_{22}g^{22}\big) - \big(b_{11}g^{12} + b_{12}g^{22}\big)\big(b_{12}g^{11} + b_{22}g^{12}\big)\Big)\, \partial_1 f \times \partial_2 f \\
&= \Big(b_{11}b_{22}\big(g^{11}g^{22} - (g^{12})^2\big) + (b_{12})^2\big((g^{12})^2 - g^{11}g^{22}\big)\Big)\, \nu\sqrt{g} \\
&= \det b \det(g^{-1})\nu\sqrt{g} = K\nu\sqrt{g}
\end{aligned}$$

So altogether $\frac{1}{3}\int_U t^3(IV)\, dxdy = \frac{1}{3}\int_U t^3u^3K\sqrt{g}\, dxdy = \frac{1}{3}t^3\int_U u^3K\, dS.$

`6. Lecture, Thursday 29.4.10`

2.7. **Characterization of constant mean curvature surfaces.** We introduce the functional

$$J_U^h(f) := A_U(f) + 2hV_U(f) = \int_U \sqrt{g} + \frac{2}{3}h\langle f, f_x \times f_y\rangle \, dxdy.$$

Combining (21) with (27) we find the following expansion:

$$\begin{aligned}
J_U^h(f^t) &= A_U(f^t) + 2hV_U(f^t) \\
&= J_U^h(f) + 2t\int_U (h-H)u\, dS + t^2\int_U \frac{1}{2}\|\nabla u\|^2 + u^2(K - 2hH)\, dS + O(t^3)
\end{aligned} \tag{29}$$

That is, if f has constant mean curvature H,

$$J_U^H(f^t) = J_U^H(f) + t^2\int_U \frac{1}{2}\|\nabla u\|^2 + u^2(K - 2H^2)\, dS + O(t^3) \tag{30}$$

Physical interfaces, like soap films, often minimize area for given volume. Using only the first order terms of expansions for area A and volume V, we show that they have constant mean curvature, and they are critical for the functional J for reasons we explain below.

Theorem 13. *For an immersion $f\colon \Omega^2 \to \mathbb{R}^3$, the following statements are equivalent:*
(i) f has constant mean curvature H.
(ii) f is critical for area for any compactly supported variation which keeps the enclosed volume fixed up to first order, that is,

$$\delta_{u\nu} A_{\operatorname{supp} u}(f) = 0 \qquad \text{for all } u \in C_0^1(\Omega, \mathbb{R}) \text{ with } \int_{\operatorname{supp} u} u \, dS = 0.$$

(iii) There is $H \in \mathbb{R}$ such that f is critical for J^H, that is,

$$\delta_{u\nu} J^H_{\operatorname{supp} u}(f) = 0 \qquad \text{for all } u \in C_0^1(\Omega, \mathbb{R}).$$

Remarks. 1. If the volume of f^t is constant, $V_U(f^t) = V_U(f)$ where $U := \operatorname{supp} u$ then certainly $\int_U u \, dS = 0$, but not conversely.
2. Statement (*ii*) means that the surface f is a critical point of the functional $A_U(f)$ under the constraint $V_U(f) = \text{const}$. Statement (*iii*) says that for f the first variation of A and the first variation of V are parallel, $\delta J(f) = \delta A(f) + 2H\delta V(f) = 0$, meaning that $2H$ can be considered a Lagrange parameter of our variational problem under a constraint.

Proof. Throughout the proof we set $U := \operatorname{supp} u$ and write δ for $\delta_{u\nu}$.

"(*i*) $\Rightarrow$ (*iii*):" This follows from the fact that (30) has no first order term.

"(*iii*) $\Rightarrow$ (*ii*):" Consider $u \in C_0^1(\Omega, \mathbb{R})$ with $\int u \, dS = 0$. Then u is volume preserving in the sense $\delta V(f) = \int u \, dS = 0$ and so, as desired,

$$\delta A(f) = \delta J^H(f) - 2H \, \delta V(f) = \delta J^H(f) = 0$$

"(*ii*) $\Rightarrow$ (*i*):" Suppose that

$$0 = \delta A(f) = -2 \int uH \, dS \qquad \text{for all } u \text{ with } \int u \, dS = 0. \tag{31}$$

We claim that the mean curvature $H(x)$ of f at x must be constant on Ω. If not, there are $x_1, x_2 \in \Omega$ such that $H(x_1) > H(x_2)$. By continuity, we can assume that for some $r > 0$, $\varepsilon > 0$ we have

$$H(x) > c+\varepsilon \quad \text{for all } x \in B_1 := B_r(x_1) \qquad \text{and} \qquad H(x) < c-\varepsilon \quad \text{for all } x \in B_2 := B_r(x_2),$$

and that the closure of these balls is contained in Ω.

Pick a bump function $\varphi \in C_0^1\big(B_r(0), [0, \infty)\big)$ which is positive at 0 and let $\varphi_i(x) := \varphi(x - x_i)$ for $i = 1, 2$. Then

$$u(x) := \frac{1}{\sqrt{\det g}} \, \varphi_1(x) - \frac{1}{\sqrt{\det g}} \, \varphi_2(x)$$

is in $C_0^1(\Omega, \mathbb{R})$ and satisfies on the one hand

$$\int_\Omega u \, dS = \int_{B_1} \varphi_1 \, d\lambda - \int_{B_2} \varphi_2 \, d\lambda > 0$$

and on the other hand

$$\begin{aligned}\int_\Omega uH\,dS &= \int_{B_1} H(x)\varphi_1(x)\,d\lambda - \int_{B_2} H(x)\varphi_2(x)\,d\lambda\\ &> (c+\varepsilon)\int_{B_1}\varphi_1(x)\,d\lambda - (c-\varepsilon)\int_{B_2}\varphi_2(x)\,d\lambda = 2\varepsilon\int_{B_r(0)}\varphi(x)\,d\lambda \neq 0,\end{aligned}$$

in contradiction to (31) □

Remark. Consider a soap bubble which is in equilibrium. Variations of the bubble arise physically by small fluctuations caused by air streams, little motions, etc. The mean curvature H of a soap film also has a physical meaning. To see this, consider first an elastic curve. Its curvature $\kappa(t)$ corresponds to a force per length. Similarly, for a surface with surface tension, there is a resulting force (per area) at a given point p, which is given by the sum of the principal curvatures $\kappa_i(p)$,

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) \sim \frac{\text{force}}{\text{area}}.$$

To make this precise, we would need to integrate normal curvatures over direction space and see that the integral equals the mean curvature. So $H(p)$ is a force per area, or a pressure, created by the geometry. If the surface is in equilibrium, the pressure must be balanced by an ambient pressure difference. Thus $H(p)$ agrees with the pressure difference to the two side of the interface. If the interface separates gases or fluids, this pressure difference is constant at each point of the surface, and so H must be constant.

2.8. **Stability.** Our expansions for area and volume contain second order information. We can use this information to decide if a surface is a local minimum for the area or for J. To deal with both cases, let F stand for either A or J; it could be more general.

Definition. A surface $f \in C^1(\Omega, \mathbb{R}^{n+1})$ is called a *local minimum* of a functional F if for all $u \in C^1_0(\Omega, \mathbb{R})$ there exists $t_0 > 0$ such that the normal variation $f^t = f + tu\nu$ satisfies for $U := \operatorname{supp} u$

$$F_U(f) \leq F_U(f + tu\nu) \qquad \text{for all } |t| < t_0. \tag{32}$$

In particular, (32) holds for f an *absolute minimum* of F, that is in case

$$F_U(f) \leq F_U(\tilde f) \qquad \text{for all } \tilde f \in C^1(\Omega, \mathbb{R}^{n+1}) \text{ with } U := \operatorname{supp}(f - \tilde f) \subset\subset \Omega.$$

However, absolute minima are difficult to detect with the methods of analysis, while local minima are standard: From one-dimensional analysis it is known that for a critical point with $f'(x) = 0$ the condition $f''(x) > 0$ implies local minimality, while a local minimum implies $f''(x) \geq 0$. Similarly for our infinite dimensional function space $C^1(U, \mathbb{R})$:

Theorem 14. *(i) If f is critical for the functional F, and the second variation satisfies*

$$\delta^2_{u\nu} F_U(f) := \frac{d^2}{dt^2} F_U(f + tu\nu)\Big|_{t=0} > 0 \qquad \text{for all } 0 \not\equiv u \in C^1_0(\Omega, \mathbb{R}) \text{ where } U = \operatorname{supp} u,$$

then f is a local minimum for F.
(ii) At a local minimum f of the functional the second variation satisfies $\delta^2_{u\nu} F_U(f) \geq 0$ for all $u \in C^1_0(\Omega, \mathbb{R})$.

Proof. The proof follows immediately from the expansion

$$F_U(f^t) = F_U(f) + t\delta_{u\nu} F_U(f) + \tfrac{1}{2} t^2 \delta^2_{u\nu} F_U(f) + O(t^3).$$

Since $\delta F_U(f) = 0$, the second order term dominates the expansion. □

Theorem 15. *Suppose f has constant mean curvature H and let $U = \operatorname{supp} u$.*
(i) Then the second variation of area at f is

$$\delta^2_{u\nu} A_U(f) = \int_U \left\|\nabla u\right\|^2 + u^2 K \, dS \tag{33}$$

(ii) The second variation of the functional J at f is

$$\delta^2 J^H_U(f) = \int_U \|\nabla u\|^2 \, dS - 2 \int_U (2H^2 - K) u^2 \, dS \tag{34}$$

for all $u \in C^\infty_0(\Omega, \mathbb{R})$, where H, K are mean and Gauss curvature of f.

The proof is immediate from (21) and (29).

Remarks. 1. Note the obvious sign of the gradient term: $\|\nabla u\|^2 = \frac{1}{\det g} g\left(\binom{-\partial_1 u}{\partial_2 u}, \binom{-\partial_1 u}{\partial_2 u}\right) \geq 0$. On the other hand $-u^2(\kappa_1^2 + \kappa_2^2) \leq 0$.
2. We can rewrite $2H^2 - K$ in terms of principal curvatures:

$$2H^2 - K = \frac{1}{2}\left(\kappa_1 + \kappa_2\right)^2 - \kappa_1 \kappa_2 = \frac{1}{2}\left(\kappa_1^2 + \kappa_2^2\right)$$

and so (34) also reads

$$\delta^2 J^H_U(f) = \int_U \|\nabla u\|^2 - u^2(\kappa_1^2 + \kappa_2^2) \, dS = \int \|\nabla u\|^2 - \|B\|^2 u^2 \, dS. \tag{35}$$

The second form of the integral, which is true in general dimension, involves the squared Riemannian norm of the second fundamental form

$$\|B\|^2 = \kappa_1^2 + \cdots + \kappa_n^2 = \sum_{ij} (g^{ij} b_{ij})^2. \tag{36}$$

In a general ambient manifold the right hand side of (34) must also contain the term $-u^2 \operatorname{Ric}(\nu)$, see, e.g., a paper by Barbosa/do Carmo.

Example. A plane is a minimal surface which is a local minimum for area as

$$\delta^2 A = t^2 \int \|\nabla u\|^2 + u^2 K \, dS = t^2 \int \|\nabla u\|^2 \, dS > 0 \qquad \text{for } u \neq 0$$

It does not follow from our variation methods, but it is true that it is an absolute minimum for area (why?).

2.9. **Outlook on stability.** Here I collect some interesting material which I should have presented systematically and with proofs.

For a surface f let us impose the constraint that it encloses a fixed volume. As we know, then f is a critical point of the area functional if and only if it has constant mean curvature. The second variation of area (33) can then be computed to be (how?)

$$\delta^2_{u\nu} A_U(f) = \int_U \|\nabla u\|^2 - \|B\|^2 u^2 \, dS \qquad \text{for } u \in C_0^\infty(U, \mathbb{R}) \text{ with } \int_U u \, dS = 0. \tag{37}$$

That is, its form agrees with (34); note that the condition on u means that the variation is volume preserving up to first order. (To understand the problem, consider a finite dimensional analogy: Suppose we have a function $A\colon \mathbb{R}^n \to \mathbb{R}$ and we know the second derivative, i.e., the Hessian $d^2 A$. If we restrict the function A to a constraint hypersurface $S := \{x \in \mathbb{R}^n : \varphi(x) = 0\}$, then the Hessian $d^2 A|_S$ will change due to curvature of the constraint hypersurface S. In our case, S would be defined by the volume constraint.)

Using integration by parts we can rewrite the second variation formulas (34) or (33): Let Δ_f be the Laplace-Beltrami operator for f, see the definition in Sect. 7.2 below. Then $\int \|\nabla u\|^2 dS = -\int u \Delta_f u \, dS$, and so

$$\delta^2_{u\nu} A_U(f) = -\int_U u \Delta_f u + \|B\|^2 u \, dS \tag{38}$$

A surface f is called *stable* for area under a volume constraint, if (37) or (38) is ≥ 0 for all admissable u.

This sense of stability allows to explain the results of soap bubble experiments. First, the only observed single bubbles are round spheres:

Theorem (Barbosa/Do Carmo)**.** *Spheres are the only compact surfaces which are a minimum of area for given volume.*

One part of this claim is that on the sphere, (38) ≥ 0 for all volume preserving u. This follows from decomposing u into eigenfunctions (spherical harmonics) of the operator $\Delta_f + \|B\|^2 = \Delta_f + n$ on the sphere. The first eigenfunctions on the sphere are the constants; they violate the volume constraint. The second eigenspace of dimension $n+1$ comes from restricting linear functions to the sphere. These functions are induced by translations of the sphere and so should not change area but satisfy the volume constraint. Indeed, they have eigenvalue n and so (38) vanishes. All other eigenfunctions have larger eigenvalues, meaning that (38) is positive. Using the fact that eigenfunctions for different eigenvalues are pairwise orthogonal, any volume preserving u must be a linear combination of eigenfunctions with eigenvalue $\geq n$, and so (38) is nonnegative. The other part of the proof is to show that no other compact surface has the property that (38) is positive for volume preserving u.

Another interesting surface to consider is the cylinder of radius $1/(2H)$ with mean curvature $H > 0$ and $\|B\| = 1/(2H)$. From soap bubble experiments it can be observed that long cylinders are not stable for area under a volume constraint: Cylinders separate into spheres (bubbles). This instability is called the *Rayleigh instability.* In fact, this phenomenon is also observed when water comes out of a pipe, tap, or fountain: First the shape is cylindrical, but quickly drops separate. Since on a cylinder, the Laplace-Beltrami operator agrees with the standard Laplacian, this is particularly easy to check by computation: If the cylinder has height 2π, then the variation function $u(x, y) = \sin(\lambda y)$, for $y \in [0, 2\pi]$ has 0 boundary values, satisfies the volume contraint, and proves the claim (exercise!).

Remark. The first variation of H can also be computed. It turns out that the mean curvature $H^t = H(f + tu\nu)$ is

$$H^t = H + \frac{1}{2}(\Delta_f + \|B\|^2)tu + O(t^2)$$

(see, for instance a paper by Böhme and Tomi, 1973). This is the starting point for various recent constructions of surfaces with constant mean curvature by so-called perturbation techniques. In the simplest case, when the operator $\Delta_f + \|B\|^2$ is positive, the implicit mapping theorem in Banach spaces does the job. If the operator has kernel, this strategy becomes more delicate, but is still managable.

2.10. **Problems.**

Problem 2 – Parallel surfaces of a cylinder:

Let $C(r)$ be a cylinder in $\mathbb{R}^3$ with radius r.

a) Show that for any pair of points $p, q \in C(r)$ there is an isometry of $\mathbb{R}^3$ which maps p to q (is it unique?). Conclude that the Gauss curvature is constant.
b) Consider the cylinder $C_h(r)$ of radius r with height h (without the bounding disks). Insert the area of $C_h(r + t)$ and $C_h(r)$ into the expansion of area for parallel surfaces and conclude that K must vanish.

Problem 3 – Determinant expansion:

Which orders of t can the term $O(t^3)$ denote in the expansion of the determinant (23)?

Problem 4 – Graphs and minimality:

Let the graph $(x, y, u(x, y))$ represent a minimal surface. Examine which of the following graphs $(x, y, \tilde{u}(x, y))$ are also minimal:

a) $\tilde{u} = u + c$ for $c \in \mathbb{R}$,

b) $\tilde{u} = cu$ for $c \in \mathbb{R}$,

c) $\tilde{u} = cu(cx, cy)$ or $\tilde{u} = cu\big(\frac{x}{c}, \frac{y}{c}\big)$ for $c \neq 0$, where the domain is chosen suitably.

Problem 5 – Minimal Graphs:

a) Differentiate the divergence form of the mean curvature equation for graphs to obtain a second order equation in the standard form

$$nH = \sum_{1 \le i,j \le n} a^{ij}(x, u, Du)\, \partial_{ij} u.$$

Compare the result with the formula for $n = 2$ obtained in class.

b) Prove that the equation is elliptic in the following sense: Suppose $u \colon \Omega \to \mathbb{R}$ satisfies $|\nabla u| < K$. Then there exists $\lambda = \lambda(K)$ such that

$$\sum_{1 \le i,j \le n} a^{ij}(x, u, Du)\, \xi_i \xi_j > \lambda |\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\} \text{ and } x \in \Omega.$$

Problem 6 – Alternative derviation of the first variation of a graph:

Consider a hypersurface $f \colon U \to \mathbb{R}^{n+1}$ which is a graph, $f(x) = \big(x, u(x)\big)$. For simplicity, we assume U is bounded and $u \in C^2(\overline{U}, \mathbb{R})$. We set

$$J(u) := \int_U \sqrt{1 + |\nabla u|^2}\, dx + n \int_U H u\, dx,$$

where for now $H = H(x)$ is an arbitrary continuous function.

a) Prove that

$$\int_U \partial_i \eta\, dx = 0 \qquad \text{for all } \eta \in C_0^1(U, \mathbb{R}) \text{ and each } i = 1, \dots, n.$$

Conclude the law of *integration by parts* in several variables,

$$\int_U \varphi\, \partial_i \eta\, dx = -\int_U (\partial_i \varphi) \eta\, dx,$$

provided one of the two scalar functions η, φ has compact support and, for φ vector valued,

$$\int_U \sum_i \varphi_i \partial_i \eta\, dx = -\int_U \eta\, \mathrm{div} \varphi\, dx.$$

b) Calculate the first variation of J, that is,

$$\delta_\eta J(u) := \frac{d}{dt} J(u + t\eta)\Big|_{t=0}$$

for $\eta \in C_0^1(U, \mathbb{R})$. You obtain the mean curvature equation for a graph in divergence form.
Remark: This is the most elegant derivation of the mean curvature equation for graphs, which for the case of minimal surfaces, $H \equiv 0$, goes back to Lagrange. However, this derivation leaves open that H agrees with the mean curvature.

Problem 7 – Expansions of volume and J for a cylinder:

As in Problem 2, let $C_l(r)$ be a cylinder of height $l > 0$ and radius $r > 0$ (ignore the top and bottom disks).

a) Compute both sides of the volume expansion for the variation $t \mapsto C_l(r+t)$.
b) Compute similarly the expansion for the functional J^H where H is the mean curvature of $C_l(r)$.

Problem 8 – Catenary:

Consider a rope whose endpoints are fixed at two points of $\mathbb{R}^3$, under the influence of gravity, such as an electrical power line between two posts, or a railway catenary. We want to determine the shape of the rope, the so-called *catenary* (*Kettenlinie*).

We consider the vertical $\mathbb{R}^2$ containing the two points, and suppose the points are not related by a vertical translation. Moreover, we suppose the curve can be represented as a graph

$$\big\{(x, f(x)), x \in [0, b]\big\}.$$

For $0 \le t \le b$ we consider the portion $(x, f(x))$ of the curve with $0 \le x \le t$. The tangent vectors at its endpoints,

$$T_0 := \big(1, f'(0)\big) \qquad \text{and} \qquad -T_t := -\big(1, f'(t)\big),$$

correspond to the forces which pull tangentially on the catenary (sketch!). Note that the lengths of $T_0, -T_t$ are chosen in a way that the horizontal components $1, -1$ of the forces balance.

a) Formulate the force balance for the vertical components of the forces: The gravity force of the rope corresponds to $\rho > 0$ times the length of the portion of the rope considered. It agrees with the sum of the two vertical components of T_a and T_s.
b) Deduce a differential equation of second order.
c) Solve the differential equation by separation of variables (substition with a hyperbolic function!). Was Galileo correct, when he claimed in 1638 that the solution curve is a parabola?
d) Here are some suggestions for further thought:
   - Does the catenary of a suspension bridge have the same shape? Assume for simplicity that all the weight is concentrated on the bridge deck, meaning that the wiring is weightless.
   - At the mathematics museum at Giessen there is an exhibit modelling the Gateway Arch at St. Louis, Missouri, see `http://en.wikipedia.org/wiki/Gateway_Arch`. It can be assembled from building blocks without any glue. Explain its shape and how the faces of the building blocks must be chosen.
   - Will a self-supporting dome have the same cross-section? (It is claimed that St. Paul's Cathedral in London is close to be self-supporting.)

`7. Lecture, Tuesday 4.5.10 (Julia Plehnert)`

3. EXAMPLES OF MINIMAL SURFACES

To analyze some examples of minimal surfaces it is convenient to introduce another curvature concept:

Definition. Let f be a hypersurface. The quantity $\int_U K\,dS \in [-\infty, \infty]$ is called the *total curvature [Totalkrümmung]* of f.

To give the total curvature a more geometric meaning, we conclude from the definition of the shape operator

$$K(p) = \det S_p = \det\big(-(df_p)^{-1} d\nu_p\big) = (-1)^n \frac{\det d\nu_p}{\det df_p} \tag{39}$$

and therefore

$$\det d\nu_p = \pm K(p) \det df_p \qquad \Rightarrow \qquad \det(d\nu_p^T d\nu_p) = K^2(p) \det(df_p^T df_p).$$

We conclude that the area of the Gauss image in $\mathbb{S}^n$ is

$$A_U\big(\nu\big) = \int_U \sqrt{\det(d\nu^T d\nu)}\,dx = \int_U |K| \sqrt{\det(df^T df)}\,dx = \int_U |K|\,dS.$$

If f is a two-dimensional minimal surface, the Gauss curvature $K = \kappa_1\kappa_2 \le 0$, hence

$$-A_U\big(\nu\big) \overset{f \text{ minimal}}{=} \int_U K\,dS. \tag{40}$$

Note that our integrals count the area with multiplicity, i.e., taken as often as the spherical image is attained. Thus the total curvature of a minimal surface is the negative area of the spherical image, taken with multiplicity. For a general, not necessarily minimal surface, $\int K\,dS$ counts the oriented area of the Gauss image with multiplicity.

3.1. **Minimal surfaces of revolution: The catenoid.** We recall some properties about surfaces of revolution, which were discussed in detail in the lecture *Differentialgeometrie*.

Let

$$(r, h)\colon I \to \mathbb{R}_+ \times \mathbb{R} \tag{41}$$

be a regular curve. We place it in the (x, z)-plane and rotate about the z-axis. The result is a *surface of revolution [Rotationsfläche]*

$$f\colon I \times \mathbb{R} \to \mathbb{R}^3, \qquad f(t, \varphi) := \big(r(t)\cos\varphi,\ r(t)\sin\varphi,\ h(t)\big). \tag{42}$$

Lemma 16. *For a surface of revolution (42), meridians $t \mapsto f(t,\varphi)$ and latitude circles $\varphi \mapsto f(t,\varphi)$ are curvature lines, with principal curvatures*

$$\kappa_1 = \frac{r'h'' - h'r''}{\sqrt{r'^2 + h'^2}^3} \qquad \textit{and} \qquad \kappa_2 = \frac{1}{r}\frac{h'}{\sqrt{r'^2+h'^2}}\,. \tag{43}$$

Proof. We compute:

$$\partial_1 f = \begin{pmatrix} r'\cos\varphi \\ r'\sin\varphi \\ h' \end{pmatrix} \quad \text{and} \quad \partial_2 f = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix},$$

and so the first fundamental form g is

$$g = \begin{pmatrix} r'^2 + h'^2 & 0 \\ 0 & r^2 \end{pmatrix} \quad \Rightarrow \quad g^{-1} = \begin{pmatrix} \frac{1}{r'^2+h'^2} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}. \tag{44}$$

The normal of the curve, $\binom{-h'}{r'}$, normed and rotated about the z-axis, gives the surface normal

$$\nu = \frac{1}{\sqrt{r'^2+h'^2}} \begin{pmatrix} -h'\cos\varphi \\ -h'\sin\varphi \\ r' \end{pmatrix}; \tag{45}$$

it is the inner normal if h is increasing. The second derivatives of f are

$$\partial_{11} f = \begin{pmatrix} r''\cos\varphi \\ r''\sin\varphi \\ h'' \end{pmatrix}, \qquad \partial_{22} f = \begin{pmatrix} -r\cos\varphi \\ -r\sin\varphi \\ 0 \end{pmatrix}, \qquad \partial_{12} f = \partial_{21} f = \begin{pmatrix} -r'\sin\varphi \\ r'\cos\varphi \\ 0 \end{pmatrix}.$$

We obtain

$$b_{11} = \langle \partial_{11} f, \nu\rangle = \frac{r'h'' - h'r''}{\sqrt{r'^2+h'^2}}, \qquad b_{22} = \langle \partial_{22} f, \nu\rangle = \frac{rh'}{\sqrt{r'^2+h'^2}}, \qquad b_{12} = \langle \partial_{12} f, \nu\rangle = 0.$$

Note that both g and b are diagonal, so that $g^{-1}b$ is diagonal as well. Hence the coordinate directions $v_1 = \frac{\partial}{\partial t}$ and $v_2 = \frac{\partial}{\partial \varphi}$ are principal curvature directions, and their principal curvatures are $\kappa_1 = g^{11}b_{11}$ and $\kappa_2 = g^{22}b_{22}$. □

Thus a surface of revolution is minimal, $0 \equiv H = \frac{\kappa_1+\kappa_2}{2}$, if and only if

$$0 = r(r'h'' - h'r'') + h'(r'^2 + h'^2). \tag{46}$$

We call a surface M *complete [vollständig]* if there is no connected surface (of the same dimension) containing it as a proper subset. For instance, a plane in $\mathbb{R}^3$ is complete, but an open or closed disk (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is not. We say a complete surface is a surface of revolution if it is invariant under rotation (we allow for $r = 0$).

Theorem 17 (Bonnet 1860)**.** *Each complete minimal surface of revolution is either a plane or a* catenoid

$$f\colon \mathbb{R}\times[0,2\pi)\to\mathbb{R}^3,\quad f(t,\varphi):=\big(r(t)\cos\varphi,\, r(t)\sin\varphi,\, t\big) \tag{47}$$
$$\text{with}\quad r(t):=a\cosh\Big(\frac{t-t_0}{a}\Big)\ \text{for } t_0\in\mathbb{R},\quad a>0.$$

The catenoid was first described by Euler in 1744. The meridian curve cosh is a *catenary* (see problem session): This means that we can obtain the catenoid by rotating the shape of a hanging chain about an axis of a certain distance.

Proof. The special case $h\equiv$ const in (42) solves the zero mean curvature equation (46). It corresponds to a (horizontal) plane without the point $P:=(0,0,h)$ on the axis; this surface becomes complete by taking the union with P.

In case h is not identical to a constant there is a point t_0 such that $h'(t_0)\neq 0$. Then h is locally monotone, and by a reparameterization of our generating curve (41) we can assume that locally $h(t)=t$. In this case the differential equation (46) becomes

$$rr''=1+r'^2,\qquad r>0. \tag{48}$$

This ODE can be solved by a separation of variables, see [O, p.64]. Here, we will only check that (47) solves the ODE (48). Indeed, $a\cosh(.)\frac{1}{a}\cosh(.)=1+\sinh^2(.)$.

Let us consider the initial values $r(0)>0$, $r'(0)\in\mathbb{R}$ for (48). Writing the initial values in the form $r'(0)=\sinh(\frac{-t_0}{a})$ with $t_0\in\mathbb{R}$ and $r(0)=a\cosh\frac{-t_0}{a}$ with $a>0$ we see that they are satisfied by the solution family (47). By the uniqueness part of the theorem of Picard-Lindelöf these are all solutions (the system $r'=R$ and $R'=(1+R^2)/r$ is Lipshitz for $r>\varepsilon>0$). Moreover, these solutions are maximal, that is, defined for all $t\in\mathbb{R}$. Consequently, any solution of the intitial value problem with $h'(t_0)\neq 0$ for some t_0 satisfies in fact $h'(t)\neq 0$ for all t.

We have solved all initial value problems for the system (46) and $r'^2+h'^2=1$; thus all solutions are of the claimed type. □

Let us discuss two more properties of the catenoid. First we consider a boundary value problem. Take a symmetric catenoid, that is, $t_0=0$ in (47). A subset of type $f\big([-\frac{d}{2},\frac{d}{2}],[0,2\pi)\big)$ of this catenoid is bounded by two circles of radius $\cosh\tau$ which have distance d. Does any such boundary configuration bound a connected subset of some catenoid?

We call two boundary circles *coaxial* if they are contained in parallel planes, have the same radius $r>0$ and the straight line through their midpoints is perpendicular to the plane; let the distance be $d>0$. A soap film experiment tells us that for the two coaxial circles to bound a catenoid, they cannot be too far apart, or, the radius cannot be too large. As follows from Prop. 2 it is sufficient to study the case $r = 1$. In our catenoid family (47), the parameter a corresponds to scaling.

So let us assume two coaxial unit circles are spaced $d > 0$ apart. By studying the one-parameter family of scaled catenoids the following is proven in the problem session:

Proposition 18. *There is a number $d_{\max} = 1.32\ldots$ such that two coaxial unit circles in distance d bound connected subsets of the following minimal surfaces: two different catenoids for $0 < d < d_{\max}$, just one for $d_{\max}$, and none for $d > d_{\max}$.*

Consider the following surfaces of revolution, bounded by the two coaxial unit circles in distance d: A catenoid of smaller area $C_1(d)$ and a catenoid of larger area $C_2(d)$ exist for each $0 < d < d_{\max}$; they coincide when $d = d_{\max}$. Moreover, let D be two (disconnected) disks; in this context the disks are refered to as the *Goldschmidt solution.* For $d < 1.05\ldots$, the catenoid $C_1(d)$ has smaller area than the disk area 2π. The area of the catenoid $C_2(d)$ is always larger than the disk. The catenoid $C(d_{\max})$ is a regular surface with area $2\pi \cdot 1.19\ldots$; nevertheless, when d is increased further, a soap film will pop. See Nitsche [N, § 515 f] and Oprea [O, 5.6] for more information.

Let us also determine the total curvature of the catenoid. By (40) we have to consider the Gauss map:

Proposition 19. *The Gauss map of a catenoid with axis direction $e_3 \in \mathbb{S}^2$ is bijective to $\mathbb{S}^2 \setminus \{\pm e_3\}$.*

Proof. By (45), $\nu(t,\varphi) = \frac{1}{\cosh t}\big(-\cos\varphi, -\sin\varphi, \sinh t\big)$. The xy-projection of $\nu(t,\varphi)$ has angle $\varphi + \pi$. The t lines go on the sphere from the south pole ($t \to -\infty$) to the north pole $t \to \infty$; they are 1-1 since the third component $\tanh t$ is strictly monotone. Thus the t-lines have an open semicircle as their Gauss image. It follows that $t \in \mathbb{R}$, $\varphi \in [0, 2\pi)$ parameterizes $\mathbb{S}^2$ except for $\pm(0, 0, 1)$. $\square$

Therefore the total curvature of the catenoid is

$$\int_{\mathbb{R}\times[0,2\pi]} K(t,\varphi)\, dS_{(t,\varphi)} = -A_{\mathbb{R}\times[0,2\pi]}\big(\nu\big) = -4\pi.$$

3.2. **Ruled minimal surfaces: The helicoid.** Let I be an interval and $c, v \colon I \to \mathbb{R}^3$ be two curves. Then the mapping

$$f \colon \mathbb{R} \times I \to \mathbb{R}^3, \qquad f(s,t) = c(t) + sv(t). \tag{49}$$

is called a *ruled surface [Regelfläche]*. From a strict point of view, we should say ruled mapping, since f need not be a surface, i.e. immersion, at each (s,t). The curve c is called the *directrix [Leitkurve]*. The straight lines $s \mapsto f(s,t)$ are the *rulings [Regelgeraden]* of f;

they are asymptote lines. Thus a ruled surface has Gauss curvature $K \leq 0$ (wherever defined). The notion *ruled* refers to a ruler; the German *Regel* is a mistranslation of the French term *règle* (ruler).

Examples. Many classical surfaces are ruled: So is the circular cylinder or, more generally, a cylinder over an arbitrary curve, and similarly the cone over a circle or over a general curve. Note that cones are immersions only excluding their tip. It is known that the shapes a bent piece of paper in space can attain are locally ruled.

The partial derivatives of f are

$$\partial_1 f(s,t) = v(t), \qquad \partial_2 f = c'(t) + sv'(t). \tag{50}$$

If $c'(t)$ and $v(t)$ are linearly independent for all $t \in I$ then f is an immersion on an open neighborhood U of $\{s = 0\}$ in $\mathbb{R} \times I$. Nevertheless, we will not assume this condition – instead we will derive it from minimality. We call a point (s,t) of a ruled mapping nonsingular if $f(s,t)$ is an immersion in a neighbourhood of (s,t); we call f nonsingular, if it is an immersion for all $s \in \mathbb{R}$.

Discovered by Meusnier in 1776, the helicoid is a ruled surface. For instance, we can let c trace out the z-axis and v rotate in a horizontal plane. This gives an embedding $f\colon \mathbb{R}^2 \to \mathbb{R}^3$,

$$f(s,t) := c(t) + sv(t) = \begin{pmatrix} 0 \\ 0 \\ ht \end{pmatrix} + s \begin{pmatrix} \sin t \\ -\cos t \\ 0 \end{pmatrix} = \begin{pmatrix} s\sin t \\ -s\cos t \\ ht \end{pmatrix} \quad \text{for } h \neq 0. \tag{51}$$

Here, we assumed that v starts on the negative y-axis at $t = 0$. The rotation is anticlockwise for $h > 0$ ("right helicoid") and clockwise for $h < 0$ ("left helicoid").

One can compute directly that the helicoid (51) is minimal. (Do so if you have never done it before!) Let us also give a geometric argument, which works without any calculation. At any given point on the helicoid, the horizontal ruling is an ambient geodesic and hence an asymptotic direction of the surface. But the orthogonal direction must also be an asymptotic direction, as by construction the helicoid is invariant under 180°-rotation about the horizontal geodesic. We note that this argument, based on symmetries, works for helicoids in any ambient space for which 180°-rotation about the geodesic ruling is an isometry: for instance for helicoids in the space forms $\mathbb{S}^3$ or $\mathbb{H}^3$.

There is a uniqueness theorem for the helicoid:

Theorem 20 (Catalan 1842)**.** *Each complete ruled minimal surface is nonsingular and either the plane or the helicoid.*

Due to time constraints we skip the proof, but it may be found in [DKHW].

The helicoid (51) has the following properties:

• It is *simply periodic*, that is, any translation by $k(0,0,2\pi h)$, $k \in \mathbb{Z}$, leaves it invariant.
• It is symmetric under rotation by any angle $\varphi \in \mathbb{R}$ followed by a vertical translation by $(0,0,h\varphi)$.
• The helicoid has infinite total curvature, but total curvature -4π in a fundamental domain $f(\mathbb{R},[0,2\pi])$.

All but the last property are straightforward. To calculate the total curvature, we point out a stunning relationship between catenoid and helicoid. To introduce it, we use the conformal representations.

Theorem 21. *Each surface in the* associated family

$$f_\vartheta\colon \mathbb{R}^2 \to \mathbb{R}^3, \qquad f_\vartheta(x,y) = \cos\vartheta \begin{pmatrix} \cosh x \cos y \\ \cosh x \sin y \\ x \end{pmatrix} + \sin\vartheta \begin{pmatrix} \sinh x \sin y \\ -\sinh x \cos y \\ y \end{pmatrix},$$

where $\vartheta \in \mathbb{R}$, is minimal. Moreover, f_0 parametrizes a catenoid (of waist radius 1) and $f_{\pi/2}$ a helicoid (with vertical period 2π); for $y \in [0,2\pi)$ the entire catenoid and a period of the helicoid is parameterized. All surfaces in the family are isometric, that is the first fundamental form is independent of ϑ, and the normal is independent of ϑ, too.

All our claims can be verified by calculation. Since catenoid and a fundamental piece of the helicoid are isometric, they have the same Gaussian curvature and hence the same total curvature

$$\int_{[0,2\pi]\times\mathbb{R}} K(t,\varphi)\, dS_{(t,\varphi)} = -4\pi.$$

The simple periodicity implies that the total curvature of the entire helicoid must be infinite.

`8. Lecture, Thursday 6.5.10 (Julia Plehnert)`

3.3. **Enneper's surface has an intrinsic rotation.** The minimal surface

$$f\colon \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(x,y) = \begin{pmatrix} x - \frac{1}{3}x^3 + xy^2 \\ -y + \frac{1}{3}y^3 - yx^2 \\ x^2 - y^2 \end{pmatrix}$$

was discovered by Enneper in 1864. To see it is minimal, let us calculate

$$f_x = \begin{pmatrix} 1 - x^2 + y^2 \\ -2xy \\ 2x \end{pmatrix}, \quad f_y = \begin{pmatrix} 2xy \\ -1 + y^2 - x^2 \\ -2y \end{pmatrix}.$$

Thus

$$\begin{aligned}|f_x(x,y)|^2 &= \big(1+x^4+y^4-2x^2+2y^2-2x^2y^2\big)+4x^2y^2+4x^2\\ &= 1+x^4+y^4+2x^2+2y^2+2x^2y^2=(1+x^2+y^2)^2=|f_y(x,y)|^2\end{aligned}$$

and $\langle f_x, f_y\rangle = 0$. We conclude

$$g=\begin{pmatrix}(1+x^2+y^2)^2 & 0\\ 0 & (1+x^2+y^2)^2\end{pmatrix}=\begin{pmatrix}(1+r^2)^2 & 0\\ 0 & (1+r^2)^2\end{pmatrix}, \tag{52}$$

where $r := \sqrt{x^2+y^2}$. Thus f is conformal, and hence to check it is minimal, by (9) it suffices to assert $\Delta f = (-2x+2x, 2y-2y, 2-2) = 0$.

To understand the geometry of the Enneper surface, let us first point out that

$$f(\pm x, x) = \Big(\pm(x+\frac{2}{3}x^3), -(x+\frac{2}{3}x^3), 0\Big),$$

meaning that the surface contains the two horizontal diagonals. On the other hand, $f(x,0) = (x-\frac{1}{3}x^3, 0, x^2)$; at $x=0$ is asymptotic to the graph of a parabola, while at infinity it is asymptotic to the graph of $(.)^{2/3}$. Since $f(\pm\sqrt{3},0) = (0,0,3)$, the Enneper surface is not embedded. In fact, the selfintersections make it hard to visualize (see [DHKW], p.146 for images, or generate them by yourself!).

To analyse Enneper's surface at 0 and infinity, we use the polar coordinate representation

$$f\big(r\cos t, r\sin t\big) = \begin{pmatrix} r\cos t - \frac{1}{3}r^3\cos^3 t + r^3\cos t\sin^2 t\\ -r\sin t + \frac{1}{3}r^3\sin^3 t - r^3\cos^2 t\sin t\\ r^2(\cos^2 t - \sin^2 t)\end{pmatrix} = \begin{pmatrix} r\cos t - \frac{1}{3}r^3\cos 3t\\ -r\sin t - \frac{1}{3}r^3\sin 3t\\ r^2\cos 2t\end{pmatrix}.$$

To see this, remember $\cos 3t = \operatorname{Re} e^{3it} = \operatorname{Re}(e^{it})^3 = \operatorname{Re}(\cos t + i\sin t)^3 = \cos^3 t - 3\cos t\sin^2 t$ and $\sin 3t = \operatorname{Im} e^{3it} = \operatorname{Im}(e^{it})^3 = \operatorname{Im}(\cos t + i\sin t)^3 = -\sin^3 t + 3\cos^2 t\sin t$.

Consider the circle $\gamma_r(t) = (r\cos t, r\sin t)$ of radius $r>0$. First, when r is small, we have

$$f\big(\gamma_r(t)\big) = \big(r\cos t, -r\sin t, r^2\cos 2t\big) + O(r^3) \qquad \text{as } r\to 0,$$

meaning that the image circle is asymptotically in the horizontal xy-plane. It is parameterized clockwise. The sign of the z-values alternates four times or every 90°; it is zero only on the diagonals.

Second, to study the image of the circle $\gamma_r(t)$ for large radius r, note that

$$f\big(\gamma_r(t)\big) = \Big(-\frac{1}{3}r^3\cos 3t,\ -\frac{1}{3}r^3\sin 3t,\ r^2\cos 2t\Big) + O(r) \qquad \text{as } r\to\infty.$$

In particular, $|f(\gamma_r(t))| = \frac{1}{3}r^3 + O(r^2)$, meaning that the image "circles" are asymptotically round. Consequently, the radial projection of the image circle onto $\mathbb{S}^2$ reads

$$\frac{f(\gamma_r(t))}{|f(\gamma_r(t))|} = \Big(-\cos 3t + O(\tfrac{1}{r}), -\sin 3t + O(\tfrac{1}{r}),\ \frac{3}{r}\cos 2t + O\big(\tfrac{1}{r^2}\big)\Big) \qquad \text{as } r\to\infty.$$

The third component is small, and so the image of a large circle is again asymptotically a horizontal round circle. However, it runs 3 times round anticlockwise! The sign of the third component indicates to which side of the horizontal xy-plane the surface sits: this time it takes exactly 270° for the image to change sign.

To explain an important property of Enneper's surface, we note that the first fundamental form $(1+r^2)^2 \operatorname{id}$ depends on r alone. In geometric language this yields:

Proposition 22. *Let R_α denote the rotation of $\mathbb{R}^2$ by an angle $\alpha \in \mathbb{R}$. The Enneper surface has an intrinsic rotation, that is f and $f \circ R_\alpha$ are isometric for each $\alpha \in \mathbb{R}$.*

Imagine a plaster model of Enneper's surface as well as a copy of the actual surface made from thin metal; they must represent an embedded portion of the surface, of course. Then the metal copy can be rotated on the surface; while doing so, it will change shape. This describes the associated family; it is obtained by rotating Enneper's surface around the z-axis.

Brian Smyth in the 1980's determined all minimal surfaces admitting intrinsic isometries; precisely Enneper's surface, as well as certain analogues with higher dihedral symmetry, have an intrinsic rotation.

Let us mention two more properties:
1. The Enneper surface is algebraic, i.e. there is a a polynomial of 9-th order (see [N, p.77]), such that the Enneper surface is its zero set. Note that any surface can be represented implicitely; the point here is that the function is not transcendental but a polynomial.
2. The Gauss map is injective; it misses only the point $-e_3 \in \mathbb{S}^2$. The total curvature is -4π.

References. [N], p.75–81

3.4. **Scherk's doubly periodic surface.** What are the interesting examples of minimal graphs, $f(x,y) = (x, y, u(x,y))$? As proven by Bernstein, if u is defined on all of $\mathbb{R}^2$, the only such surfaces are planes (u is linear). Nevertheless there are interesting graphs defined over certain subsets of the plane.

Along with four other minimal surfaces, Scherk in 1835 described a graph defined over the open square $Q := \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by

$$u\colon Q \to \mathbb{R}, \qquad u(x,y) := \log \frac{\cos y}{\cos x}.$$

To prove Scherk's surface is minimal, let us check it satisfies the mean curvature equation for graphs (16),

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0.$$

Indeed, since

$$u_x = \frac{\sin x}{\cos x}, \quad u_y = -\frac{\sin y}{\cos y}, \quad u_{xx} = \frac{1}{\cos^2 x}, \quad u_{yy} = -\frac{1}{\cos^2 y}, \quad u_{xy} = 0$$

we have

$$\frac{1 + \frac{\sin^2 y}{\cos^2 y}}{\cos^2 x} - \frac{1 + \frac{\sin^2 x}{\cos^2 x}}{\cos^2 y} = \frac{\cos^2 y + \sin^2 y - (\sin^2 x + \cos^2 x)}{\cos^2 x \cos^2 y} = 0.$$

The only intersection of the surface with the xy-plane, that is, the only zeros of u, are the two horizontal diagonals $(\pm x, x, 0)$. When (x, y) tends to one of the four (open) boundary edges of the square Q, the function u has limiting values $+\infty$ or $-\infty$. Indeed, if $(x, y) \to (\frac{\pi}{2}, y_0)$, where $|y_0| < \frac{\pi}{2}$, then $\log\cos y - \log\cos x \to \log\cos y_0 + \infty = \infty$.

It remains to discuss limits at the four vertices. We claim that the closure of the graph contains the four vertical lines $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2}) \times \mathbb{R}$. To see that, fix $\lambda > 0$ and note

$$u\Big(\frac{\pi}{2} - \eta, \frac{\pi}{2} - \xi\Big) = \log\frac{\sin\xi}{\sin\eta} = \log\frac{\xi - \frac{1}{3!}\xi^3 + \dots}{\eta - \frac{1}{3!}\eta^3 + \dots} \;\to\; \log\lambda \qquad \text{as } \xi = \lambda\eta \searrow 0.$$

But $\log\colon (0, \infty) \to \mathbb{R}$ is surjective, implying that this limit can attain any value in $\mathbb{R}$. This shows that any point on the four vertical lines is a limit point of a sequence of points on the graph.

The surface extends to all (x, y) such that $0 < \frac{\cos y}{\cos x} < \infty$, that is, to the black squares of a suitable chequerboard. Let us consider the closure of this surface. This means to join the set of vertical lines over the vertices of this chequerboard. The closure is complete. One can show that the extended surface is smooth at the vertical lines, and that its mean curvature vanishes. Thus we have constructed a complete minimal surface. This surface is *doubly periodic*, i.e. invariant under any translation by $(k \cdot 2\pi, l \cdot 2\pi, 0), (k, l) \in \mathbb{Z}^2$.

References. [N] p.63/64, [DHKW] p.151–159

3.5. **Problems.**

Problem 9 – Geometric Intuition:

a) Consider a catenoid with vertical axis. How does it intersect horizontal planes, vertical planes through the origin, (centered at the origin)?
b) How does a helicoid with vertical axis intersect horizontal planes, vertical planes through the origin, a sphere of radius r (centered at the origin)?
c) How does the xy-plane intersect Enneper's surface?

Problem 10 – Scaled catenoids:

Consider a catenoid with vertical axis,

$$f\colon \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(t,\varphi) := \big(\cosh t\cos\varphi,\, \cosh t\sin\varphi,\, t\big),$$

positioned symmetrically w.r.t. the horizontal xy-plane. Then we scale the catendoid, $f_\lambda(t,\varphi) := \lambda f(t,\varphi)$, where $\lambda > 0$, and denote the image surface by $M_\lambda := f_\lambda\big(\mathbb{R},[0,2\pi)\big) \subset \mathbb{R}^3$.

a) Determine an open cone $K \subset \mathbb{R}^3$ about the z-axis with tip at the origin, which is maximal regarding the following property: For all $\lambda > 0$ and $x, y \in \mathbb{R}^2$ is $f_\lambda(x,y) \notin K$. What is the aperture angle α of K?
b) Determine the number $d_{\max} > 0$ such that any pair of coaxial unit circles is the boundary of a connected set of some catenoid M_λ precisely if the distance satisfies $d \le d_{\max}$. Express $d_{\max}$ in terms of α.
c) Do the catenoids M_λ have a limit for $\lambda \to \infty$, that is, can a sequence of points $\lambda f(t_\lambda, \varphi_\lambda)$ converge?
d) Let L_x be the vertical line $L_x := \{(x,0,z), z \in \mathbb{R}\}$. Prove: The intersectin point $(x,0,z)$ of the upper catenoid half $f_\lambda\big([0,\infty),0\big)$ with the line L_x defines a unique solution $z = z(\lambda,x) \ge 0$ for $0 < \lambda \le x$.
e) The function $\lambda \mapsto z(\lambda,x)$ can be extended continuously to 0 by $\lim_{\lambda\to 0} z(\lambda,x) = 0$.
f) Two coaxial unit circles in distance $d < d_{\max}$ are the boundary of a connected subset of a catenoid M_λ for exactly two values of λ.
g) Use part e) to discuss the limit of M_λ for $\lambda \to 0$. That is, find a maximal subset M of $\mathbb{R}^2$ with the property: For each $p \in M$ there is a sequence $p_\lambda \in M_\lambda$, such that $\lim_{\lambda\to 0} p_\lambda = p$.
h) Consider the convergence of the previous part. Do the normals converge as well (so-called *C^1-convergence*)? *Hint:* How does the normal of M_λ behave at the intersection with L_x?

9. Lecture, Tuesday 11.5.10: Rob Kusner (Univ. of Mass. at Amherst, USA)

Problem 11 – Force balancing for the catenoid:

Let M be a minimal surface, and $K \subset M$ be a compact subset with piecewise smooth boundary (and non-empty interior); the boundary ∂K then is the union of smooth loops. Rob Kusner discussed in his lecture the principle of *force balancing [Kräftegleichgewicht]*: the total *force*

$$F(\partial K) := \int_{\partial K} \eta\, ds = 0.$$

Here η is the *exterior conormal*, that is, η is a unit tangent vector of M which is normal to ∂K. We now apply this principle specifically to minimal surfaces of revolution.

a) Specify $K = K(t)$ in a way to prove that for a minimal surface of revolution $f(t,\varphi) = \big(r(t)\cos\varphi, r(t)\sin\varphi, h(t)\big)$ the function $t \mapsto \int_{C(t)} \eta\, ds$ over the circle $C(t) = \{f(t,\varphi) : 0 \le \varphi < 2\pi\}$ is constant.

b) Check with the explicit representation of the catenoid $r(t) = a\cosh(t/a)$ that the force is independent of the circle chosen.
c) Conversely, derive the ODE and perhaps the representation of the catenoid (or plane) from the principle of force balancing.

The same principle for nonzero constant mean curvature can be used to give a qualitative discussion of the Delaunay surfaces. It also applies to higher dimensions or other ambient spaces.

Problem 12 – Force balancing for Scherk's doubly periodic surface:

Consider Scherk's doubly periodic surface over one square, that is, the graph $u(x, y) = \log(\cos y/\cos x)$ over $(x, y) \in (-\pi/2, \pi/2)^2$.

a) What is the vertical component of the force over the intersection of a horizontal plane P_z at height z with the surface (choose η with $\eta^3 \geq 0$)? Do not engage in a calculation! What can you say about the horizontal component of the force for the intersection curve with a vertical axis-parallel plane?
b) Scherk's surface is a graph over the square $(-\pi/2, \pi/2)^2$ with boundary values alternating between $+\infty$ and $-\infty$. Similar surfaces exist over more general quadrilaterals *[Vierecke]*. Derive a necessary condition the quadrilaterals must satisfy from force balancing. (The condition is sufficient by work of Jenkins and Serrin from the 60's.)

`10. Lecture, Tuesday 18.5.10`

4. CONSTANT MEAN CURVATURE SURFACES OF REVOLUTION

We discuss the surfaces of revolution with nonzero constant mean curvature which were first described by Delaunay 1841. I used lecture notes by K. Steffen to prepare this section. Other references include: Oprea, Differential Geometry, Prentice Hall 1997, Sect. 3.6; J. Eells, The surfaces of Delaunay, Math. Intelligencer 1997; and web sites (see below).

4.1. **Analytic discussion of the ODE.** From (43) we recall that a surface of revolution

$$f\colon I \times \mathbb{R} \to \mathbb{R}^3, \qquad f(t,\varphi) := \big(r(t)\cos\varphi,\, r(t)\sin\varphi,\, h(t)\big), \qquad r>0,$$

has mean curvature

$$2H = \frac{r'h'' - h'r''}{(r'^2+h'^2)^{3/2}} + \frac{1}{r}\frac{h'}{\sqrt{r'^2+h'^2}}\,. \tag{53}$$

For physics reasons, namely by the principle of force conservation,

$$t \mapsto \left(2\pi r \frac{h'}{\sqrt{r'^2+h'^2}}\right)' \tag{54}$$

is constant for $H \equiv 0$. Indeed, the generating curve (r,h) has tangent vector $(r',h')/\sqrt{r'^2+h'^2}$ and along a circle of radius $r(t)$ the vertical component $\frac{h'}{\sqrt{r'^2+h'^2}}$ is constant. The integral of the conormal over the circle of length $2\pi r$ agrees with the integral over the constant vertical component, and so $2\pi r \cdot \frac{h'}{\sqrt{r'^2+h'^2}}$ is constant in r.

Let us calculate (54) under the assumption that H is constant:

$$\begin{aligned}\left(\frac{rh'}{\sqrt{r'^2+h'^2}}\right)' &= \frac{(r'h'+rh'')(r'^2+h'^2) - rh'(r'r''+h'h'')}{(r'^2+h'^2)^{3/2}}\\ &= \frac{r'h'(r'^2+h'^2) + rh''r'^2 - rh'r'r''}{(r'^2+h'^2)^{3/2}}\\ &= rr'\Big(\frac{r'h''-h'r''}{(r'^2+h'^2)^{3/2}} + \frac{h'}{r\sqrt{r'^2+h'^2}}\Big) = 2rr'H = (r^2H)'\end{aligned}$$

This implies, upon integration,

$$\frac{rh'}{\sqrt{r'^2+h'^2}} - Hr^2 = \mathrm{const}\,. \tag{55}$$

Thus from the second order equation (53) we have derived the first order equation (55); it is called a *first integral* of the second order ODE.

Remark. We can identify (55) with a geometric quantity. After multiplying with 2π the first term becomes the integral of the conormal over the circle $C(r)$ generated by $(r(t), h(t))$, while the second term becomes $2H$ times the integral of the normal over the disk $D(r)$ of area πr^2 spanned by this circle. That is,

$$r \mapsto 2\pi \cdot (55) = \int_{C(r)} \eta \, ds - 2H \int_{D(r)} \nu \, dA$$

This result can be shown to hold in much larger generality. Note also that for $H \equiv 0$ our formula reduces to the integral of the conormal being preserved.

Let us get back to (55). As long as $h' \neq 0$, we could parameterize solution curves as $h(t)$, say with $h' > 0$, such that $\frac{r}{\sqrt{r'^2+h'^2}} - Hr^2$ becomes a constant, say a, and so the equation will be separable. However, only for $H \equiv 0$ does this lead to the explicit solutions $r(t) = a\cosh((t-t_0)/a)$, which are catenoids, or to the planes $h \equiv \text{const}$.

Instead we will choose the curve (r, h) parameterized such that

$$h'(t) = r(t),$$

and we assume the mean curvature $H \in \mathbb{R} \setminus \{0\}$ is constant. This is possible on solution arcs with $h'(t) \neq 0$. Note that the case $h' \equiv 0$ will not arise for $H \not\equiv 0$. Assuming the special parameterization, we take the derivative of (55),

$$0 = \left(r\frac{r}{\sqrt{r'^2+r^2}} - Hr^2 \right)' = r\left(\left(\frac{r}{\sqrt{r'^2+r^2}} - 2Hr \right)' + r' \frac{1}{\sqrt{r'^2+r^2}} \right). \tag{56}$$

On the other hand,

$$\begin{aligned} \left(\frac{r'}{\sqrt{r'^2+r^2}} \right)' &= \frac{r''(r'^2+r^2) - r'(rr'+r'r'')}{(r'^2+r^2)^{3/2}} = \frac{r''r^2 - r'^2 r}{(r'^2+r^2)^{3/2}} = r\frac{r''h' - h''r'}{(r'^2+h'^2)^{3/2}} \\ &\overset{(53)}{=} \frac{h'}{\sqrt{r'^2+h'^2}} - 2Hr = \frac{r}{\sqrt{r'^2+h'^2}} - 2Hr \end{aligned} \tag{57}$$

To eliminate h, let us now take one more derivative of (57) and replace the right hand side using (56) divided by r:

$$\left(\frac{r'}{\sqrt{r'^2+r^2}} \right)'' = -\frac{r'}{\sqrt{r'^2+r^2}}.$$

Up to translation in t, this second order ODE takes the solutions

$$\frac{r'}{\sqrt{r'^2+r^2}} = -\varepsilon \sin t, \tag{58}$$

where $\varepsilon \in \mathbb{R}$ is an integration parameter. Differentiating (58) and rewriting the left hand side in terms of (57) gives

$$\frac{r}{\sqrt{r'^2+r^2}} = 2Hr - \varepsilon \cos t. \tag{59}$$

To eliminate r' from the equation we add the squares of (58) and (59):

$$1 = \frac{r'^2 + r^2}{(\sqrt{r'^2 + r^2})^2} = \varepsilon^2 \sin^2 t + (2H)^2 r^2 + \varepsilon^2 \cos^2 t - 2(2H) r\varepsilon \cos t$$

$$\Leftrightarrow \qquad 0 = 2Hr^2 - 2\varepsilon(\cos t)r + \frac{1}{2H}(\varepsilon^2 - 1).$$

This quadratic equation in r has the solutions

$$r_{1,2} = \frac{\varepsilon \cos t}{2H} \pm \frac{1}{2H}\sqrt{\varepsilon^2 \cos^2 t + 1 - \varepsilon^2} = \frac{\varepsilon}{2H} \cos t \pm \frac{1}{2H}\sqrt{1 - \varepsilon^2 \sin^2 t}. \tag{60}$$

We may assume that $\varepsilon \geq 0$ and $H > 0$, since a sign change of the first term can as well be affected by $t \mapsto t + \pi$. We want to discuss maximal solution branches in terms of ε and H.

Please check that the above calculations can be reversed so that $r(t)$ indeed solves the equation for constant mean curvature. Except for the special cases $\varepsilon = 0$ or 1, the solutions $h(t) = \int^t r(s)\, ds$ have no elementary representation. Hence we can only give a qualitative discussion.

• For $\varepsilon = 0$ we must have the +-sign in (60). The solution $r(t) \equiv \frac{1}{2H}$ is defined for $t \in \mathbb{R}$ and generates a cylinder of radius $\frac{1}{2H}$.

• For $\varepsilon = 1$ we also must have the +-sign, and a solution with $r > 0$ can only be defined on intervals such as $t \in (-\pi/2, \pi/2)$:

$$r(t) = \frac{1}{2H} \cos t + \frac{1}{2H} |\cos t| = \frac{1}{H} \cos t$$

From $h'(t) = r(t)$ we obtain $h(t) = \frac{1}{H} \sin t$, so that a sphere of radius $\frac{1}{H}$ is parameterized.

• For $0 \leq \varepsilon < 1$ we have again the +-sign, and the solution (60) is defined for all $t \in \mathbb{R}$. The solutions generate the *unduloids*. They are simply periodic and, due to the fact h is increasing on the ODE solution, the unduloids are embedded. The value of r varies from $r_{\min} = (1 - \varepsilon)/2H$ (at $t = \pm\pi$) to $r_{\max} = (\varepsilon + 1)/2H$ (at $t = 0$), that is, the difference is $r_{\max} - r_{\min} = \varepsilon/H$. The unduloid with $\varepsilon = 0$ agrees with a cylinder.

• For $\varepsilon > 1$ there are still smooth solution curves $\big(r(t), h(t)\big)$ generating smooth periodic, surfaces of revolution with self-intersections, called *nodoids*. Unfortunately, our approach does not yield a continuous parameterization of the generating ODE-curves; instead, different arcs must be patched together. Indeed, for $\varepsilon > 1$ the map $t \mapsto h(t)$ has critical points and so no longer is monotone along solution curves. There are two geometrically different solution branches, namely for $t \in (-\arcsin(1/\varepsilon), \arcsin(1/\varepsilon))$ and for $t \in (-\arcsin(1/\varepsilon) + \pi, \arcsin(1\varepsilon) + \pi)$; the first term in (60) has a different sign on these two intervals. If a singular point with $|t| = \arcsin(1/\varepsilon)$ is approached, then $h' \to 0$, which means the tangent (r', h') becomes horizontal. Given our description, we need to piece together smooth arcs, after vertical translation. This gives a complete generating curve. For $\varepsilon \to \infty$ the generating curve converges to the circle of radius $1/|H|$ at ∞ (check details!).

Using the local uniqueness theorem shows that zeros of h' are isolated when $H \neq 0$, and that through each point there is at most one solution arc. This shows that we have arrived at a complete description of all constant mean curvature surfaces of revolution.

`11. Lecture, Thursday 20.5.10` ______________________________

4.2. **Geometric discussion of the ODE.** Although the ODE for the generating curve of a constant mean curvature surface of revolution has no explicit solution, there is a very explicit geometric description of its solutions, which is due to Delaunay and Sturm.

To set it up, we need the notion of a roulette *[Rollkurve]*. Consider a curve $c\colon I \to \mathbb{R}^2$, together with the origin of the $\mathbb{C}$-plane. When the curve roles (tangentially) along the y-axis of the (r,h)-plane, the origin traces out another curve $\gamma(t)\colon I \to \mathbb{R}^2$ which we want to analyze now.

To set up the problem, we describe the roling in terms of a family of motions. We decompose the motion into translation and rotation; to describe the rotation it is convenient to use a complex model for the plane. Thus we have

$$B_t\colon \mathbb{C} \to \mathbb{C}, \qquad B_t(w) := \gamma(t) + e^{i\varphi(t)} w. \tag{61}$$

The motion B_t is subject to the following conditions:
1. The point $c(t)$ is mapped to some point $i\sigma(t) \in i\mathbb{R}$ on the y-axis. Since the roling is without slip *[Schlupf]*, the distance travelled by σ agrees with arclength of c.
2. The image of c is tangential to the y-axis.
Thus B is defined by

$$B_t(c(t)) = i\sigma(t) := i\int^t |c'|, \qquad \text{subject to} \qquad e^{i\varphi(t)}c'(t) \in i\mathbb{R}.$$

Our goal is to determine $\gamma(t) := B_t(0)$.

It becomes easier to describe γ if we assume that c has a curvature with one sign, that is, on locally strictly convex arcs. Then $c'(t)$ is locally injective on $\mathbb{S}^1$ and we can reparameterize such that

$$c'(t) = |c'(t)|e^{-it},$$

by invoking the one-dimensional inverse function theorem. Clearly, $\varphi(t) = \pi/2 + t$ and so

$$\gamma(t) \overset{(61)}{=} B_t(c(t)) - e^{i(\pi/2+t)}c(t) = i\big(\sigma(t) - e^{it}c(t)\big) \quad \Rightarrow \quad e^{it}c(t) = \sigma(t) + i\gamma(t). \tag{62}$$

From this we conclude

$$\gamma'(t) = i\sigma' + e^{it}c - ie^{it}c' = i|c'| + e^{it}c - i|c'| = i\gamma + \sigma.$$

Taking real and imaginary parts yields the first formula of the following statement:

Lemma 23. *Let $c\colon I \to \mathbb{R}^2$ be a curve with positive curvature, parameterized such that $c'(t) = e^{-it}|c'(t)|$, and let $\sigma(t) = \int^t |c'|$ be arclength of the curve $c(t)$. Then the roulette $\gamma(t) = (r(t), h(t))\colon I \to \mathbb{R}^2$ of the point $0 \in \mathbb{R}^2$ satisfies the ODE system*

$$h' = r, \qquad r' = -h + \sigma$$

and c has the representation

$$c(t) = r'e^{-it} + ire^{-it}. \tag{63}$$

Proof. It only remains to prove the second representation:

$$c(t) \overset{(62)}{=} ie^{-it}\gamma + e^{-it}\sigma = ie^{-it}(r + ih) + e^{-it}\sigma = (-h + \sigma)e^{-it} + ire^{-it}$$

□

Theorem 24. *The generating planar curves for the Delaunay surfaces of revolution with constant mean curvature $H \neq 0$ are roulettes of a focus of the conic sections with long semi major-axis $a = \frac{1}{|H|}$ and excentricity $\varepsilon = \frac{1}{a}\sqrt{a^2 + b^2}$.*

Hence the unduloids arise from ellipses and the nodoids from hyperbolas. In the latter case, the branches of the hyperbola switch in the asymptotic position, see
`http://www.mathcurve.com/courbes2d/delaunay/delaunay.shtml`

Examples. 1. A roling circle of radius $a = \frac{1}{2H}$ generates the cylinder of mean curvature H.
2. In the case $b = 0$, the ellipse degenerates to a segment of length $2a = \frac{1}{H}$ with the focus in one of the endpoints. The roulette is a chain of circles. As long as the segment rotates about the focus, the roulette is constant.

Proof. The polar representation $c(t) = \rho(\vartheta(t))e^{i\vartheta(t)}$ of a conic section with a focus at the origin is

$$\rho(\vartheta) = \frac{\ell}{1 \pm \varepsilon \cos\vartheta}. \tag{64}$$

Here $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$ is the equation of the ellipse $(+)$ or hyperbola $(-)$, with two semiaxes of lengths $a \geq b$. The eccentricity is $\varepsilon = \sqrt{a^2 \pm b^2}/a$ and $\ell = b^2/a$. See, for instance, `http://en.wikipedia.org/wiki/Conic_section` Note that replacing $\cos\vartheta$ by $\sin\vartheta$ in (64) only means a rotation of the conic by 90 degrees.

We will verify by calculation that the generating curve of a Delaunay surfaces is a roulette of a curve with the above polar representation.

First we calculate

$$\frac{1}{\sqrt{r'^2+r^2}} \overset{(59)}{=} 2H - \frac{\varepsilon\cos t}{r} \overset{(60)}{=} 2H - \frac{\varepsilon\cos t}{\frac{\varepsilon}{2H}\cos t + \frac{1}{2H}\sqrt{1-\varepsilon^2\sin^2 t}}$$

$$= 2H - \frac{\varepsilon\cos t\Big(\frac{\varepsilon}{2H}\cos t - \frac{1}{2H}\sqrt{1-\varepsilon^2\sin^2 t}\Big)}{\frac{\varepsilon^2-1}{(2H)^2}}$$

$$= \frac{2H}{\varepsilon^2-1}\Big(\varepsilon^2 - 1 - \varepsilon^2\cos^2 t + \varepsilon\cos t\sqrt{1-\varepsilon^2\sin^2 t}\Big)$$

$$= \frac{2H}{1-\varepsilon^2}\Big(1-\varepsilon^2\sin^2 t - \varepsilon\cos t\sqrt{1-\varepsilon^2\sin^2 t}\Big). \tag{65}$$

To simplify this expression, let us consider the polar angle ϑ of c, as given by (63):

$$\vartheta(t) := \arg c(t) = \arg\big(r'e^{-it} + ire^{-it}\big).$$

For a unit vector w we have $\sin\arg(w) = \operatorname{Im} w$, and so

$$\sin\vartheta = \operatorname{Im}\arg(\ldots) = \frac{\operatorname{Im}\big(r'e^{-it}+ire^{-it}\big)}{\sqrt{r'^2+r^2}} = \frac{-r'\sin t + r\cos t}{\sqrt{r'^2+r^2}}$$

$$\overset{(58)(59)}{=} \varepsilon\sin^2 t + 2Hr\cos t - \varepsilon\cos^2 t \overset{2H\cos t\cdot(60)}{=} \varepsilon\sin^2 t + \cos t\sqrt{1-\varepsilon^2\sin^2 t},$$

which rewrites the last two terms of (65).

Let us now combine these results. In general, the polar representation of c is $c(t) = \rho(t)e^{i\vartheta(t)}$, and in our case

$$c = |c|e^{i\arg c} \overset{(63)}{=} \sqrt{r'^2+r^2}\, e^{i\vartheta} = \frac{\frac{1}{2H}(1-\varepsilon^2)}{1-\varepsilon\sin\vartheta}e^{i\vartheta}.$$

But this means we have written $c(t) = \rho(\vartheta(t))e^{i\vartheta(t)}$, and the representation for ρ is the polar representation of a conic section (64), where $\sin\vartheta$ replaces $\cos\vartheta$ and $\ell = \frac{1}{2H}(1-\varepsilon^2)$. □

Corollary 25. *The period of an unduloid is the circumference of the generating ellipse, hence an elliptic integral.*

However, arclength of the generating curve of a Delaunay surface is integrable and turns out to be π/H for a period.

4.3. **Outlook.** In this context, let us mention two theorems by Korevaar, Kusner, Solomon, which we will discuss in Sect. 7:

1. A properly embedded annulus with constant mean curvature $H \neq 0$ is a Delaunay unduloid. That is, topology implies geometry.
2. An *end* is a proper immersion of the punctured disk $D\setminus\{0\}$. If an end is properly embedded and has mean curvature $H \neq 0$ it is asymptotic to an unduloid.

Similar statements hold for $H \equiv 0$, when ends are asymptotically catenoids or planes.

While the surfaces of revolution with nonzero constant mean curvature are thus determined, we could also look for constant mean curvature analogues of the other minimal surfaces we considered:

• Cmc surfaces with an interior rotation were determined by B. Smyth. This case also reduces to an ODE, but the discussion is much more involved.

• Ruled cmc surfaces: Only the cylinder.

• There are also cmc surfaces with screw motion isometries. These are discussed in Kenmotsu's book as the solution of the appropriate ODE; there is a more elegant description as surfaces in the associated family of the Delaunay surfaces.

5. Maximum principle and Alexandrov theorem

The maximum principle is the most important tool to analyse surfaces with constant mean curvature. We introduce it here gradually, starting with harmonic functions, then generalizing to linear elliptic equations, and to constant mean curvature surfaces. The goal of this section is a theorem by Alexandrov characterizing the spheres as the unique compact embedded surfaces with constant mean curvature.

References: The various forms of the maximum principle are contained in [Gilbarg-Trudinger]. For the Alexandrov theorem see [Spivak IV, Ch.9, Add. 3] or [Hopf, Ch. VII, p.147ff].

5.1. **Harmonic functions.** Let us look first at the model case for an elliptic equation. This is the Laplace equation $\Delta u = 0$, whose solutions $u \in C^2$ are called *harmonic*.

Examples of harmonic functions include:
1. Constant and linear functions,
2. Real or imaginary parts of holomorphic functions,
3. functions like $x^2 - y^2$ are harmonic,
4. coordinate functions of minimal surfaces in conformal parameterization, by (9).

A crucial property of harmonic functions is the (weak) *maximum principle*:

Proposition 26. *Let $U \subset \mathbb{R}^n$ be a bounded domain and $u \in C^2(U) \cap C^0(\overline{U})$. If u is* subharmonic*, $\Delta u \geq 0$, the maximum principle holds,*

$$\sup_{\partial U} u = \sup_{U} u.$$

Similarly, for u superharmonic*, $\Delta u \leq 0$, the minimum principle $\inf_{\partial U} u = \inf_U u$ holds.*

Note first that for a continuous function $\sup_{\overline{U}} u = \sup_U u$. Second, the boundedness of U is essential – find a counterexample for U unbounded!

Proof. Step 1: Consider for $\varepsilon > 0$ the auxiliary function

$$v(x) := u(x) + \varepsilon |x|^2.$$

Then $\Delta v = \Delta u + 2n\varepsilon > 0$ for all $x \in U$.

Step 2: On the compact set $\overline{U}$, the continuous function v takes a maximum at $y = y(\varepsilon) \in \overline{U}$. Suppose that y is an interior point of U. Then for each i the restriction $v(y + te_i)$ has a maximum and so $\partial_{ii} v(y) \leq 0$. Summing, we find $\Delta v(y) \leq 0$, in contradiction to $\Delta v > 0$. Consequently, $y \in \partial U$.

Step 3: Since $\varepsilon|x|^2$ is non-negative we have

$$\sup_U u \leq \sup_U \left(u + \varepsilon|x|^2\right) \leq \sup_{\partial U} \left(u + \varepsilon|x|^2\right) \qquad \text{for all } \varepsilon > 0.$$

But $\varepsilon|x|^2$ is bounded on U, and so the claim follows. Similarly for the minimum. $\square$

Another way to derive the maximum principle is by using a surprising property of harmonic functions: Each value $u(x)$ is the mean of the function u over any ball $\overline{B_r(x)} \subset U$, that is, $u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy$, see [GT, Thm.2.1]. This property does not generalize to the more general elliptic equations, so we do not discuss it any further here.

`12. Lecture, Tuesday 25.5.10` ———

The most important application of the maximum principle is to show the uniqueness of the Dirichlet problem to Poisson's equation:

Proposition 27. *Let U be a bounded domain and $f \in C^0(U, \mathbb{R})$. Then any two solutions $u_1, u_2 \in C^2(U, \mathbb{R}) \cap C^0(U, \mathbb{R})$ to the* Poisson equation

$$\Delta u(x) = f(x) \text{ for all } x \in U \qquad \text{with } u_1(x) = u_2(x) \text{ for all } x \in \partial U,$$

coincide.

Proof. The function $v := u_1 - u_2$ satisfies $\Delta v \equiv 0$ and has boundary values $v = u_1 - u_2 = 0$. Thus by Proposition 26 we have

$$\sup_U v = \sup_{\partial U} v = 0, \qquad \text{and} \qquad \inf_U v = \inf_{\partial U} v = 0,$$

so that $v = u_1 - u_2 \equiv 0$ on U. $\square$

It is a much more difficult task to construct such solutions. For special domains U, like the ball or a half-space, there are explicit formulas. In general, however, abstract existence schemes must be used ("Perron process", see [GT] Sect. 2.8).

Let us conclude this section with an example of a more general equation which does *not* satisfy a maximum principle: Consider the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{for } \lambda > 0,$$

where u has zero boundary values. If we let U be the cube $(0, \pi)^n$ then the functions

$$u(x) := \sin(k_1 x_1) \cdot \ldots \cdot \sin(k_n x_n) \qquad \text{with } k_1, \ldots, k_n \in \mathbb{N}$$

have zero boundary values and satisfy $\Delta u + \left(k_1^2 + \ldots + k_n^2\right) u = 0$. Clearly, $u \equiv 0$ also solves the equation. So whenever λ agrees with a sum of n squared natural numbers, there are non-vanishing solutions of the equation. They violate the maximum principle as well as uniqueness. Functions satisfying $\Delta u = \lambda u$ are called *eigenfunctions* of the Laplace operator Δ; we have seen that for the cube U the discrete sequence of numbers $\{\sum_{i=1}^n k_i^2 : k_i \in \mathbb{N}\} \subset \mathbb{R}$ is contained in the point spectrum of Δ.

References: [GT], Sect. 2

5.2. **Weak maximum principle for elliptic equations.** On our way to the minimal surface equation we introduce more general equations which behave much alike the Laplace equation. Our class can be shown to include the Laplacian equation after a change of variables, and so again we expect the maximum principle to hold.

Definition. A *linear partial differential operator of second order* is given by

$$Lu(x) := \sum_{i,j=1}^n a^{ij}(x)\,\partial_{ij}u(x) + \sum_{k=1}^n b^k(x)\,\partial_k u(x) = \operatorname{trace}(A\,d^2u) + \langle b, \nabla u\rangle \tag{66}$$

where $a^{ij}, b^k \in C^0(U,\mathbb{R})$ such that the matrix $A = (a^{ij})$ is symmetric and $b = (b^1,\dots,b^n)$ is bounded on U. The operator L is called *uniformly elliptic* if there exists $\lambda > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j = \xi^t A\,\xi > \lambda|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\} \text{ and } x \in U. \tag{67}$$

The ellipticity condition (67) means that the lowest eigenvalue of A is bounded below uniformly on U. The symmetry assumption on A can always be achieved (how?).

Note that linearity means $L[u+v] = L[u] + L[v]$.

Examples. 1. If A is the identity matrix and $b = 0$ then L is the Laplacian.
2. A special case is that A is constant and symmetric. Then (67) is equivalent to A being positive definite.
3. An elliptic equation essentially behaves like a distorted Laplace equation. However, if we give up positive definiteness then the equation behaves differently: For instance, for the hyperbolic equation $\Delta u - u_{tt} = f$ the Dirichlet problem is not well-posed, the maximum principle is not valid (superposition of waves!), and solutions can be less regular than their boundary/initial values (shock waves!).
4. Later, we will encounter operators in *divergence form*, that is,

$$Lu(x) = \sum_{i,j} \partial_i\big(\tilde a^{ij}\partial_j u\big) = \operatorname{div}\big(\tilde A(x)\nabla u(x)\big),$$

where $\tilde A$ is a symmetric matrix of functions in $C^1(\overline{U})$. Indeed, differentiating we find

$$Lu = \sum_{i,j} \tilde a^{ij}\partial_{ij}u + \sum_j \big(\sum_i \partial_i \tilde a^{ij}\big)\partial_j u.$$

Thus if $\tilde a^{ij}$ satisfies the ellipticity condition (67) and if we set $b^k := \sum_i \partial_i \tilde a^{ik}$ then L is elliptic.

Lemma 28 (Weak maximum principle)**.** *Let U be bounded, $u \in C^2(U,\mathbb{R}) \cap C^0(\overline{U},\mathbb{R})$, and L be uniformly elliptic. If $Lu \geq 0$ for all $x \in U$ then*

$$\sup_U u = \sup_{\partial U} u \tag{68}$$

Similarly, if $Lu \leq 0$ then $\inf_U u = \inf_{\partial U} u$.

Proof. Step 1: We consider an auxiliary function $v := u + \varepsilon e^{\gamma x_1}$ where $\varepsilon > 0$. Here, $\gamma > 0$ is chosen large enough so that the last inequality in

$$Lv \overset{L \text{ linear}}{=} Lu + \varepsilon L(e^{\gamma x_1}) \overset{Lu \geq 0}{\geq} \varepsilon \left(a^{11}\gamma^2 + b^1\gamma\right) e^{\gamma x_1} \overset{L \text{ ell.}}{\geq} \varepsilon \left(\lambda\gamma^2 - |b^1|\gamma\right) e^{\gamma x_1} \overset{|b| \text{ bdd.}}{>} 0 \tag{69}$$

holds.

Step 2: On the compact set $\overline{U}$, the function v takes a maximum at some point $y \in \overline{U}$. Suppose y is an interior point of U. Then $\nabla v(y) = 0$ so that $Lv(y) = \operatorname{trace}\big(A(y)d^2v(y)\big)$. We claim that the product of the negative semidefinite matrix $d^2v(y)$ with the positive definite matrix $A(y)$ is negative semidefinite. This implies $Lv(y) \leq 0$ contradicting (69), and so $y \in \partial U$ holds.

To prove our claim, consider arbitrary matrices M, N. Then $\operatorname{trace}(MN) = \sum_{ij} m_{ij} n_{ji} = \operatorname{trace}(NM)$ so that the trace of a matrix product is invariant of the order of the product. In particular, for each $T \in \mathsf{GL}(n)$, and with obvious notation,

$$\operatorname{trace}\left(A\, d^2v\right) = \operatorname{trace}\left((T^{-1}AT)(T^{-1}d^2v\, T)\right) = \operatorname{trace}\left(\widehat{A}\,\widehat{d^2v}\right).$$

Since A is symmetric we can choose $T \in O(n)$ such that $\widehat{A} := T^{-1}AT$ is diagonal. Note that

$$\xi^t\big(T^{-1}MT\big)\xi = (\xi^t T^t)M(T\xi) = (T\xi)^t M(T\xi).$$

So on the one hand the diagonal matrix $\widehat{A}$ is still positiv definite, meaning $\widehat{a}^{ii} > 0$ for all i. On the other hand, $\widehat{d^2v}(y) = T^{-1}d^2v(y)T$ is negative semidefinite, so that its diagonal entries are non-positive, $\widehat{d^2v}(y)_{ii} \leq 0$. Consequently,

$$Lv(y) = \operatorname{trace}\left(\widehat{A}\,\widehat{d^2v}\right) = \sum_{i=1}^n \widehat{a}^{ii}(y)\widehat{\partial_{ii}v}(y) \leq 0,$$

which establishes our claim.

Step 3: We have $\sup_U u \leq \sup_U(u + \varepsilon e^{\gamma x_1}) = \sup_{\partial U}(u + \varepsilon e^{\gamma x_1})$. Since this holds for each $\varepsilon > 0$ and $e^{\gamma x_1}$ is bounded on U, the claim follows. $\square$

As for harmonic equations, the uniqueness of solutions for elliptic equations with prescribed boundary values is again immediate:

Theorem 29. *Let U be bounded and $f \in C^0(U, \mathbb{R})$. If $u_1, u_2 \in C^2(U, \mathbb{R}) \cap C^0(\overline{U}, \mathbb{R})$ are two solutions of the elliptic boundary value problem $Lu_i = f$ with the same boundary values $u_1|_{\partial U} = u_2|_{\partial U}$ then $u_1 = u_2$.*

Proof. The function $u_1 - u_2$ has zero boundary values, satisfies $L(u_1 - u_2) = 0$, and so the maximum principle gives $u_1 - u_2 \equiv 0$. □

5.3. **Hopf Lemma and strong maximum principle for elliptic equations.** For the following lemma, the boundary of the domain needs to be suitably good: A domain U satisfies an *interior sphere condition* at $p \in \partial U$, if there is a ball $B_r(q) \subset U$ with $p \in \partial B_r(q)$. Then the interior normal ν to $B_r(q)$ at p is called an *inner normal* to U at p. If the boundary is C^1, the inner normal is unique. If a compact domain U has a C^2-boundary then the local normal form of a hypersurface together with a compactness argument prove that U satisfies an interior sphere condition uniformly, i.e., with uniform $r > 0$.

`13. Lecture, Thursday 27.5.10` ______________________________

Lemma 30 (E. Hopf Boundary Point Lemma)**.** *Suppose U satisfies an interior sphere condition at $p \in \partial U$ with an interior normal ν and let L as in (66) be uniformly elliptic with coefficient matrix A bounded on U. Suppose that the function $u \in C^2(U, \mathbb{R}) \cap C^0(\overline{U}, \mathbb{R})$ satisfies $Lu \geq 0$ and takes a strict boundary maximum at $p \in \partial U$, that is, $u(x) < u(p)$ for all $x \in U$. Then, provided the normal derivative exists,*

$$\frac{\partial u}{\partial \nu}(p) < 0.$$

Proof. Let $B = B_r(y) \subset U$ be a ball with $p \in \partial B$. Without loss of generality, $y = 0$ and $u(p) = 0$. Then for each $c > 0$ the auxiliary function

$$\varphi(x) := e^{-c|x|^2} - e^{-cr^2}$$

is positive on the annulus $R = B_r(0) \setminus B_{r/2}(0)$, it vanishes at p, and, as we will use at the end of the proof, has positive directional derivative at p w.r.t. the inner normal of ∂R. Moreover, we have $\partial_i \varphi(x) = -2cx_i e^{-c|x|^2}$ and hence

$$L\varphi(x) = e^{-c|x|^2} \Big(4c^2 \sum_{i,j} a^{ij} x_i x_j - 2c \sum_i (a^{ii} + b^i x_i) \Big) \geq 2ce^{-c|x|^2} \Big[2c\lambda |x|^2 - \textstyle\sum_i a^{ii} - |b||x| \Big].$$

Since $A, b, |x|$ are bounded on R, and $|x|^2 > r^2/4$, we can choose c large enough to conclude $[\ldots] \geq 0$ on R and hence $L\varphi \geq 0$.

Our function u is negative in the interior, so we can choose $\varepsilon > 0$ with $u + \varepsilon\varphi \leq 0$ on $\partial B_{r/2}$; on ∂B_r the same holds anyway. Since $L(u + \varepsilon\varphi) \geq 0$ the weak maximum principle (68) applies to give $u + \varepsilon\varphi \le 0$ on R, while $(u + \varepsilon\varphi)(p) = 0$. Thus the normal derivative of this function at p is ≤ 0 which implies

$$\frac{\partial u}{\partial \nu}(p) \le -\varepsilon \frac{\partial \varphi}{\partial \nu}(p) < 0.$$

□

Example. In the situation of the Hopf Boundary Point Lemma the Taylor expansion of a harmonic function (or a solution to $Lu \ge 0$) at a maximum has a nonzero linear term. However, at a corner point of a general boundary, the interior sphere condition is not satisfied, and the linear term of the Taylor expansion can vanish: The harmonic function $u(x, y) = -xy$ on the quadrant $(0, \infty) \times (-\infty, 0)$ agrees with its Taylor expansion at the origin, and so starts with a quadratic term.

We can now conclude:

Theorem 31 (Strong maximum principle)**.** *Let U be a domain and L be uniformly elliptic with $Lu \ge 0$ on U. If u achieves an interior maximum then u is constant.*

Proof. Suppose, contrary to the statement, that u achieves a maximum $m \ge 0$ at an interior point but that u is non-constant. Then the open set $U_< := \{x \in U, u(x) < m\}$ is non-empty. Pick a point $p \in U_<$ that is closer to $\partial U_<$ than to ∂U, and consider the largest open ball $B(p) \subset U_<$. Then $u(x) = m$ for some point $x \in \partial B(p)$ while $u < m$ on $B(p)$. Also, $\overline{B}(p) \subset\subset U$, and so the matrix A is bounded on $B(p)$. By the Hopf Boundary Point Lemma, $\nabla u(x) \ne 0$, contradicting the fact that m is the maximum of u. □

References: [GT], Sect. 3.1 and 3.2

5.4. **Maximum principle for graphs with prescribed mean curvature.** Our starting point is the mean curvature equation in its divergence form (13): Let

$$Qu := \sum_{i=1}^{n} \partial_i \left(\frac{\partial_i u}{\sqrt{1 + |\nabla u|^2}} \right) - nH, \tag{70}$$

then a graph $\big(x, u(x)\big)$ of mean curvature $H = H(x)$ satisfies $Qu = 0$.

The equation $Qu = 0$ is nonlinear, that is, $Qu = Qv = 0$ does not imply $Q(u + v) = 0$. Nevertheless there is a maximum principle, saying that two different graphs with the same mean curvature cannot have a one-sided tangential touching:

Theorem 32. *Let U be a domain and $u, v \in C^2(U, \mathbb{R}) \cap C^0(\overline{U}, \mathbb{R})$ describe two graphs which have the same mean curvature function $H \in C^0(U, \mathbb{R})$ with respect to the upper normal. Then we have:*

(i) Interior maximum principle: *If $u \le v$ and $u(p) = v(p)$ at some interior point $p \in U$, then $u \equiv v$.*

(ii) Boundary maximum principle: *Suppose there exists a boundary point* $p \in \partial U$ *such that the following holds:*
• ∂U *satisfies an interior sphere condition at* p *and* $u, v \in C^2(U \cup \{p\}) \cap C^0(\overline{U}, \mathbb{R})$,
• $u(p) = v(p)$,
• $\frac{\partial u}{\partial \nu}(p) = \frac{\partial v}{\partial \nu}(p)$, *where* ν *is an inner normal at* p,
• $u(x) \leq v(x)$ *for all interior points* $x \in U$.
Then $u \equiv v$.

Example. Once again, boundary regularity is necessary for (ii) to hold: The Enneper surface is graph in a neighbourhood of the origin 0 and has the xy-plane as its tangent plane. In particular, the Taylor expansion does not have a linear term at 0. On the other hand, on a rotated quadrant U (violating the sphere condition at 0), the Enneper surface takes zero boundary values but it does not agree with the xy-plane.

Proof. It will be convenient to define

$$\alpha^i \in C^\infty(\mathbb{R}^n, \mathbb{R}), \quad \alpha^i(p) := \frac{p_i}{\sqrt{1 + |p|^2}}, \qquad 1 \leq i \leq n,$$

so that $Qu = \operatorname{div} \alpha(\nabla u) - nH$.

As we have seen in the uniqueness proofs for Laplace's or elliptic equations, it is useful to consider the difference $u - v$. If the graphs u, v have the same mean curvature $H(x)$, then at each $x \in U$

$$\begin{aligned}
0 = Qu - Qv &= \sum_i \frac{\partial}{\partial x_i} \alpha^i(\nabla u) - \frac{\partial}{\partial x_i} \alpha^i(\nabla v) = \sum_i \frac{\partial}{\partial x_i} \alpha^i \Big(t \nabla u + (1-t) \nabla v\Big)\Big|_{t=0}^{t=1} \\
&= \int_0^1 \frac{d}{ds} \left[\sum_i \frac{\partial}{\partial x_i} \alpha^i \big(s \nabla u + (1-s) \nabla v\big) \right]_{s=t} dt \\
&= \sum_{i,j} \int_0^1 \frac{\partial}{\partial x_i} \left(\frac{\partial \alpha^i}{\partial p_j} \big(t \nabla u + (1-t) \nabla v\big) \big(\partial_j u - \partial_j v\big) \right) dt \\
&= \sum_{i,j} \frac{\partial}{\partial x_i} \left(\left[\int_0^1 \frac{\partial \alpha^i}{\partial p_j} \big(t \nabla u + (1-t) \nabla v\big) dt \right] \big(\partial_j u - \partial_j v\big) \right).
\end{aligned}$$

Thus in terms of

$$a^{ij} \in C^1(U, \mathbb{R}), \quad a^{ij}(x) := \int_0^1 \frac{\partial \alpha^i}{\partial p_j} \big(t \nabla u(x) + (1-t) \nabla v(x)\big)\, dt$$

we can rewrite our result to say that $w := u - v$ satisfies

$$Lw := \sum_{i,j} \partial_i \big(a^{ij}(x) \partial_j w\big) = 0.$$

Here L is a linear second order partial differential operator in divergence form which puts us into the position to apply the maximum principle. We have achieved linearity of L at the expense of having its coefficients a^{ij} depend on our given functions u, v. No complications arise since, within the present proof, these two functions are fixed once and for all.

We claim that L is elliptic on any compact subset $K \subset\subset U$. We differentiate to find

$$\sum_{i,j} \partial_i\big(a^{ij}(x)\partial_j w(x)\big) = \sum_{i,j} a^{ij}(x)\partial_{ij}w(x) + \sum_k \big(\textstyle\sum_i \partial_i a^{ik}(x)\big)\partial_k w(x). \tag{71}$$

First of all, the chain rule gives that the linear coefficients $\sum_i \partial_i a^{ik}$ are bounded on K: this follows from α smooth and $u, v \in C^2(K, \mathbb{R})$. Also, (71) means that the ellipticity condition amounts to showing $\sum a^{ij}\xi_i\xi_j > \lambda|\xi|^2$. To see this, set $P(t,x) := t\nabla u(x) + (1-t)\nabla v(x)$. Then

$$\frac{\partial \alpha^i}{\partial p_j}(P) = \frac{\partial}{\partial p_j}\frac{P_i}{\sqrt{1+|P|^2}} = \frac{\delta_{ij}}{\sqrt{1+|P|^2}} - \frac{P_iP_j}{\big(1+|P|^2\big)^{3/2}}, \qquad 1 \le i,j \le n.$$

The continuous function $|P(t,x)|$ takes a maximum over the compact set $[0,1]\times K$. Hence we can define a positive number $\lambda = \lambda(K,u,v)$ such that the following holds:

$$\begin{aligned} \sum_{i,j}\frac{\partial \alpha^i}{\partial p_j}(P)\xi_i\xi_j &= \frac{1}{\big(1+|P|^2\big)^{3/2}}\Big(\big(1+|P|^2\big)|\xi|^2 - \langle P,\xi\rangle^2\Big) \\ &\overset{\text{Schwarz}}{\ge} \frac{1}{\big(1+|P(t,x)|^2\big)^{3/2}}|\xi|^2 > \lambda|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^n\setminus\{0\} \end{aligned} \tag{72}$$

Integration of this equation with respect to t verifies the ellipticity condition for L,

$$\sum_{i,j} a^{ij}(x)\xi_i\xi_j = \sum_{i,j}\Big[\int_0^1 \frac{\partial\alpha^i}{\partial p_j}\big(P(t,x)\big)\,dt\Big]\xi_i\xi_j > \int_0^1 \lambda|\xi|^2\,dt = \lambda|\xi|^2 \qquad \forall\, x \in K.$$

Let us prove (i). The strong maximum principle, Thm. 31, implies $w = u - v \equiv 0$ for any compact set $K \subset\subset U$ which contains the point $p \in U$ where u and v coincide. Consequently, $u \equiv v$ on all of U.

14. Lecture, Tuesday 1.6.10

We now prove (ii). The interior sphere condition provides us with a ball B such that ∂B contains only the boundary point $p \in \partial U$. Except for a neighbourhood of p, the set B is a set compactly contained in U. Thus as before and using the assumed differrentiability of u, v at p, the functions $\partial_i a^{ik}$ and $|P|^2$ are bounded on B, and so $Lw = L(u-v)$ is elliptic on B. We have $w = u - v \le 0$ on B. In fact, invoking the interior maximum principle we may assume $w = u - v < 0$ on B. Then the Hopf Boundary Point Lemma 30, applied to $w = u - v$, contradicts our assumption $\frac{\partial w}{\partial \nu}(p) = 0$. $\square$

5.5. **Uniqueness of surfaces with prescribed mean curvature.** We discuss applications of the maximum principle. To consider surfaces globally, beyond the parameterized local pieces $f\colon U \to \mathbb{R}^n$, we define as follows.

A *hypersurface* M is an n-dimensional submanifold of $\mathbb{R}^{n+1}$. A *hypersurface with boundary* is locally either parameterized as $f\colon B^n \to \mathbb{R}^{n+1}$ (for interior points), or $f\colon B^n \cap \{x^n \geq 0\} \to \mathbb{R}^{n+1}$ (for boundary points). The boundary can be empty.

If you know abstract manifolds, then whenever we say hypersurface M you can take instead the image of an n-manifold Σ immersed into $\mathbb{R}^{n+1}$, i.e., M denotes an immersion $\varphi\colon \Sigma^n \to \mathbb{R}^{n+1}$. Similarly, a manifold with boundary has charts either to B^n or to $B^n \cap \{x^n \geq 0\}$.

The second definition is more general than the first in that it allows for self-intersections. Indeed, when the immersion of the second definition happens to be an embedding then we are in the situation of the first definition, and could actually take the inclusion map of a submanifold Σ for φ.

With this notion of a surface we first assert the *convex hull property* of minimal surfaces:

Theorem 33. *Let $M \subset \mathbb{R}^{n+1}$ be a minimal hypersurface M with boundary which is bounded. Then*
(i) M is contained in the convex hull of its boundary values, $M \subset \operatorname{conv}(\partial M)$, and
(ii) if M touches the boundary $\partial \operatorname{conv}(\partial M)$ of its convex hull at a point interior to M, then M is contained in a plane.

Here, be definition the convex hull of a set $A \subset \mathbb{R}^n$ is the intersection of all half-spaces containing A.

Proof. (ii) Suppose there is a hyperplane P which is tangent to M at an interior point, such that only one component of $\mathbb{R}^3 \setminus P$ contains points of M. We can represent M locally as a graph over P. Then by the maximum principle, Thm. 32(i), locally M must agree with P. But the set of points for which M agrees with P is closed (consider sequences) and open (by the interior maximum principle). Since M is connected, all of M must be a subset of P.

(i) By applying a suitable motion we can assume that $\operatorname{conv}(\partial M)$ is contained in the lower halfspace $\{z \leq 0\}$. Suppose M intersects the upper halfspace $\{z > 0\}$. Then consider planes $P(h) = \{x^{n+1} = h\}$ at height h such that $P(h) \cap M \neq \emptyset$. The set of such heights h is bounded above (since M is bounded), and so there is a supremum $h_0 < \infty$ of these values. Moreover, $P(h_0)$ still intersects M, by the compactness of $\overline{M}$. We have arrived at a contradiction to the interior maximum principle, Thm. 32(i). $\square$

Clearly, the convex hull property fails for unbounded minimal surfaces: think of a catenoid half, bounded by a circle. Also for $H \neq 0$ it cannot hold: A circle bounds a spherical cap.

If the boundary of a surface is empty then the convex hull property says that the surface must be the empty set, that is:

Corollary 34. *A complete minimal surface (with no boundary) cannot be compact.*

We have not yet addressed the existence question for graphs with prescribed boundary values (the so-called Dirichlet problem). We can, however, address uniqueness for this problem:

Theorem 35. *Let U be a bounded domain and $u \in C^2(U, \mathbb{R}) \cap C^0(\overline{U}, \mathbb{R})$. Suppose $f(x) = \big(x, u(x)\big)$ is a graph of mean curvature $H \in C^0(U, \mathbb{R})$.*
(i) Then any other graph $\big(x, v(x)\big)$, $v\colon U \to \mathbb{R}$ with the same mean curvature function $H(x)$ and the same boundary values coincides with u.
(ii) If U is convex and f minimal, then f parameterizes the unique bounded minimal surface attaining its boundary values (no matter if graph or not).

Property (ii) fails for nonzero mean curvature. An example with arbitrary constant mean curvature $H \not\equiv 0$ is a spherical cap, not equal to a hemisphere, considered as a graph over a disk. Then the complementary spherical cap, reflected in the plane of the disk (to have the same sign of H), is another solution with the same mean curvature.

Thus the strategy of the following indirect proofs is to create an interior one-sided touching of two minimal surfaces, contradicting the interior maximum principle.

Proof. (i) Suppose u, v are two solutions with the same boundary values. Let $m \geq 0$ be the maximum of $u - v$. Since $u = v$ on ∂U, the maximum is attained at an interior point $p \in U$, and $u \leq v + m$ on U. The interior maximum principle Thm. 32(i), applied to u and $v + m$, proves $u \equiv v + m$. The boundary values then prove $m = 0$.

(ii) Let N be a bounded minimal surface whose boundary values agree with the graph $\big(x, u(x)\big)$. The set $\overline{U} \times \mathbb{R}$ is convex, and so Thm. 33(i) gives that N is also contained in $\overline{U} \times \mathbb{R}$. In fact, $N \subset U \times \mathbb{R}$ since else N would be planar by Thm. 33(ii), contradicting the fact that U has non-empty interior.

Now consider the foliation with translated graphs, $f(t, x) = \big(x, u(x) + t\big)$ for $t \in \mathbb{R}$. Since N is bounded, there exist $t_{\max} := \sup\{t \in \mathbb{R}, f(t, U) \cap N \neq \emptyset\}$ and $t_{\min} := \inf\{\ldots\}$. Suppose now that N is different from the graph f so that one of $t_{\max}, t_{\min}$ is nonzero, say $t_{\max}$.

The two surfaces $\overline{N}$ and $f(t_{\max}, \overline{U})$ are compact, hence they intersect at a point $p = f(x)$; as $t_{\max} > 0$ the point p cannot be a boundary point of N. Thus the intersection point p of N and $f(t_{\max}, U)$ is interior to both surfaces.

In order to apply the (interior) maximum principle, we claim that N can be represented as a graph in a neighbourhood of p. To see this, note that if the normal $\pm\nu(p)$ of N at p was different from the normal to the graph f at x, then there would be points of N to both sides of the graph, contradicting the maximality of $t_{\max}$. $\square$

In case (ii), existence for arbitrary boundary values will follow from the solution of the Plateau problem in the next part, together with the present statement. However, in the general case (i), existence does not hold for arbitrary boundary values.

Example. We want to show that convexity of the domain is necessary for uniqueness in part (ii) to hold. Our counterexample is a piece of a catenoid M, considered as graph over a suitable annulus in $\mathbb{R}^2$. For the outer radius $R > 0$ of the annulus take any radius larger than the waist of M. Consider the family of scalings $M_\lambda := \lambda M$ for $0 < \lambda < 1$. The scaled catenoids M_λ also contain a circle of radius R. Assume that the M_λ are translated along the axis such that these circles are contained in the xy-plane and such that the waist circle occur with positive z-coordinate. Then any M_λ intersects M in a second circle of radius $\rho(\lambda) < R$. In particular, there are two minimal surfaces bounded by the two circles of radius $\rho(\lambda)$ and R; here the z coordinate is 0 for the outer one, and a suitable positive number for the inner one. The subset of M bounded by the two circles, is a graph, while the subset of M_λ bounded by the same circles contains the catenoid waist and hence is not graph. Consequently, over the annulus $U := B_R \setminus B_{\rho(\lambda)}$ there is a minimal graph, such that a distinct minimal surface has the same boundary.

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5.6. **Short account on existence results.** For minimal surfaces, there are three different methods to prove existence.

1. Dirichlet problem for constant mean curvature graphs over a domain:

Theorem. *Let $U \subset \mathbb{R}^n$ be a bounded C^2-domain, $H \in \mathbb{R}$. Then the* Dirichlet problem

$$\begin{cases} \operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = nH & \text{on } U \\ u = \varphi & \text{on } \partial U \end{cases}$$

is solvable for all boundary values $\varphi \in C^0(\partial U)$, if and only if the mean curvature H' of ∂U (or curvature H' if $n = 2$) satisfies

$$(n-1)H'(y) \geq n|H| \qquad \text{for all} \quad y \in \partial U. \tag{73}$$

Let us mention two special cases:
1. If U is convex it satisfies $H' \geq 0$ and so there is a minimal graph (x, u).

However, for domains violating (73) for some boundary values the Dirichlet problem is solvable, e.g., $\varphi = \text{const}$, while some other boundary values exist, such that there is no solution (see problems).

2. We can find a graph in $\mathbb{R}^3$ with mean curvature $H = 1$ for arbitrary boundary values over the circle of curvature ≥ 2, that is, for radius $R \leq 1/2$.
On the other hand, for $R > 1$, the maximum principle shows that there is no graph with $H = 1$ for any boundary values (compare with a sphere!). For $1/2 \leq R \leq 1$ graphs with $H = 1$ can still exist for certain φ.

See [GT, ch. 16.3].

2. The Weierstrass-Enneper representation formula gives a minimal surface in terms of two complex functions.

Theorem. *If $h\colon U \subset \mathbb{C} \to \mathbb{C}$ is holomorphic and $g\colon U \to \overline{\mathbb{C}}$ meromorphic then $f\colon U \to \mathbb{R}^3$*

$$f(z) = \operatorname{Re} \int_{z_0}^{z} h\Big(\frac{1}{2}\Big(\frac{1}{g} - g\Big), \frac{i}{2}\Big(\frac{1}{g} + g\Big), 1\Big)\, dw$$

is an immersed minimal surface, provided certain conditions on the pole and zero order of g and h are met. Conversely, any minimal surface can be locally represented this way.

The mapping $g = \mathrm{st} \circ \nu$ is stereographic projection of the Gauss map ν from $\mathbb{S}^2$ into the complex plane. There is also a global version of the theorem where U is a Riemann surface.

Examples (see problems): a) Catenoid: $U = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $g(z) = z$, $h(z) = \frac{1}{z}$
b) Enneper $U = \mathbb{C}$, $g(z) = z$, $h(z) = 2z$

There is a generalization to constant mean curvature which is much more involved, by Dorfmeister, Pedit, and Wu from 1998.

3. Plateau's problem: The Belgian physicist Joseph Plateau studied soap films. In his treatise *Statique expérimentale et théorique des liquides soumis aux seules forces moléculaires* from 1873 he postulated that every wire frame bounds at least one soap film; this is nowadays called the *Plateau problem.* Mathematically formulated:

Theorem. *A Jordan (injective) curve $\Gamma\colon \mathbb{S}^1 \to \mathbb{R}^n$ bounds an immersed minimal 2-disk $f \in C^2(D, \mathbb{R}^n) \cap C^0(\overline{D}, \mathbb{R}^n)$, i.e., $f|_{\partial D}$ is a parameterization of Γ.*

The minimal disk is not unique, and there may be other minimal surfaces, perhaps of smaller area, having a different topology. There is a generalization to other ambient manifolds N by Morrey, if N is noncompact a further condition must be assumed. A good source is the book by Jost.

Mathematicians tried unsuccessfully to prove this statement by establishing Weierstrass data since the late 19th century. Only in the 1930's, and Douglas and Rado found independently a solution with different methods. In 1936, Douglas was awarded the first Fields medal for his solution.

Let us give an idea of the lengthy and complicated proof. The proof finds the solution by variational methods, that is, by taking a sequence with area decaying to the infimum and then establishing that it converges to a nice minimal surface. However, minimizing area directly leads to degeneracies – thin "hairs" do not contribute much to area, but could add one-dimensional structures to the limiting object. The remedy is to consider a problem which penalizes poor parameterizations at the same time: This is done by minimizing the Dirichlet energy $\int |df_n|^2$ where $f_n \in C^2(D, \mathbb{R}^n) \cap C^0(\partial D, \mathbb{R}^n)$ has boundary values parameterizing Γ, similar to the proof of the Riemann mapping theorem. A suitable limit $f = \lim f_n$ in the same class will turn out to be the solution to the Plateau problem; convergence is established by the Arzela-Ascoli theorem. Then f is also harmonic. Moreover, since it minimizes energy among all maps whose boundary values parameterize Γ, it can be shown to be conformal. But a harmonic map f which also is conformal is at the same time critical for area, i.e., a minimal surface. Unfortunately, this only establishes f as a differentiable map, but not as an immersion. By work started by Osserman in the 1960's, *branch points* with $df_p = 0$ can be ruled out.

For constant mean curvature, additional constraints must be imposed: Clearly, spherical caps can only bound boundary circles with radius $R < 1/H$. The maximum principle shows that such circles cannot be the boundary of any embedded surface with $H \geq 1$, see problems. However, given a condition of this kind, Hildebrandt proved the solvability of Plateau's problem for constant mean curvature in 1970:

Theorem. *Let $H > 0$ and Γ be a piecewise smooth Jordan curve in $\mathbb{R}^3$, contained in the closed ball $\overline{B}_{1/H}$ of radius $1/H$. Then there exists a map $f \in C^2(D, \overline{B}_{1/H}) \cap C^0(\overline{D}, \overline{B}_{1/H})$ such that*
- *f restricted to $\mathbb{S}^1$ parameterize Γ injectively,*
- *f is conformal in D with $\Delta f = 2H\partial_1 f \times \partial_2 f$.*

Note that f is not established as an immersion; this follows from further theorems. However, at points f is an immersion the mean curvature is H.

Furthermore, there are geometric methods which produce constant mean curvature $H > 0$ surfaces in $\mathbb{R}^3$ from minimal surfaces in the sphere $\mathbb{S}^3_{1/H}$ of radius $1/H$, discovered by Lawson 1970. This way, solutions of the Plateau problem in $\mathbb{S}^3$ can be used to construct mean curvature H surfaces.

5.7. **Alexandrov theorem.** We now apply the maximum principle to prove the symmetry of compact solutions of $H \equiv 1$, and hence the uniqueness of $\mathbb{S}^2$. The proof uses reflection of the portion of the solution to one side of a plane. This is often called the *Alexandrov reflection*; since only the maximum principle is essential it applies to more general PDE's as well (Gidas, Ni, Nirenberg).

Theorem 36 (Alexandrov, 1956)**.** *Spheres are the only compact embedded hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvature.*

In case of constant nonzero H the radius of the sphere is $1/H$. We have formulated the theorem in a way that it also applies to the case $H \equiv 0$, in which case there are no compact solutions. In that case, however, a much simpler proof can be given, based on the fact that there is a sphere enclosing the surface and touching it from one side; this proof also works for immersed compact minimal surfaces.

The following fact will be used in the proof: A complete embedded hypersurface M decomposes $\mathbb{R}^{n+1}$ into two connected components. Here, a *(connected) component* of set A (of a Euclidean or, more generally, a topological space) is a nonempty subset $C \subset A$ which is relatively open and closed, and does not contain a proper nonempty subset with the same property. It is equivalent to require that C has the property that any differentiable function to the real numbers with vanishing gradient takes constant values on C.

In dimension $n = 2$, our statement is the *Jordan curve theorem*, and holds for any injective continuous curve.

Our statement is not as straightforward as it might appear on the first sight: It is, for instance, not true that an injective continuous map of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$ has a connected component in its complement which is homeomorphic to the ball B^n: The *Alexander horned sphere* provides a counterexample for the case of $\mathbb{S}^2 \to \mathbb{R}^3$. As long as we assume that the manifold is embedded, that is, a homeomorphism onto its image, such pathological examples cannot occur.

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Let us now indicate how our statement can be proven in the case that $M \subset \mathbb{R}^{n+1}$ is a compact smooth submanifold of dimension n. For our special argument to work, we also assume M to be oriented by a normal mapping ν. Since M is immersed and ν normal, the mapping

$$\varphi \colon M \times \mathbb{R} \to \mathbb{R}^{n+1}, \qquad \varphi(p,t) = p + t\nu(p)$$

has rank $n+1$ on $M \times \{0\}$. By continuity and since M is compact, there is $\varepsilon > 0$ such that φ has full rank on the thickened neighbourhood $M_\varepsilon := M \times (-\varepsilon, \varepsilon)$. If M is embedded there is a homeomorphism of a neighbourhood of $M \times \{0\} \subset M \times \mathbb{R}$ into $\mathbb{R}^{n+1}$, meaning that for ε small enough also $\varphi\colon M_\varepsilon \to \mathbb{R}^{n+1}$ is an embedding. We claim there are at most two components of $\mathbb{R}^{n+1} \setminus M$. Let us denote with $M_\varepsilon^\pm$ dthe two sets $M \times (0, \varepsilon)$ and $M \times (-\varepsilon, 0)$. Then clearly the set $\varphi(M_\varepsilon) \setminus M$ has exactly two components $\varphi(M_\varepsilon^\pm)$. On the other hand, the boundary of any further connected component of $\mathbb{R}^{n+1} \setminus M$ must be contained in M, ruling out that any such component exists. This proves our claim.

Now we prove there are at least two components, defining a generalized winding number. To see the two sets $\varphi(M_\varepsilon^\pm)$ are actually in different components of $\mathbb{R}^{n+1} \setminus M$, we construct a locally constant continuous function on $\mathbb{R}^{n+1} \setminus M$ which takes at least two different values. To define it we need a concept of differential topology. If $f \in C^1(M^n, N^n)$ is a differentiable mapping from a compact manifold to a target manifold of the same dimension, then at each point $y \in N$ we define a *degree* by counting the number of its preimages with a sign given by their orientation,

$$\deg_y(f) := \sum_{x \in f^{-1}(y)} \operatorname{sign}(\det df_x) \in \mathbb{Z}.$$

For almost all $y \in N$ the number of preimages is finite and the differential of all preimages x of y is nonzero, by *Sard's Theorem.* Hence $\deg_y$ is defined for y in a dense subset of N. The sum can be empty; in particular, if f is not surjective in the sense that a set of positive measure is not attained then the degree is 0. The simplest examples worth considering are maps from $\mathbb{S}^1$ to $\mathbb{S}^1$.

It is a fact that the degree is independent of the point y considered, $\deg(f) := \deg_y(f)$. Consequently, for N compact (or with finite measure), we can also represent the degree by averaging, $\deg(f) = \frac{1}{\lambda(N)} \int_M \det df_x \, d\lambda_M(x)$, that is, we compute the oriented area of the image of M.

Let us now specifically consider a function which also depends on $p \in \mathbb{R}^{n+1} \setminus M$:

$$\pi_p\colon M \to \mathbb{S}^n, \quad \pi_p(x) := \frac{x - p}{|x - p|}.$$

Geometrically, this is radial projection of $x \in M$ onto a sphere with midpoint p. Using that $\deg_y(\pi_p)$ does not depend on $y \in \mathbb{S}^n$, we now consider dependence of $\deg(\pi_p)$ on the parameter p. We claim that $p \mapsto \deg(\pi_p)$ is a continuous function into the integers $\mathbb{Z}$. To see that, consider the representation of $\deg(\pi_p)$ as the oriented area of the image π_p, divided by $A(\mathbb{S}^n)$. Obviously, this depends continuously on the midpoint p of the projection sphere. For p large, we have $\deg(\pi_p) = 0$. Indeed, for such p there is an open set in the target $\mathbb{S}^n$ which is not covered by π_p, and so the degree must vanish. On the other hand, the degree changes by ± 1 when M is crossed, which gives the claim. See [Spivak I, ch.8, problem 22].

The preceding sketch of proof in particular gives the Jordan curve theorem in the plane for differentiable simple curves.

Let us now give the proof of the Alexandrov theorem:

Proof. Let M be a compact embedded hypersurface with $H \equiv 1$, and denote the component of $\mathbb{R}^{n+1} \setminus M$ with compact closure by V.

Consider the planes $\Pi^s := \{x_{n+1} = s\}$ bounding the halfspaces $H_+^s := \{x_{n+1} > s\}$ and $H_-^s := \{x_{n+1} < s\}$. We write $M_\pm^s$ or $V_\pm^s$ for the intersection of M or V with the open halfspaces $H_\pm^s$, respectively.

Let σ_s be reflection in Π^s. Let us decrease s from ∞ to a minimal value a such that the reflection of the upper portion M_+^s is strictly inside the surface M in the following sense:

$$a := \inf\{s_0 \in \mathbb{R}\colon \text{ for all } s \geq s_0 \text{ holds } \sigma_s(M_+^s \cup V_+^s) \subset V_-^s\} \tag{74}$$

The x_{n+1}-coordinate of M is bounded above, and so large positive numbers s are certainly contained in the above set. On the other hand, since x^{n+1} is also bounded below on M all negative numbers s below this bound cannot be contained in the set, meaning the set is bounded below. We conclude that $a \in \mathbb{R}$ exists.

We claim that at $s = a$ one of the following cases must occur:
(i) Interior contact: There exists $p \in M_-^s \cap \sigma_s(M_+^s)$.
(ii) Boundary contact: At a point $p \in \Pi^s \cap M$, the tangent plane of T_pM is vertical so that M and $\sigma_s(M)$ have the same tangent plane at p.

To prove the claim we show that if none of these conditions holds for some s then $s > a$.

Let ν be the inner normal of M. Then for $s > a$ the inner normal points downwards on the upper half, $\nu_{n+1} \leq 0$ on M_+^s. Indeed, if $\nu_{n+1} > 0$ at some point $p \in M \cap \Pi^s$, then $\sigma_s(M)$ and M are transverse at p, i.e., their tangent spaces at p are distinct such that a neighbourhood of $\sigma_s(M_+^s)$ at p cannot be contained in V_-^s.

If (i) does not hold then the two sets M_-^s and $\sigma_s(M_+^s)$ have a positive distance within the lower halfspace $H_-^{s-\varepsilon}$ for any $\varepsilon > 0$. If in addition (ii) does not hold then $\nu_{n+1} < 0$ on $M \cap \Pi^s$, and so for some $\delta > 0$ actually $\nu_{n+1} < -\delta < 0$ on the compact set $M \cap \Pi^s$. Since M is C^2 this in turn implies $\nu_{n+1} < -\delta/2$ on the thickened set $M \cap \{s - \varepsilon \leq x^{n+1} \leq s + \varepsilon\}$. These two facts mean that there is a neighbourhood of s such that (i) and (ii) will not occur. Hence s must be larger than the infimum a of (74).

We now apply the interior or boundary maximum principles to show that Π^a is a symmetry plane for M.

In case (i), the tangent planes $T_pM_-^a$ and $T_p\sigma(M_+^a)$ at the point p must coincide (otherwise $\sigma(V_+^a)$ was not contained in V_-^a). But then each of the two surfaces, M_-^a and $\sigma_a(M_+^a)$ can locally be represented as a graph $h_\pm$ over the common tangent plane. The condition $\sigma\big(\overline{V_+^a}\big) \subset \overline{V_-^a}$ implies $h_- \leq h_+$ w.r.t. the upper normal, and so h_- and h_+ coincide at p.

The interior maximum principle, Thm. 32(i), then proves that the two local graphs $h_\pm$ coincide. In fact, $\sigma_a(M_+^a) = M_-^a$ since the set $\sigma_a(M_+^a) \cap M_-^a$ is relatively closed in M_-^a, and also relatively open by the interior maximum principle.

In case (ii) we can again write M_-^a and $\sigma_a(M_+^a)$ as graphs $h_\pm$ over their common vertical tangent plane at p. Since both surfaces have boundary in Π_a, the domain of the local graph is bounded by a straight arc containing p, and so the interior sphere condition holds at p. Also the graphs are C^2 up to the boundary: they extend to the complete C^2-surface M. Finally, the tangency of (ii) means that the normal derivatives of the two functions $h_\pm$ coincide at p. Again this shows that locally $\sigma(M_+^a)$ and M_-^a agree. An openness/closedness-argument as before then proves globally $\sigma_a(M_+^a) = M_-^a$.

We have shown that Π^a is a symmetry plane for M. The same argument works with respect to any direction, meaning that M has a symmetry plane with any prescribed normal. By the next lemma, M must be a sphere. □

Reflections through non-parallel planes generate rotations:

Lemma 37. *If a bounded set $S \subset \mathbb{R}^n$ has a plane of mirror symmetry in every direction then there is a point p such that S is invariant under arbitrary rotations about p; in particular, S is a union of spheres.*

Proof. Take n mutually orthogonal symmetry planes $P_1, \ldots, P_n$ for S. They intersect in just one point $\{x\} := P_1 \cap \ldots \cap P_n$. Note that reflections in the P_i preserve distance to x.

Now let P be another plane of symmetry. Suppose P does not contain x, so that the distance d of P to x is positive. Then P must miss one of the 2^n generalized quadrants $\overline{Q}$ of $\mathbb{R}^n$.

Let a be a point of S (if S is empty there is nothing to prove). By reflections in the planes P_i we obtain $a_0 \in \overline{Q} \cap S$. Reflecting a_0 in P gives a point $a_1 \in S$ with $\|a_1 - x\| \geq 2d$. We can reflect in the P_i's to obtain a point $a_2 \in \overline{Q} \cap S$, preserving the distance to x. Iterating reflections in P and the P_i's, we obtain a sequence of points $a_n \in S$ with norm $\|a_n - x\| \geq (n+1)d \to \infty$, contradicting the boundedness assumption. □

For a number $\lambda > 0$, the *isoperimetric problem* in $\mathbb{R}^n$ is the problem to determine a domain S with prescribed measure $\lambda(S)$, such that the smooth hypersurface ∂S has least area (or $(n-1)$-dimensional measure).

Corollary 38. *The solution of the isoperimetric problem in $\mathbb{R}^n$ are balls bounded by spheres.*

It is interesting –in fact geometrically more interesting– to consider the isoperimetric problem in other ambient spaces. In particular, for compact spaces there are nontrivial solutions: Even for the standard 3-torus, $T^3 := \mathbb{R}^3/\mathbb{Z}^3$, the solution family $\lambda \mapsto M_\lambda$ is not known. Only in the class of sufficiently symmetric solutions, Ros could determine the solutions for any $0 \le V(S) \le 1$. See the problems for the simpler case T^2.

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The existence of a solution to the isoperimetric problem is not a priori clear, and involves nontrivial analytical work. Taking it for granted, the rest of the proof is trivial:

Classification proof. Consider a solution S such that $M := \partial S$ is smooth, i.e., M is an embedded hypersurface. Then M minimizes area for given volume $V(S)$. Hence by the n-dimensional version of Thm. 13, a solution has constant mean curvature H. Hence Alexandrov's theorem applies to show that M is a sphere; in particular H must be nonzero. $\square$

There are various other settings where Alexandrov reflection applies. We discuss some of them.

1. What are the compact embedded constant mean curvature surfaces $M \subset \mathbb{R}^{n+1}$ bounded by a (planar) circle in $\mathbb{R}^n \times \{0\}$? An yet unsolved conjecture says that these must be spherical caps. There are counterexamples in case M is not compact (Delaunay) or compact but immersed (by Kapouleas). Alexandrov reflection gives that the statement is true if the constant mean curvature surface is contained in one of the two halfspaces defined by the hyperplane containing the circle [source?].

2. The technique generalizes to other ambient spaces:
- Compact constant mean curvature hypersurfaces in $\mathbb{H}^{n+1}$ are distance spheres.
- Compact hypersurfaces with constant H contained in open hemispheres of $\mathbb{S}^{n+1}$ are distance spheres, in particular hemispheres cannot contain minimal surfaces.

To prove these assertions, the reflection is applied w.r.t. to a foliation of the spaces with totally geodesic hypersurfaces, namely hyperbolic hyperplanes or great spheres. Reflection in these hypersurfaces preserves mean curvature. However, the method does not work to prove anything about hypersurfaces in the entire sphere $\mathbb{S}^n$. In fact there are compact embedded minimal surfaces in $\mathbb{S}^n$ with any genus $g \ge 0$. The *Lawson conjecture* says that any embedded minimal torus in $\mathbb{S}^3$ is the Clifford torus $\{(z,w) \in \mathbb{S}^3 : |z|^2 = |w|^2 = 1/2\}$. Alexandrov reflection had long been considered a candidate for a proof, but a proof suggested by Kilian and Schmidt in 2008 employs integrable systems methods.

3. The reflection technique has been applied to non-compact surfaces, under the assumption of controlled asymptotics of the non-compactness:
- Minimal embedded annuli with two embedded ends of finite total curvature in $\mathbb{R}^3$ (R.

Schoen: Uniqueness, symmetry, and embeddedness of minimal surfaces, J. Diff. Geom. 18, 791–809, 1983).
• Properly embedded constant mean curvature annuli in $\mathbb{R}^3$ are Delaunay unduloids; see Sect. 7 below for this and further results.

Corollary 39. *Suppose* Π *decomposes a constant mean curvature surface* M *into two halves* $M_\pm$ *as in the proof of the Alexandrov reflection. Then*
(i) *the Gauss map of each half* $M_\pm$ *(inner normal) takes values in a hemisphere of the* 2*-sphere,* $\nu(M_\pm) \subset \mathbb{S}^2_\mp$*, with* $\nu(\partial M_\pm)$ *being contained in the equator, and*
(ii) *each half* $M_\pm$ *is graph over the domain* $\Pi \cap M$*.*

4. A *capillary surface* M in a set $N \subset \mathbb{R}^3$ is a surface of constant mean curvature which meets ∂N with constant *contact angle*, that is, the normal of N and the normal of M make a constant angle. J. McCuan generalized the Alexandrov technique to work with spheres instead of planes and proved: If N is a wedge, then any compact capillary surface is a *spherical drop*, so no drops of annular type exist in a wedge.

5. There are spaces like the Heisenberg group which do not admit any reflection. However, there are still rotations about vertical geodesics, and so compact embedded surfaces of constant mean curvature can be obtained as ODE solutions. An interesting problem is to confirm that these surfaces are the unique compact embedded constant mean curvature surfaces, even though the technique of Alexandrov reflection does not work.

There is also an interesting generalization of the embeddedness to which the Alexandrov reflection technique can still be applied.

Definition. A compact hypersurface $\varphi\colon M^n \to \mathbb{R}^{n+1}$ is *Alexandrov embedded* if the n-manifold M is the boundary of an $(n+1)$-manifold N and there is an immersion $\Phi\colon N \to \mathbb{R}^{n+1}$ extending φ in the sense $\Phi|_M = \varphi$.

Example. A figure-8-curve is not Alexandrov embedded, but an overlapping circle is.

A careful inspection of the proof we gave establishes the following corollary.

Corollary 40 (Alexandrov 1960's)**.** *The conclusion of the theorem extends to Alexandrov embedded compact hypersurfaces with constant mean curvature.*

Note that in this version of the Alexandrov theorem the proof does no longer rely on the topological fact that a compact surface bounds a compact component!

5.8. **Problems.**

Problem 13 – Test:

a) Why is it no loss of generality to assume that the matrix A in $Lu = \operatorname{trace} A\, d^2u = \sum a^{ij}\partial_{ij}u$ is symmetric?
b) Prove that if u is subharmonic ($\Delta u \geq 0$) on a bounded domain U, and h is a harmonic function ($\Delta h = 0$) with the same boundary values, $u|_{\partial U} = h|_{\partial U}$, then $u(x) \leq h(x)$ for all $x \in U$. Discuss also the equality case $u(p) = h(p)$ for an interior point $p \in U$.
c) A standard linear algebra result is that a linear map $L\colon V \to W$ with $\ker L = 0$ gives $Lx = b$ has at most one solution x. Draw the analogy to the uniqueness theorem for the Poisson equation $Lu = f$. (What are the vector spaces V, W?)

Problem 14 – Maximum of harmonic functions on unbounded domains:

Exhibit an unbounded domain $U \subset \mathbb{R}^n$ with non-empty boundary and a harmonic function $u\colon U \to \mathbb{R}$ such that u does not take a maximum on the boundary.

Problem 15 – Versions of the maximum principle:

Let $U \subset \mathbb{R}^n$ be bounded.

a) Prove for a harmonic function $u \in C^2(U,\mathbb{R}) \cap C^0(\overline{U},\mathbb{R})$:
$$\sup_U |u(x)| = \sup_{\partial U} |u(x)|$$
b) A mapping $u \in C^2(U,\mathbb{R}^n) \cap C^0(\overline{U},\mathbb{R}^n)$ is called *harmonic* if each component is a harmonic function. Prove that harmonic mappings satisfy the above maximum principle.

Problem 16 – Uniqueness and symmetry of solutions:

Suppose σ is reflection in the hyperplane $\{x_n = 0\} \subset \mathbb{R}^n$,
$$\sigma\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad \sigma(x_1,\dots,x_n) := (x_1,\dots,x_{n-1},-x_n).$$
We call a domain $U \subset \mathbb{R}^n$ *mirror symmetric* if $\sigma(U) = U$. For a bounded mirror symmetric domain U, consider a function $u \in C^2(U,\mathbb{R}) \cap C^0(\overline{U},\mathbb{R})$ whose boundary values are invariant under σ, that is, $u(x) = u\big(\sigma(x)\big)$ for all $x \in \partial U$.

Consider the following cases:

1. u is harmonic, or
2. u solves a uniformly elliptic equation $Lu = 0$.

a) Decide for each of the two cases if u respects the symmetry σ, i.e.,
$$u(x_1,\dots,x_n) = u(x_1,\dots,x_{n-1},-x_n) \qquad \text{for all } x \in U.$$

b) On the other hand, find a solution v of the equation $\Delta v + v = 0$ which has symmetric boundary values, but is not invariant under σ (it suffices to consider $n = 1$).

c) Consider the cases for which the answer under a) is in the affirmative. Prove the same statement more generally for isometries $A \in \mathsf{O}(n)$, for instance for rotations.

Problem 17 – Maximum principle with exceptional points:

Let us first state two facts:
1. $\log|x| \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is harmonic.
2. If $f \colon \Omega^2 \to \mathbb{R}^2$ is conformal then $\Delta(u \circ f) = (\Delta u) \circ f$.
Use these facts to prove the following:

a) Let $D \subset \mathbb{R}^2$ be the unit disk, and set $D^* := \overline{D} \setminus \{(1,0)\}$, $S^* := \mathbb{S}^1 \setminus \{(1,0)\}$. Find a harmonic function $u \in C^2(D,\mathbb{R}) \cap C^0(D^*,\mathbb{R})$ with boundary values $u|_{S^*} = 0$ such that u is not constant.
Hint: Exhibit a nonzero harmonic function with zero boundary values on the upper halfplane.

b) Prove that each bounded harmonic function $u \in C^2(D,\mathbb{R}) \cap C^0(D^*,\mathbb{R})$ is constant.
Hint: Compare with $\varepsilon \log|z-1|$.

c) Generalize: Can you admit more than just one exceptional point? Can you replace the boundedness assumption on u by a growth condition at the exceptional points? What is the n-dimensional generalization?

d) Prove the two facts stated above by calculation.

Problem 18 – Dirichlet problem over a non-convex domain:

The following example indicates that the solvability of the Dirichlet problem for arbitrary boundary data requires convexity of the domain.

a) Let C be a catenoid whose axis of revolution is the z axis. Find a closed simple (Jordan) curve $\Gamma \subset C$ with the following properties:
1. The projection $\pi(\Gamma)$ into the xy-plane is injective.
2. $\pi(\Gamma)$ is the boundary of a domain U in the xy-plane which is not convex.
3. Γ bounds an open bounded subset $M \subset C$ such that M is not graph over U.

b) Prove that M is the unique minimal surface bounded by Γ; perhaps you need to modify Γ suitably. Hence Γ cannot bound a minimal surface which is a graph over U.
Hint: Which theorem of the lecture can only prove this claim?

c) Can you prescribe other non-constant boundary values over ∂U, such that there is a unique minimal graph over U?

Problem 19 – Constant mean curvature surfaces bounded by circles:

a) Suppose $M \subset \mathbb{R}^3$ be an embedded surface
   - with mean curvature 1,
   - the boundary ∂M is a circle of radius $R > 1$,

- M is contained in a halfspace (determined by the plane of the circle).

However, we do not assume that M is bounded. Prove that M cannot exist.

Hint: You can assume that such an M would decompose the closed upper halfspace $\{z \geq 0\}$ into two connected components U and V.

b) Suppose $M \subset \mathbb{R}^3$ is a surface
- with mean curvature 1,
- M is a bounded,
- the boundary of M is a circle of radius $R \leq 1$,
- M is contained in a halfspace (determined by the plane of the circle).

Prove that (as stated in class) Alexandrov reflection works to show that M is a spherical cap of a unit sphere.

c) Generalize the statements to arbitrary dimension – are they true?

Problem 20 – Quiz:

True or false?

a) A constant mean curvature surface with boundary and compact closure is always contained in the convex hull of its boundary values.
b) If two surfaces of constant mean curvature are one-sided tangent at an interior point then they agree locally.

Problem 21 – Mean curvature of entire graphs:

Determine the assumption on ε you need for the following assertions to become true, and give a counterexample if the bound for ε is attained.

a) There is no graph over the entire plane $\mathbb{R}^2$ with constant mean curvature $H > \varepsilon$.
b) There is no graph over the entire plane $\mathbb{R}^2$ with variable constant mean curvature $H(x) > \varepsilon$.

Problem 22 – Alexandrov reflection:

Suppose M is a compact hypersurface, not necessarily with constant H. If a is defined as in (74), can there exist $s > a$ such that there is a point p in $M \cap \Pi^s$ with vertical tangent plane T_pM? Is your answer left unchanged if we assume that H is constant in addition?

Problem 23 – Alexandrov embedding in dimension 2*:*

Consider a map $\Phi \in C^0(\overline{D}, \mathbb{R}^2)$, which is an immersion on D; let $\varphi := \Phi|_{\mathbb{S}^1}$ be the boundary restriction.

a) Suppose $\varphi\colon \mathbb{S}^1 \to \mathbb{R}^2$ is injective. Show that there is a unique immersion Φ extending φ to D, up to diffeomorphism.

Hint: Consider the number of preimages of Φ on $\mathbb{R}^2 \setminus \varphi(\mathbb{S}^1)$.

b) We show that the extension Φ of φ if φ is not injective. That is, there exist immersions Φ, Ψ of the disk with continuous boundary values $\Psi|_{\mathbb{S}^1} = \Phi|_{\mathbb{S}^1} = \varphi$, but there is no diffeomorphism $\sigma\colon D \to D$, such that $\Psi \circ \sigma = \Phi$.
   To see that, find a mirror symmetric polygon in $\mathbb{R}^2$ with self-intersections, for which the extension Φ is not be mirror symmetric. It is sufficient to consider a hexagon (or pentagon).

Problem 24 – The nodoid is not Alexandrov embedded:

Prove that the nodoid is not Alexandrov embedded, that is, there is no immersion

$$F\colon B^3 \setminus \{(0,0,\pm 1)\} \to \mathbb{R}^3,$$

whose restriction to the boundary $\mathbb{S}^2 \setminus \{(0,0,\pm 1)\}$ parameterizes a nodoid.
Hint: Apply the Gauss-Bonnet formula to a suitable planar slice.

Problem 25 – Isoperimetric sets in 2-tori:

In class we showed that isoperimetric sets are bounded by constant mean curvature surfaces, or bounded by constant curvature curves in the case of 2-dimensional domains. For the present problem we can also assume that the solution domains are connected.

a) Determine explicitely isoperimetric sets in a square 2-torus, say: with unit area. To do so, plot the function $L(A)$, giving the length of the boundary of a set with area A for various candidates. Note that in the torus there is no difference between inside and outside.
b) Discuss the same problem for a general 2-torus.
c) Do you have conjectures about the analogous problem for 3-tori? Plot $A(V)$ for some obvious candidates.

Problem 26 – Weierstrass data:

The Enneper-Weierstrass representation formula is

$$f(z) = \operatorname{Re} \int_{z_0}^{z} h(w) \left(\frac{1}{2}\Big(\frac{1}{g(w)} - g(w) \Big), \frac{i}{2}\Big(\frac{1}{g(w)} + g(w) \Big), 1 \right) dw.$$

a) Prove that on $U = \mathbb{C}$ the Weierstrass data $g(z) = z$, $h(z) = 2z$ give the Enneper surface f.
b) Prove that on $U = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $g(z) = z$, $h(z) = \frac{1}{z}$ the function f parameterizes a catenoid.
   Hint: After integration, use conformal polar coordinates $z = \exp(r + i\varphi)$, that is, calculate $f\big(\exp(r + i\varphi)\big)$.

`18. Lecture, Thursday 17.6.10`

6. Hopf's uniqueness theorem

The theorem by Heinz Hopf says that any immersion of $\mathbb{S}^2$ with mean curvature 1 is a round unit sphere. Thus we have the same conclusion as in Alexandrov's theorem, but now the embeddedness assumption is replaced by the asumption of an immersed 2-dimensional sphere. The proof relies on complex analysis and topology; it is genuinely 2-dimensional.

To present the proof, we need to discuss a couple of concepts: immersions of a sphere or general surface, and indices of vector fields on surfaces.

References: [Spivak IV, Ch.9, Add.2], [Hopf], [Jost]

6.1. **Submanifolds and surfaces.** In order to formulate Hopf's theorem, we need the notion of an immersed sphere. As a first step, we introduce the domain of this map. In contrast to the local theory of differential geometry, this amounts to introducing the concept of a surface globally. We do this rightaway in the smooth class, not in the topological category.

If you know about manifolds then a surface is a 2-manifold, you can skip the folowing subsection.

Recall that an n-dimensional *submanifold* $\Sigma^n \subset \mathbb{R}^m$ can locally be described as the zero set of a smooth mapping φ, that is, for each $p \in \Sigma$ there is a neighbourhood $V \subset \mathbb{R}^m$, and a smooth mapping $\varphi\colon V \to \mathbb{R}^{m-n}$, such that
- $V \cap \Sigma = \{p \in V : \varphi(p) = 0\}$ and
- $d\varphi$ has rank $m-n$ on V.

The second condition serves to avoid non-differentiable points, bifurcations, and to make Σ truly n-dimensional. We call a submanifold of dimension $n = 2$ a *surface.*

A topology on Σ is defined by declaring open sets in Σ as sets which are obtained by intersecting Σ with ambient open sets. This allows to introduce notions like continuous functions on a surface, or continuous functions between surfaces (give the definitions!).

In order to introduce differentiable terminology we need the parametric picture. For each point $p \in \Sigma$ there is an open neighbourhood $V^\alpha \subset \mathbb{R}^m$ such that there is a differentiable homeomorphism $f^\alpha\colon U^\alpha \to \Sigma \cap V^\alpha$, where df^α has rank n; then f^α is called a *parameterization.* Here α is in some index set A, such that all parameterizations cover, $\Sigma = \bigcup_{\alpha\in A} f_\alpha(U_\alpha)$. Clearly, if Σ is compact then finitely many parameterizations suffice.

Examples. 1. By the implicit mapping theorem, each point has a neighbourhood such that, up to reindexing, the surface can locally be represented as a graph $f(x,y) := (x, y, h(x,y))$;

thus parameterizations of submanifolds always exist.
2. For surfaces, we will need conformal parameterizations f where $\partial_1 f \perp \partial_2 f$ and $|\partial_1 f|^2 = |\partial_2 f|^2 \neq 0$.
3. The two conformal *stereographic projections*

$$f_\pm(x,y) = \frac{\pm 1}{1+x^2+y^2}\big(2x, 2y, x^2+y^2-1\big) \tag{75}$$

suffice to parameterize all of $\mathbb{S}^2$. (What is the similar formula for $\mathbb{S}^n$?)

Given a pair of parameterizations f^α, f^β we can define a change of coordinates or *transition map*

$$\tau^{\beta\alpha} := (f^\beta)^{-1} \circ f^\alpha \colon U^\alpha \cap (f^\alpha)^{-1}(U^\beta) \to U^\beta \cap (f^\beta)^{-1}(U^\alpha).$$

Then $\tau^{\beta\alpha}$ is a diffeomorphism. This important property is usually proved as follows: By extending f^α, f^β from an n-dimensional domain to an m-dimensional domain, a diffeomorphism of m-dimensional sets is defined; its restriction to the n-dimensional subsets remains a diffeomorphism. Let us also mention that for abstract manifolds, differentiability of the transition maps becomes a definition.

This allows us to define:
• If Σ has a covering with parameterizations $\{f^\alpha : \alpha \in A\}$ such that all transition maps have positive determinant then Σ is called *oriented*.
• A *differentiable map* $g \colon \Sigma \to \mathbb{R}$ is a map such that composition with all parameterizations, $g \circ f^\alpha$, is differentiable.
• Similarly, a map between surfaces $\varphi \colon \Sigma_1 \to \Sigma_2$ is differentiable when its composition with parameterizations is differentiable: $f_2^{-1} \circ \varphi \circ f_1$ is differentiable where f_i are arbitrary local parameterizations of Σ_i.

Remark. One may explicitely define particular reference surfaces $\Sigma_g \subset \mathbb{R}^3$ of *genus* $g \in \{0, 1, 2, \ldots\}$, where the genus is the number of holes or handles attached to $\mathbb{S}^2$.

Theorem (Classification of surfaces, Möbius 1870 (idea))**.** *Each compact oriented surface is diffeomorphic to a surface Σ_g, where $g \in \{0, 1, 2, \ldots\}$.*

6.2. **Vector fields on surfaces and indices.** The tangent space of a submanifold Σ at a point is easy to define for an implicit representation φ in a neighbourhood of p:

$$T_p\Sigma = \ker d\varphi_p.$$

Parametrically, if $p = f^\alpha(x)$, then $T_p\Sigma = df^\alpha_x(\mathbb{R}^n)$, independently of the parameterization f^α chosen. In fact

$$df^\alpha(X^\alpha) = df^\beta(X^\beta) \quad \Leftrightarrow \quad d\tau^{\beta\alpha}(X^\alpha) = (df^\beta)^{-1} df^\alpha(X^\alpha) = X^\beta \tag{76}$$

(where the differentials are taken at which point?). That is, tangent vectors transform with the Jacobian of the transition map. A tangent vector can always be represented as a tangent vector to a curve.

Definition. A *vector field* on Σ is a smooth map $Y\colon \Sigma \to \mathbb{R}^m$ such that $Y(p) \in T_p\Sigma$.

In terms of parameterizations, we can always write $Y = df^\alpha(X^\alpha)$, where X^α transforms as in (76).

To analyse singularities of vector fields on surfaces we will make use of the following property:

Lemma 41. *If $h\colon I \to \mathbb{S}^1 \subset \mathbb{R}^2$ is continuous then there is a function $\tilde h\colon I \to \mathbb{R}$, called the* lift *of h, such that $h(t) = (\cos \tilde h(t), \sin \tilde h(t))$. The mapping $\tilde h$ is unique up to addition of integer multiples of 2π.*

This lemma is a special case of the path liftling property of covering spaces. We now want to address of a non-simply connected domain. In the one-dimensional case such a domain is $\mathbb{S}^1$ which we also view as the quotient space of $[0, 2\pi]$ where endpoints identified, i.e.

$$\mathbb{S}^1 = [0, 2\pi] \Big/ \{0\} \sim \{2\pi\}.$$

Then there is no continuous lift, but only a lift with a "gap" at a point, say at $t = 0$:

Definition. The *degree* of $h\colon \mathbb{S}^1 \to \mathbb{S}^1$ is the integer

$$\deg(h) := \frac{1}{2\pi}\Big(\lim_{t\nearrow 2\pi} \tilde h(t) - \lim_{t\searrow 0} \tilde h(t)\Big) \quad \in \mathbb{Z},$$

where $\tilde h$ is the lift of the function h restricted to $I := (0, 2\pi)$.

Examples. 1. The maps $h(t) := \big(\cos(2\pi kt), \sin(2\pi kt)\big) = e^{2\pi ikt}$ have degree $k \in \mathbb{Z}$.

2. For a closed curve of length L, parameterized with constant speed $L/2\pi$ we can consider $h(t) := c'(t)/|c'(t)|$. Then $\deg(h)$ is called the *turning number*. A theorem, first proved rigorously by Hopf, says that a simple closed curve has turning number ± 1.

3. The degree is the sum of the images of h, signed by orientation.

`19. Lecture, Tuesday 22.6.10` ________________________________

Definition. (i) A vector field Y with *isolated singularities* on $U \subset \mathbb{R}^n$ or on a submanifold Σ is a vector field which is defined on U or Σ except for a set of isolated points.
(ii) A vector field Y has *isolated zeros* if Y is nonzero except at isolated points.

Note that on a compact surface an isolated set is finite.

Example. If X has isolated zeros, then the unit vector field $Y := X/|X|$ has isolated singularities.

Definition. (i) [$\mathbb{R}^2$-case:] Suppose $X\colon U \to \mathbb{R}^2$ is a vector field with isolated singularities and zeros. Then the *index* of X at $x \in U$ is the degree of the map

$$\operatorname{ind}_x X := \deg\left(t \mapsto \frac{X\big(x+\varepsilon(\cos t,\sin t)\big)}{\big|X\big(x+\varepsilon(\cos t,\sin t)\big)\big|}\right), \qquad t \in [0,2\pi]\Big/\{0\} \sim \{2\pi\}, \tag{77}$$

where $\varepsilon > 0$ is chosen such that X has no zero in $B_\varepsilon(x) \setminus \{x\}$.
(ii) [Surface case:] On an oriented surface Σ, the index of the vector field Y at $p = f^\alpha(x)$ is the index of X^α where $Y = df^\alpha(X^\alpha)$.

Examples. A constant vector field has index 0.
The radial vector field in $\mathbb{R}^2$, $X(x) = x/|x|$ has index 1 at 0.

Let us state a simple but important property: Two vector fields

$$X, Y\colon \{0 < |x| < r\} \subset \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\}$$

are called *homotopic* if there exists a continuous map

$$H\colon I \times \{0 < |x| < r\} \to \mathbb{R}^n \setminus \{0\} \quad \text{with } H(0,x) = X(x) \text{ and } H(1,x) = Y(x).$$

Lemma 42. *The index is invariant under homotopies of vector fields.*

Proof. The function $t \mapsto \operatorname{ind}_x H(t,x)$ is continuous and integer valued, hence constant. $\square$

Consequences of this property include:
• Any continuous vector field X at a point p where $X(p) \neq 0$ has index 0. Indeed, the continuity of X implies that $H(t,x) := tX(p) + (1-t)X(x)$ is nonzero on small balls about p.
• Definition (i) is independent of ε: Consider $H(t,x) := X(tx)$, where this time the homotopy runs over $t \in [\varepsilon_1, \varepsilon_2]$, say.
• The index (77) can be defined more generally with respect to a curve $c(t)$, replacing the circle of radius ε about p, i.e., $\operatorname{ind}_x X := \deg(\frac{X}{|X|} \circ c(t))$. The curve c can be any smooth curve homotopic to φ contained in a punctured disk about p where X has no zeros or singularities. That is, the curve c must only have winding number $+1$ w.r.t. p.

We also note that (ii) is independent of the parameterization; indeed, the index is invariant of orientation preserving diffeomorphisms.

We give another example for the index which we need lateron. Suppose $\Phi\colon \mathbb{C} \to \mathbb{C}$ is holomorphic. We can consider Φ as a vector field on $\mathbb{C} = \mathbb{R}^2$. If Φ is not constant, then its zeros are isolated and so we can calculate the index of the vector field.

Lemma 43. *Let $n \in \mathbb{N}_0$.*
(i) The function $\Phi\colon \mathbb{C} \to \mathbb{C}$, $\Phi(z) := z^n$, has index n at 0.
(ii) Suppose a complex power series $\Phi(z) := \sum_{k\geq n} a_k z^k$ with $a_n \neq 0$ converges in a neighbourhood of 0. Then Φ has index n at 0.

Proof. (i) We have $h(t) := \Phi(\varepsilon e^{it})/|\Phi(\ldots)| = (\varepsilon e^{it})^n/\varepsilon^n = e^{int}$. Then $\tilde{h}(t) = int$ and $\operatorname{ind}\Phi(0) = \deg h(0) = n$.

(ii) We write

$$\Phi(z) = a_n z^n \left(1 + z\Big(\frac{a_{n+1}}{a_n} + z\frac{a_{n+2}}{a_n} + \ldots\Big)\right) =: a_n z^n \big(1 + r(z)\big),$$

where r also converges in a neighbourhood of 0. But $0 < |r| < 1$ for $0 < |z|$ sufficiently small, and so $H(t,z) = a_n z^n\big(1+tr(z)\big)$ is a nonzero homotopy from $a_n z^n$ to Φ. Since z^n has degree n, also the rotated (and dilated) field $a_n z^n$ has degree n, and so the result follows again from the homotopy invariance of the index. ☐

6.3. **Poincaré-Hopf index theorem.** Later, we will make use of the Poincaré-Hopf index theorem although we cannot provide the complete proof here. We need a property of compact surfaces, namely that they can be triangulated. In the language of algebraic topology, a surface is a simplicial complex. We assume this property here.

Definition. Suppose a surface Σ has a finite triangulation with F faces, E edges and V vertices. Then the *Euler characteristic* of Σ is the number

$$\chi(\Sigma) := V - E + F \quad \in \mathbb{Z}.$$

The Poincaré-Hopf Theorem can be considered a quantitative version of the hairy ball theorem, which says that on the even-dimensional spheres all vector fields have zeros.

Theorem 44 (Poincaré-Hopf)**.** *For an oriented compact surface Σ, the sum of the indices of any vector field Y with isolated zeros agrees with the Euler characteristic,*

$$\chi(\Sigma) = \sum_{p:Y(p)=0} \operatorname{ind}_p Y.$$

The Euler characteristic is purely topological, i.e., the concept of a continuous, topological surface is sufficient to define it. Although the index can be defined for continuous vector fields as well, the proof of the theorem represents it with an integral over curvature quantities, and so requires differentiability. Thus the right hand side belongs to the setting of differentiable analysis.

Easy part of proof. Given a triangulation, let us define a particular vector field Y on each triangle: It has zeros at the vertices, the midpoints of the edges, and at a distinguished point in the interior of each triangle.

To define Y, take a source of index 1 at the point in the interior of the triangle, and a sink of index 1 at each vertex, while on each edge midpoint there is a “saddle point” of index -1. On incident triangles such a vector field can match continuously. Thus the index sum of Y coincides with $\chi(\Sigma) = V - E + F$. □

The hard part of the proof is to verify that the same index arises for any vector field. It depends on a local version of the Gauss-Bonnet theorem.

The Euler characteristic of a surface can also be expressed in terms of the genus $g \in \mathbb{N}_0$: It is $\chi(\Sigma) = 2 - 2g$ (see problems).

Remark. The theorem generalizes to arbitrary dimension. To define the Euler characteristic, the manifold must be represented as a simplicial complex, having cells between dimension 0 (vertices) and n. Then

$$\chi(\Sigma) := \#(0\text{-cells}) - \#(1\text{-cells}) \pm \ldots + (-1)^n \#(n\text{-cells}).$$

On the other hand, the degree of a vector field can be defined in arbitrary dimensions as the integral over the oriented area of the normalized vector field, divided by the area of the image sphere. The Gauss-Bonnet theorem in the proof can be replaced by Stoke’s theorem, see [Spivak I, ch.11, Thm.28/Cor.29].

`20. Lecture, Thursday 24.6.10`

6.4. **Line fields.** In order to consider curvature lines, we want to generalize the index from vector fields to line fields.

Definition. (i) *Real projective space* is $\mathbb{R}P^{n-1} := \mathbb{S}^{n-1}/\pm = \{1\text{-dimensional subspaces of } \mathbb{R}^n\}$.
(ii) A *line field* on $U \subset \mathbb{R}^n$ is a map from $P\colon U \to \mathbb{R}P^{n-1}$.
(iii) A line field on a submanifold $\Sigma \subset \mathbb{R}^m$ is a map $P\colon \Sigma \to \mathbb{R}P^{m-1}$ such that $P(p) \subset T_p\Sigma$.

Again, we will consider line fields with isolated singularities. In the context of integrability, a line field is also called a *one-dimensional distribution.*

Example. 1. Each vector field X with isolated zeros induces a line field with isolated singularities by setting $P(x) := \operatorname{span}\{X(x)\}$ for x such that $X(x)$ is nonzero. However, conversely there are many line fields which do not come from vector fields.
2. On a surface of negative curvature immersed into $\mathbb{R}^3$, the principal curvatures satisfy $\kappa_1 < 0 < \kappa_2$, say. Each of the respective principal curvature directions defines a line field.

In the case $n = 2$, we consider the *projective line* $\mathbb{R}P^1$ as the semicircle $\mathbb{S}^1$ with opposite points identified. Hence $h\colon I \to \mathbb{R}P^1$ again has a lift

$$\tilde{h}\colon I \to \mathbb{R}, \qquad \text{with } h(t) = \big(\cos \tilde{h}(t), \sin \tilde{h}(t)\big),$$

this time unique up to adding inter multiples of π. The degree is

$$\deg h := \frac{1}{2\pi}\Big(\lim_{t \nearrow 2\pi} \tilde{h}(t) - \lim_{t \searrow 0} \tilde{h}(t)\Big) \quad \in \mathbb{Z}/2,$$

where now half-integer values can occur. Thus, as for vector fields the index is

$$\operatorname{ind}_x P := \deg\Big(t \mapsto P\big(x + \varepsilon(\cos t, \sin t)\big)\Big) \in \mathbb{Z}/2.$$

Let us discuss the case of curvature lines specifically. An *umbilic [Nabelpunkt]* of a surface $\Sigma^2 \subset \mathbb{R}^3$ is a point where both principal curvatures agree. Exactly at umbilic points are all directions principal curvature directions.

Examples. 1. Spheres have all points umbilical.
2. Suppose an embedded surface $\Sigma \subset \mathbb{R}^3$ is invarinat under a k-fold rotation, where $k \geq 3$. Then the axis of rotation ℓ meets Σ perpendicularly and all points of $\Sigma \cap \ell$ are umbilical.

If an umbilic point $p \in \Sigma$ is isolated, then the two principal curvatures are distinct in a neighbourhood U of p. Thus the direction for the smaller principal curvature, say, defines a line field P_1 in U with a certain index $\operatorname{ind}_p P_1$. The larger principal curvature defines another line field P_2. But P_2 is orthogonal to P_1, and so $\tilde{h}_2 = \tilde{h}_1 + \pi/2$ or $\tilde{h}_2 = \tilde{h}_1 - \pi/2$, and so the index of P_1 and P_2 agrees,

$$\operatorname{ind}_p P_1 = \operatorname{ind}_p P_2.$$

We call this number the *index of the curvature line field* of Σ at p.

The Poincaré-Hopf Theorem 44 extends to line fields:

Theorem 45. *Suppose a compact oriented surface Σ has a line field P with isolated singularities $\{p_1, \ldots, p_k\}$. Then*

$$\chi(\Sigma) = \sum_{i=1}^{k} \operatorname{ind}_{p_i} P.$$

We want to indicate how this result can be derived from the Poincaré-Hopf Theorem for vector fields, but we will be sketchy on covering spaces. See [Spivak III, Ch.4, Addendum 2] for more details.

Sketch of proof. The idea is to construct a surface $\tilde{\Sigma}$ so that the line field P on Σ lifts to a vector field V on $\tilde{\Sigma}$. The basic observation is that going around a singularity twice we always return with the same direction of the line field. So for $\tilde{\Sigma}$ we take the *branched double* *cover* of Σ, which has a $2:1$ projection to Σ, except at the k singular *branch points* which project $1:1$ to the singular points $p_1, \dots, p_k \in \Sigma$.

Let us first describe how $\tilde{\Sigma}$ is obtained from Σ locally, in a neighbourhood of a singularity p with index $\operatorname{ind}_p P = n/2 \in \mathbb{Z}/2$:
• We cut Σ along a ray to p
• and shrink the angle at p from 2π to π; the boundary consists of two copies of the ray.
• We glue in a second copy of the same surface piece along the boundary.

Then globally we can think of extending the cuts to a system of cuts between pairs of singular points, such that the cuts do not intersect. The resulting abstract surface is independent of the cuts chosen. This description of a branched cover is traditional in Riemann surface theory and goes back to Riemann.

To compute the index of the vector field V on $\tilde{\Sigma}$, consider the angle function $\tilde{h}$. Suppose P has index $\operatorname{ind}_p P$. After shrinking the angle, we are left with $\operatorname{ind}_p P - 1/2$ as the angle difference, so after gluing in the second copy we get twice of that, namely $2\operatorname{ind}_p P - 1$. Thus V has index

$$\sum \operatorname{ind} V = \sum (2\operatorname{ind} P - 1) = \Big(2\sum \operatorname{ind} P\Big) - k. \tag{78}$$

On the other hand, consider a triangulation of Σ whose vertex set includes $p_1, \dots, p_k$. Then a triangulation of $\tilde{\Sigma}$ is obtained by doubling the faces, edges, and the vertices different from the singular set, so that

$$\chi(\tilde{\Sigma}) = \tilde{V} - \tilde{E} + \tilde{F} = (2V - k) - 2E + 2F = 2\chi(\Sigma) - k. \tag{79}$$

Comparing (78) with (79) we see that indeed $\chi(\Sigma) = \sum \operatorname{ind} P$. $\square$

It is amazing that the following is not yet known:

Conjecture (Loewner)**.** The index of the curvature line field P of a smoothly immersed surface $\Sigma^2 \subset \mathbb{R}^3$ at an isolated singularity $p \in \Sigma$ satisfies $\operatorname{ind}_p P \leq 1$.

Since a sphere has Euler characteristic 2, the Poincaré-Hopf theorem then implies that any immersed sphere should have at least two umbilics. This consequence is the so-called *Carathéodory conjecture* (from 1924). The two conjectures have only been proved for analytic immersions or for surfaces which are boundaries of convex sets.
See `http://en.wikipedia.org/wiki/Carath%C3%A9odory_conjecture`

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6.5. **Integrability conditions for hypersurfaces.**

Definition. An immersed n-dimensional submanifold in $\mathbb{R}^{n+k}$ is a mapping $\varphi\colon \Sigma \to \mathbb{R}^{n+k}$, where $\Sigma^n \subset \mathbb{R}^m$ is a submanifold and φ has a differential of rank n.

By the chain rule, the local representations $F^\alpha := \varphi \circ f^\alpha\colon U \to \mathbb{R}^{n+k}$, $\alpha \in A$, are differentiable mappings with a Jacobian of rank n. We will often write M for an immersed surface $\varphi(\Sigma)$, but we should keep in mind that M is not a subset of $\mathbb{R}^{n+k}$, but denotes a mapping. Only when φ is an embedding, M can equally well be considered a subset, namely a submanifold.

If the codimension k equals 1 and submanifold is oriented we can choose a normal ν. Hence all the data of local differential geometry is defined for each chart, such as the fundamental forms

$$g^\alpha(X^\alpha, Y^\alpha) = \langle dF^\alpha(X^\alpha), dF^\alpha(Y^\alpha)\rangle \qquad b^\alpha(X^\alpha, Y^\alpha) = \Big\langle \sum_{ij} (X^\alpha)^i (Y^\alpha)^j\, \partial_{ij} F^\alpha, \nu \Big\rangle,$$

or the Weingarten map, etc. With respect to the standard basis $(e_1, \dots, e_n)$ on U^α, these forms again have matrix elements, for instance

$$g^\alpha_{ij} := \langle \partial_i F^\alpha, \partial_j F^\alpha\rangle, \qquad b^\alpha_{ij} := \langle \partial_{ij} F^\alpha, \nu\rangle \qquad \text{for } 1 \le i,j \le n.$$

Due to the invariance under the change of coordinates, principal curvatures and the mean curvature are invariant of parameterization, and the images of curvature lines agree. In the following, we will not keep track of the index $\alpha \in A$. In order to have notation consistent with the differential geometry class, we allow ourselves to write $f\colon U \to \mathbb{R}^m$ in place of a specific parameterization F^α.

A standard technique in elementary differential geometry is to decompose into tangential and normal part. The normal part of the second derivatives is $\langle \partial_{ij} f, \nu\rangle = b_{ij}$. To decompose $d^2 f \in Tf \oplus Nf$, let us assign a name to the tangential part (denoted with $^\top$):

Definition. The *Christoffel symbols* of an immersion $f\colon U \to \mathbb{R}^m$ are the functions $\Gamma^k_{ij}\colon U \to \mathbb{R}$ for $1 \le i,j,k \le n$, defined by

$$\big(\partial_{ij} f(p)\big)^\top = \sum_k \Gamma^k_{ij}(p) \partial_k f(p). \tag{80}$$

The Schwarz lemma gives $\Gamma^k_{ij} = \Gamma^k_{ji}$. For an affine hyperplane, the Christoffel symbols vanish if and only if the parameterization is by an affine map. What would the analogous statement be for a cylinder?

Lemma 46. *The Christoffel symbols can be computed from the first fundamental form g and its derivatives alone:*

$$\Gamma_{ij}^{l} = \frac{1}{2}\sum_{k=1}^{n} g^{lk}\big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\big) \qquad \textit{for } 1 \le i, j, l \le n \tag{81}$$

Proof. We calculate the terms on the right hand side:

$$\begin{aligned} \partial_i g_{jk} &= \partial_i \langle \partial_j f, \partial_k f\rangle = \langle \partial_{ij} f, \partial_k f\rangle + \langle \partial_j f, \partial_{ik} f\rangle \\ \partial_j g_{ki} &= \quad \langle \partial_{jk} f, \partial_i f\rangle + \langle \partial_k f, \partial_{ji} f\rangle \\ -\partial_k g_{ij} &= -\langle \partial_{ki} f, \partial_j f\rangle - \langle \partial_i f, \partial_{kj} f\rangle \end{aligned}$$

The sum is

$$\frac{1}{2}\big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\big) = \langle \partial_{ij} f, \partial_k f\rangle \overset{\partial_k f \in Tf}{=} \big\langle (\partial_{ij} f)^\top, \partial_k f\big\rangle = \big\langle \textstyle\sum_\mu \Gamma_{ij}^{\mu} \partial_\mu f, \partial_k f\big\rangle = \displaystyle\sum_\mu \Gamma_{ij}^{\mu} g_{\mu k}.$$

Hence, multiplying with the inverse matrix g^{-1} gives the desired representation

$$\sum_k \frac{1}{2}\big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\big) g^{kl} = \sum_{k,\mu} \Gamma_{ij}^{\mu} g_{\mu k} g^{kl} = \sum_\mu \Gamma_{ij}^{\mu} \delta_\mu^l = \Gamma_{ij}^l. \qquad \square$$

Invoking also (5), we can express all surface quantities in terms of the fundamental forms:

Theorem 47. *Let $f\colon U^n \to \mathbb{R}^{n+1}$ be an immersed hypersurface with normal $\nu\colon U \to \mathbb{S}^n$. Then (f,ν) satisfies a system of partial differential equations on U, called* hypersurface equations*: The* Gauss equation *[Gaussche Ableitungsgleichung]*

$$\partial_{ij} f = \sum_k \Gamma_{ij}^k \, \partial_k f + b_{ij}\nu \qquad \textit{for } 1 \le i, j \le n \tag{82}$$

and the Weingarten formula

$$\partial_j \nu = -\sum_{i,k} g^{ik} b_{kj} \partial_i f \qquad \textit{for } 1 \le j \le n. \tag{83}$$

Conversely we can ask if the fundamental forms g (thus Γ) and b determine a hypersurface f. In general this is not true. However, it turns out that provided two more equations are satisfied, the problem can be solved. Let us first state these so-called *integrability* or *compatibility conditions*:

Theorem 48. *Each solution (f,ν) of the hypersurface equations* (82)(83) *satisfies the* Gauss equations

$$\partial_i \Gamma_{jk}^s - \partial_j \Gamma_{ik}^s + \sum_{r=1}^n \Gamma_{jk}^r \Gamma_{ir}^s - \Gamma_{ik}^r \Gamma_{jr}^s = \sum_{r=1}^n \big(b_{jk} b_{ir} - b_{ik} b_{jr}\big) g^{rs} \quad \textit{für } 1 \le i, j, k, s \le n, \tag{84}$$

and the (Mainardi-)Codazzi equations

$$0 = \partial_i b_{jk} - \partial_j b_{ik} + \sum_{s=1}^n \Gamma_{jk}^s b_{is} - \Gamma_{ik}^s b_{js} \quad \textit{für } 1 \le i, j, k \le n. \tag{85}$$

It is sufficient to consider $i < j$ for both equations.

These conditions are not implied by the hypersurface equations. The left hand side of the Gauss equation represents the Riemann curvature tensor R^s_{ijk} and is the starting point of Riemannian geometry.

Proof. The proof is by calculation. It is based on a simple idea: The Schwarz lemma implies at each point of U

$$\partial_{ijk} f = \partial_{jik} f \qquad \text{for all } 1 \le i, j, k \le n; \tag{86}$$

this is nontrivial for $i \neq j$, so we may restrict to $i < j$. We compute these derivatives in terms of the frame $\partial_1 f, \ldots, \partial_n f, \nu$:

$$\begin{aligned}
\partial_i \partial_{jk} f &\overset{(82)}{=} \partial_i \Big(\sum_s \Gamma^s_{jk}\, \partial_s f + b_{jk} \nu \Big) \\
&= \sum_s \Big(\partial_i \Gamma^s_{jk}\, \partial_s f + \Gamma^s_{jk} \partial_{is} f \Big) + \partial_i b_{jk}\, \nu + b_{kj} \partial_i \nu \\
&\overset{(82)(83)}{=} \sum_s \partial_i \Gamma^s_{jk}\, \partial_s f + \sum_s \Gamma^s_{jk} \Big(\textstyle\sum_r \Gamma^r_{is} \partial_r f + b_{is} \nu \Big) + \partial_i b_{jk}\, \nu - \displaystyle\sum_{r,s} b_{jk} \big(b_{ir} g^{rs} \partial_s f \big) \\
&= \sum_s \Big(\partial_i \Gamma^s_{jk} + \textstyle\sum_r \big(\Gamma^r_{jk} \Gamma^s_{ir} - b_{jk} b_{ir} g^{rs} \big) \Big) \partial_s f + \Big(\textstyle\sum_s \Gamma^s_{jk} b_{is} + \partial_i b_{jk} \Big) \nu
\end{aligned}$$

To obtain the similar expression for $\partial_j \partial_{ik} f$, all we need to do is to swap i and j. We now write (86) in terms of the frame. For each i, j, k this gives n equations for the tangential part (the vectors $\partial_s f$ are linearly independent!), and one for the normal part. The n tangential equations give the Gauss equations, the normal component the Codazzi equation. $\square$

It is interesting to note that the relations $\partial_{ij} \nu = \partial_{ji} \nu$ do not yield any new equations.

To explain the significance of the compatibility equations, let us mention:

Theorem (Fundamental Theorem of Surfaces, Bonnet)**.** *Let $U \subset \mathbb{R}^n$ (with $n \ge 2$) be simply connected, $p \in U$. Suppose symmetric matrix valued functions $g, b \colon U \to \mathbb{R}^{n^2}$ are given, where g is positive definit, such that the compatibility equations* (84) *and* (85) *hold. Then there exists a solution $f \colon U \to \mathbb{R}^{n+1}$, $\nu \colon U \to \mathbb{S}^n$ of the hypersurface equations* (82) (83)*. It is unique for given initial values $f(p)$, df_p, $\nu(p)$ subject to the obvious conditions*

$$g_p(X, Y) = \langle df_p \cdot X, df_p \cdot Y \rangle, \quad |\nu(p)| = 1, \quad \nu(p) \perp df_p(X) \qquad \textit{for all } X, Y \in \mathbb{R}^n.$$

6.6. **Conformal parameterization.** Locally, we can assume a conformal parameterization, meaning the following:

Theorem. *Let $f\colon U \to M \in \mathbb{R}^m$ be a parametrization of an immersed surface and $p = f(0)$. Then the image of a neighbourhood of 0 can be parameterized conformally by $\tilde{f}\colon V \to M \in \mathbb{R}^m$, i.e., there is a function $\lambda\colon V \to (0,\infty)$ such that*

$$g_{ij}(x) = \lambda(x)\,\delta_{ij}, \qquad i = 1,2.$$

If f is already real analytic then there is a simple proof going back to Gauss, see [Spivak IV, Ch. 9, Add. 1]. In fact, surfaces of constant mean curvature always have a real analytic parameterization [Spivak V, Ch. 10,9, Thm. 13]. However, the proof of the theorem in the smooth case is a lot more involved.

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In conformal coordinates, the equations for H and K simplify to

$$H = \frac{b_{11} + b_{22}}{2\lambda}, \qquad K = \frac{b_{11}b_{22} - b_{12}^2}{\lambda^2}. \tag{87}$$

We claim the Christoffel symbols of a conformally parameterized surface are

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2\lambda}\partial_1\lambda, \qquad \Gamma^2_{22} = -\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2\lambda}\partial_2\lambda, \tag{88}$$

Indeed, we calculate from definition (81), together with $g^{-1} = (1/\lambda)\delta$:

$$\begin{aligned}
\Gamma^i_{ii} &= \frac{1}{2\lambda}\partial_i g_{ii} = \frac{1}{2\lambda}\partial_i\lambda \quad \text{for} \quad i = 1,2,\\
\Gamma^k_{ii} &= \frac{1}{2\lambda}\big(2\partial_i g_{ik} - \partial_k g_{ii}\big) = -\frac{1}{2\lambda}\partial_k g_{ii} = -\frac{1}{2\lambda}\partial_k\lambda \quad \text{for} \quad k \neq i,\\
\Gamma^k_{12} = \Gamma^k_{21} &= \frac{1}{2\lambda}\big(\partial_1 g_{2k} + \partial_2 g_{1k} - \partial_k g_{12}\big) = \begin{cases} \frac{1}{2\lambda}\partial_2\lambda, & \text{for } k = 1,\\ \frac{1}{2\lambda}\partial_1\lambda, & \text{for } k = 2. \end{cases}
\end{aligned}$$

For $n = 2$, there are two *Codazzi equations* (85), namely $(i,j,k) = (1,2,1)$ or $(1,2,2)$. Moreover, for conformal coordinates these equations simplify:

$$\begin{aligned}
-\partial_1 b_{21} + \partial_2 b_{11} &= \sum_{s=1}^{2} \Gamma^s_{21} b_{1s} - \Gamma^s_{11} b_{2s} = \Gamma^1_{21} b_{11} + (\Gamma^2_{21} - \Gamma^1_{11}) b_{12} - \Gamma^2_{11} b_{22}\\
&\overset{(88)}{=} \frac{\partial_2\lambda}{2\lambda}(b_{11} + b_{22}) \overset{(87)}{=} \partial_2\lambda\, H\\
-\partial_1 b_{22} + \partial_2 b_{12} &= \sum_{s=1}^{2} \Gamma^s_{22} b_{1s} - \Gamma^s_{12} b_{2s} = \Gamma^1_{22} b_{11} + (\Gamma^2_{22} - \Gamma^1_{12}) b_{12} - \Gamma^2_{12} b_{22}\\
&\overset{(88)}{=} -\frac{\partial_1\lambda}{2\lambda}(b_{11} + b_{22}) \overset{(87)}{=} -\partial_1\lambda\, H
\end{aligned} \tag{89}$$

In order to eliminate the derivatives of λ, we differentiate $\lambda H = (b_{11} + b_{22})/2$:

$$\partial_1\lambda\, H + \lambda\,\partial_1 H = \frac{\partial_1 b_{11} + \partial_1 b_{22}}{2} \qquad \text{and} \qquad \partial_2\lambda\, H + \lambda\,\partial_2 H = \frac{\partial_2 b_{11} + \partial_2 b_{22}}{2}$$

In the Codazzi equations (89) we substitute the derivatives of λ by the last equation and obtain

$$-\partial_1 b_{22} + \partial_2 b_{12} = \lambda\,\partial_1 H - \frac{\partial_1 b_{11} + \partial_1 b_{22}}{2} \quad \text{and} \quad \partial_2 b_{11} - \partial_1 b_{12} = -\lambda\,\partial_2 H + \frac{\partial_2 b_{11} + \partial_2 b_{22}}{2}.$$

Hence for any conformally immersed surface we have

$$\lambda\,\partial_1 H = \partial_1 \frac{b_{11} - b_{22}}{2} + \partial_2 b_{12} \qquad \text{and} \qquad -\lambda\,\partial_2 H = \partial_2 \frac{b_{11} - b_{22}}{2} - \partial_1 b_{12}. \tag{90}$$

6.7. **Hopf's theorem for immersed spheres.** For H constant, we can regard (90) as Cauchy-Riemann equations and derive geometric consequences:

Lemma 49. (i) *If $f\colon U^2 \to \mathbb{R}^3$ is a conformal local parameterization of a surface with constant mean curvature then the* Hopf function

$$\Phi\colon U \to \mathbb{C}, \qquad \Phi(z) := \frac{b_{11}(z) - b_{22}(z)}{2} - i b_{12}(z)$$

is holomorphic.
(ii) *Let M be a surface immersed to $\mathbb{R}^3$ with constant mean curvature. Then either the umbilics of M are isolated or all points of M are umbilics.*

Proof. (i) If H is constant then the left hand sides of (90) vanish, and so these equations represent the Cauchy-Riemann equations $0 = \partial_1 \operatorname{Re}\Phi - \partial_2 \operatorname{Im}\Phi = \partial_2 \operatorname{Re}\Phi + \partial_1 \operatorname{Im}\Phi$.

(ii) Consider the Hopf function Φ of a local parameterization f of M. The zeros of Φ coincide with the umbilics of the surface:

$$|\Phi|^2 = \frac{(b_{11} - b_{22})^2}{4} + b_{12}^2 = \frac{(b_{11} + b_{22})^2}{4} + b_{12}^2 - b_{11}b_{22} \overset{(87)}{=} \lambda^2(H^2 - K) = \lambda^2\Big(\frac{\kappa_1 - \kappa_2}{2}\Big)^2.$$

The analytic function Φ either vanishes identically, or has isolated zeros. Consider the subset S of those points of M which have a neighbourhood where the Hopf function of some parameterization vanishes identically. By definition, this subset is open, but it is also closed by analyticity. Since M is connected S is either all of M, or S is empty and the zeros of M are isolated. $\square$

To calculate the index of the curvature line field, let us now characterize curvature directions in terms of the local Hopf function Φ. In conformal coordinates,

$$S^i_{\ j} = \sum_k g^{ik} b_{kj} = \sum_k \frac{1}{\lambda}\delta^{ik} b_{kj} = \frac{1}{\lambda} b_{ij} \tag{91}$$

and so a vector $X = (X_1, X_2) \in \mathbb{R}^2 \setminus \{0\}$ is a curvature direction at $p = f(x)$ if and only if

$$\exists\, \mu \in \mathbb{R}: \quad SX = \frac{1}{\lambda}\begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22}\end{pmatrix}\begin{pmatrix} X_1 \\ X_2\end{pmatrix} = \frac{1}{\lambda}\begin{pmatrix} b_{11}X_1 + b_{12}X_2 \\ b_{12}X_1 + b_{22}X_2\end{pmatrix} \overset{!}{=} \mu \begin{pmatrix} X_1 \\ X_2\end{pmatrix}$$
$$\Leftrightarrow \quad b_{11}X_1X_2 + b_{12}X_2^2 = b_{12}X_1^2 + b_{22}X_1X_2 \quad \Leftrightarrow \quad -b_{12}X_1^2 + (b_{11} - b_{22})X_1X_2 + b_{12}X_2^2 = 0.$$

(Which value must be chosen for μ when we read the equivalence from the right to the left? Distinguish the cases $X_1 = 0$, $X_2 = 0$, $X_1X_2 \neq 0$.) For a general, non-conformal metric, the same condition is more involved to state; see, for instance, [Spivak III, ch.3, (D)].

Therefore the vector field X on $U \subset \mathbb{R}^2$ represents curvature directions if and only if

$$\begin{aligned} 0 &= -b_{12}X_1^2 + (b_{11} - b_{22})X_1X_2 + b_{12}X_2^2 \\ &= \operatorname{Im}\Big[\Big(\frac{b_{11} - b_{22}}{2} - ib_{12}\Big)\Big(X_1^2 + 2iX_1X_2 - X_2^2\Big)\Big] \\ &= \operatorname{Im}\big(\Phi(z)(X_1 + iX_2)^2\big). \end{aligned}$$

Remark. Note that $z \mapsto z^2$ has $\operatorname{Im} z^2 = 0$ exactly on the real and imaginary axis. So for Φ nonzero, the condition $\operatorname{Im}(\Phi(z)(X_1(z)+iX_2(z))^2) = 0$ does select two orthogonal directions at each point, which point in the directions $\arg(\Phi(z)), \arg(\Phi(z)) + \pi/2$. This indicates that the set of two curvature lines can be expected to have a description in terms of the purely real directions for a quadratic form. Moreover, there is a global form of this quadratic form, defined on the "underlying Riemann surface" for our problem, wich is the *Hopf differential* $\Phi(x)dz^2$.

For a complex number w let $\arg w$ denote an angle which w makes with the x-axis, so that $w = |w| \exp(i \arg w)$. Note that $\operatorname{Im} w = 0$ if and only if $\arg w \in \pi\mathbb{Z}$. Thus $X = X_1 + iX_2$ is a principal curvature direction at z if and only if there exists $m \in \mathbb{Z}$ such that

$$m\pi = \arg\big(\Phi(z)X^2\big) = \arg\Phi(z) + 2\arg X$$

or

$$\arg X = -\frac{1}{2}\arg\Phi(z) + \frac{m\pi}{2}. \tag{92}$$

Proposition 50. *Suppose $f\colon U \to \mathbb{R}^3$ is a conformal immersion into a surface M of constant mean curvature H with Hopf function Φ. Let $p = f(0)$ be an isolated umbilic of M, so that Φ has a zero of order n at 0, that is,*

$$\exists n \in \mathbb{N}: \quad \Phi(z) = \sum_{k\geq n} a_k z^k \qquad a_n \neq 0, \quad \textit{for } z \sim 0.$$

Then the index of the curvature line field at p is $-n/2 < 0$.

In particular, the curvature line field satisfies the Loewner conjecture.

Proof. Let $c\colon [0, 2\pi] \to \mathbb{R}^2$ be a small circle around 0. To compute the index of the curvature line field, we let $\vartheta\colon [0, 2\pi] \to \mathbb{R}$ be the angle between the curvature line field and the x-axis. Then $2\pi \operatorname{ind}(p) = \vartheta(2\pi) - \vartheta(0)$. Then by (92) we have

$$\vartheta(t) = -\frac{1}{2} \arg \Phi(c(t)) + \frac{m\pi}{2}$$

where $m \in \mathbb{Z}$ is constant, by continuity, and so

$$2\pi \operatorname{ind}(0) = \vartheta(2\pi) - \vartheta(0) = -\frac{1}{2}\big(\arg \Phi(c(2\pi)) - \arg \Phi(c(0))\big).$$

But the function Φ is holomorphic by Lemma 49. Hence it has a power series expansion, as required in Lemma 43(*ii*), and so the latter lemma gives

$$\arg \Phi(c(2\pi)) - \arg \Phi(c(0)) = \frac{\Phi(c(2\pi))}{|\Phi(c(2\pi))|} - \frac{\Phi(c(0))}{|\Phi(c(0))|} = 2\pi n. \qquad \square$$

Theorem 51 (H. Hopf 1951)**.** *Suppose M is a topological sphere, immersed into $\mathbb{R}^3$ with constant mean curvature $H \geq 0$. Then M is a sphere of radius $1/H$.*

Proof. If the umbilic points of M are dense, then M has constant principal curvatures $\kappa = H$ and so M is a distance sphere of radius $1/H$.

By Lemma 49 otherwise the umbilic points are isolated. So on our compact surface the umbilics form a finite set $\{p_1, \ldots, p_k\}$.

Consider the index of the curvature line field P at the points p_i, as defined in (6.4). For our surface with constant mean curvature we showed in Prop. 50 that each umbilic has a negative index $\operatorname{ind}_{p_i} P < 0$. So the Poincaré-Hopf Theorem 45 gives

$$\chi(M) = \sum_i \operatorname{ind}_{p_i} P \leq 0, \tag{93}$$

contradicting $\chi(M) = 2$ for a sphere. $\square$

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We can also say something about other topological types:

Theorem 52. *Suppose M is a surface immersed into $\mathbb{R}^3$ with constant mean curvature H.*
(*i*) *If M is a torus, then M does not have umbilic points.*
(*ii*) *If M is a surface of genus $g \geq 2$, then M has umbilic points.*

Proof. Again, if M were totally umbilic with H constant then M would be a sphere, which is impossible. So by Lemma 49 the umbilic points are isolated. Then (93) is defined.

By Prop. 50 the index of any umbilic is negative. Therefore, (93) is negative if and only if umbilic points exist. This gives:

(i) A torus has Euler characteristic 0 and so no umbilic points can exist.
(ii) Due to $\sum \operatorname{ind} P = \chi(M) = 2 - 2g \leq -2$ umbilics must exist. □

Note that (i) gives that the curvature lines are globally orthogonal on a torus, and so form a nice rectangular net.

Hopf's theorem makes essential use of dimension 2 –for conformal parameters– and of co-dimension 1 –for the Codazzi equation. However, it generalizes to other ambient spaces. In order to obtain suitably good Codazzi equations, the ambient space must be sufficiently symmetric:
• In the space forms $\mathbb{H}^3$ and $\mathbb{S}^3$, the Codazzi equation is simple to state, and so constant mean curvature spheres are again distance spheres. For $\mathbb{S}^3$, this was proved by Almgren in 1966 (see also "Minimal surfaces in $\mathbb{S}^3$" by Lawson 1970, Lemma 1.2). See [Jost, Cor. 1.4.7].
• For the product spaces $\Sigma(\kappa) \times \mathbb{R}$, where $\Sigma(\kappa)$ is a surface of constant curvature κ. Abresch and Rosenberg in 2004 generalized the Hopf differential, and so again the topological spheres with constant mean curvature are the rotationally invariant spheres in these spaces, which are classified as solutions of an ODE up to isometry.
• Abresch claimed the same result for the more general homogeneous 3-manifolds $E(\kappa, \tau)$, which include the products and $\mathbb{S}^3$ (but not $\mathbb{H}^3$). In this case again the spheres are characterized as surfaces invariant under rotation about the fibres of these spaces, and so are uniquely determined ODE solutions. A reference for the proof is "Constant mean curvatyure surfaces in homogeneous 3-manifolds" by Daniel/Hauswirth/Mira from 2009. There is further work by Daniel and Mira on the case of the homogeneous 3-manifold Sol. • The same technique also applies to other geometric problems. One is Nitsche's theorem that any constant mean curvature disk which meets a unit ball orthogonally must be part of the sphere. Such surfaces arise for the *partitioning problem* in the ball, where an interface of least area is sought which separates two components of given volume. See [Jost, Cor 1.4.6].

After the theorem's by Alexandrov and Hopf were achieved in the 1950's, it appeared to many people that also all compact surfaces, immersed to $\mathbb{R}^3$ with constant mean curvature, should be distance spheres. This was known as the *Hopf conjecture*, and more people tried to prove it than to find counterexamples. Only in the 1980's it became clear that the Hopf conjecture is not true:
• In 1986 Wente found constant mean curvature tori ($g = 1$), and in some sense all of these were classified by Pinkall and Sterling in 1989.
• Kapouleas in 1991 found constant mean curvature surfaces of genus $g \geq 3$ by perturbing a set of touching spheres, so that the touching is replaced by small unduloidal or nodoidal necks. In 1993 he also found examples with $g = 2$ by "fusing" two Wente tori together.
Of course, all these examples cannot be Alexandrov embedded.

6.8. **Problems.**

Problem 27 – Stereographic projection:

Let $f_\pm$ be projection of $\mathbb{S}^n$ from the north or south pole onto the equatorial plane. Check geometrically (no computation!) that the transition map $\tau(x) := (f_+^{-1} \circ f_-)(x) = x/|x|$

a) is conformal.
b) Is τ orientation preserving?

Problem 28 – Singularities of the curvature line field:

Use images of line fields with singularities of low index such as p.109 in Hopf, Differential geometry in the large.

a) Assuming these represent a curvature line field of some surface, draw the other curvature line field and convince yourself that it has the same index.
b) Can you imagine an example of a surface with the depicted curvature line field? Consider only the indices admitted by the Loewner conjecture.

Problem 29 – Index of spherical vector fields:

Draw unit vector fields on $\mathbb{S}^2$. It can helps to regard them as the stereographic projection of vector fields on $\mathbb{R}^2$ (see back side!).

a) With singularities at north and south pole (recall from class).
b) With only one singularity – what must be its index?
c) With three singularities. (A strategy is to merge singularities.)
d) Can you find similar vector fields on $\mathbb{S}^n$?

Problem 30 – Euler characteristic and genus:

a) Position a torus of revolution in space suitably, so that you can the Poincaré-Hopf theorem for the torus by choosing the gradient vector field of the height function to compute the index.
b) Now do the same for a nice model of an oriented surface $\Sigma_g \subset \mathbb{R}^3$ of genus $g \in \mathbb{N}_0$, homeomorphic to a sphere with g handles attached. Show that the Euler characteristic is

$$\chi(\Sigma) = 2 - 2g.$$

Problem 31 – Euler characteristic of spaces:

a) If you happen to know the spaces, calculate the Euler characteristic of a Klein bottle and of $\mathbb{R}P^2$.
b) If you know about stereographic projection: Calculate the Euler characteristic from the 4-dimensional cube (also called hypercube) or 4-dimensional tetrahedron.

c) Is there a vector field on $\mathbb{S}^3$ without zeros?
Hint: Try to find $X(x)$ in terms of the coordinates of x.

Problem 32 – Euler characteristic of islands:

a) Consider an island with no lakes. Show that # peaks $-$# passes $+$# sinks $= 1$, provided all these numbers are finite. Here, peaks are local maxima of height, sinks local minima, and passes critical points which are neither peaks nor sinks.
Hint: Poincaré-Hopf. Which vector field is appropriate?

b) Find an island with 3 peaks, 1 pass, and no sinks. Correct the formula in part a) (!).

c) Now admit lakes and show, with a correction yet to be included, that

$$\#\text{ peaks} - \#\text{ passes } + \#\text{ sinks } = 1 - \#\text{ lakes}\ .$$

Problem 33 – Symmetries and umbilics:

Let $f\colon U^2 \to \mathbb{R}^3$ be a two-dimensional surface. For simplicity, suppose $f(p) = 0$ for some $p \in U$ such that $T_p f$ agrees with the xy-plane.

a) Let $R_3\colon \mathbb{R}^3 \to \mathbb{R}^3$ be $120°$ rotation about the z-axis. Prove: If $f(U)$ is invariant under the rotation R, then p is an umbilic *[Nabelpunkt]*.

b) Does part a) hold for an arbitrary rotation R_k of angle $2\pi/k$ where $k = 2, 3, \ldots$?

c) Let $R_6\colon \mathbb{R}^3 \to \mathbb{R}^3$ be rotation by $60°$ about the z-axis, and S reflection in the xy-plane. Prove: If $f(U)$ is invariant under $R_{\overline{6}} := S \circ R_6$, then the principal curvatures at p vanish, that is p is a *flat point*.

Problem 34 – Example for integrability conditions:

Suppose the fundamental forms of an immersion $f\colon U \to \mathbb{R}^3$ satisfy

$$g_{ij} = \delta_{ij} \quad \text{and} \quad b_{11} = b_{12} = b_{21} = 0, \quad b_{22}(x,y) = b(x,y)$$

a) Derive a necessary condition from the Gauss and Codazzi equations.

b) Given the condition, the fundamental theorem for surfaces guarantees the existence of a surface with fundamental forms g and b. What is the Gauss curvature of this surface?

c) Give two examples of such surfaces (or characterize all such immersions $f\colon \mathbb{R}^2 \to \mathbb{R}^3$).

Problem 35 – Gauss curvature and Christoffel symbols:

Consider an immersion $f\colon U^2 \to \mathbb{R}^3$ whose Gauss curvature does not vanish identically.

a) Can all Christoffel symbols vanish identically?

b) Can the first fundamental form be constant?

Problem 36 – Gauss curvature of the hyperbolic plane:

Suppose the upper halfplane $U = \{(x, y) : y > 0\}$ parameterizes an immersion into $\mathbb{R}^3$ with the conformal first fundamental form $g = \frac{1}{y^2}\delta$.

a) What are the eight Christoffel symbols Γ_{ij}^k?
   Hint: We calculated Christoffel symbols for conformal parameterizations in class.
b) Suppose a subdomain of U immerses to $\mathbb{R}^3$ with first fundamental form g. What is its Gauss curvature? By a theorem of Hilbert the entire hyperbolic plane (U, g) does not immerse into $\mathbb{R}^3$.

Problem 37 – Totally umbilic surfaces are spheres:

We want to prove that a surface Σ for which each point is umbilic is a subset of the sphere $\mathbb{S}^2$ or the plane. (We do not assume constant mean curvature.)

a) Consider a parameterization $(f(x, y), \nu(x, y))$ and differentiate the equation for a principal curvature direction to derive the equation $\nu + \kappa f \equiv C$ where $\kappa(x, y)$ and C are constant.
b) Why is Σ contained in a plane when $\kappa \equiv 0$? Otherwise, take the equation from part a) and show that f has constant distance $1/|\kappa|$ to some point.

7. Non-compact embedded constant mean curvature surfaces

The goal of this section is to show the following uniqueness theorem, proved by Korevaar, Kusner, and Solomon (1990):

Theorem 53. (i) *A properly embedded annulus in* $\mathbb{R}^3$ *with constant mean curvature* $H \equiv 1$ *is a Delaunay unduloid.*
(ii) *There is no properly embedded plane with* $H \equiv 1$.

Here we made use of a topological term:

Definition. A continuous map $F\colon \Omega \to \mathbb{R}^n$ is *proper* if for all $R > 0$ the preimage of any ball $F^{-1}(\overline{B}_R)$ is a compact subset of Ω.

This property means that a path $\gamma\colon (0,1] \to \Omega$ which tends to the boundary, i.e., $\lim_{t\to 0}\gamma(t) \in \partial\Omega$, has an image path $F \circ \gamma$ tending to infinity in the sense $\lim_{t\to 0}|F(\gamma)| = \infty$.

Moreover, for part (i), an *embedded annulus* is a submanifold $M \subset \mathbb{R}^3$ which is the image of a parameterization $f\colon D \setminus \{0\} \to \mathbb{R}^3$, which is an embedding (homeomorphism onto its image). Similarly, for the *plane* of part (ii), the domain of f is $\mathbb{R}^2$ (or the disk).

Examples. 1. If Ω is compact, then F is proper (why?).
2. One dimension lower, a spiral is a non-proper submanifold of $\mathbb{R}^2$, homeomorphic to the real line.
3. Nadirashvili in the 1990's showed that there exist complete minimal immersions into the ball. These are highly non-proper: Any neighbourhood of the boundary of the ball contains a non-compact piece of the surface. (In particular this is a counterexample to the *Calabi-Yau conjecture*, which says that the ball cannot contain a complete surface with non-positive Gauss curvature.)

Remark. There are counterexamples to the theorem when the embeddedness assumption is replaced by immersedness. For (i) these are the *bubbletons* of Pinkall, Sterling, and Wente, and for (ii) these are the *Smyth surfaces*, generalizing Enneper's minimal surfaces (they also have an intrinsic rotation.)

Statement (ii) is restated and proven in Thm. 58, statement (i) follows from ...

7.1. Annular ends.

We will first look at ends. An *annular end* is a proper immersion $F\colon \overline{D} \setminus \{0\} \to \mathbb{R}^3$.

Lemma 54 (Plane Separation)**.** *Let* $M \subset \mathbb{R}^3$ *be a properly embedded annular end with* $H \equiv 1$*. Suppose* $P_\pm$ *are two parallel planes bounding two disjoint closed halfspaces* $H_\pm$*. If the distance of the planes satisfies* $\operatorname{dist}(P_+, P_-) > 2$*, then at least one of the halfspaces* $H_\pm$ *contains only compact connected components of* $M \cap H_\pm$.

Note that the diameter of an unduloid end is less than 2 with 2 attained at the spherical limit. Thus the lemma certainly holds for an end of an unduloid.

In the proof, we will refer to a concept of differential topology which we will not pursue in depth:

Definition. The *linking number* $\operatorname{link}(\gamma, \delta)$ of two disjoint oriented simple smooth loops $\gamma, \delta \subset \mathbb{R}^3$ is the following: Take an oriented smooth disk Δ with $\partial\Delta = \delta$ such that γ intersects Δ transversally (i.e., not tangentially) in at most finitely many points. Then $\operatorname{link}(\gamma, \delta)$ is the number of points of $\gamma \cap \Delta$ counted with a sign, depending on the relative orientation of the intersection point.

It can be shown that this definition is independent of the choice of the disk Δ, and has the following properties.
- Symmetry: $\operatorname{link}(\gamma, \delta) = \operatorname{link}(\delta, \gamma)$
- $\operatorname{link}(\gamma, \delta) = 0$ if there exists Δ disjoint from γ.
- If γ is homotopic (or better homologous) to $\tilde{\gamma}$ in $\mathbb{R}^3 \setminus \delta$ then $\operatorname{link}(\gamma, \delta) = \operatorname{link}(\tilde{\gamma}, \delta)$.

Proof of plane separation. On the contrary, suppose both halfspaces contain noncompact components of M. Choose R large enough so that $F(\mathbb{S}^1) \subset B_R$.

Assume the planes are parallel and symmetric to the xy-plane, $P_\pm = \{z = \pm C\}$ for $C > 1$. Let T be the solid torus of revolution with radius $(C+1)/2$ whose core curve c is the circle of radius $R + C$ in the xy-plane. Then T is strictly contained in the slab *[Schicht]* between P_+ and P_-, and lies in the complement of B_R.

Inside the torus T and not touching ∂T, we can place unit spheres. The strategy of the proof is to find such a sphere which touches M from one side and with the same mean curvature normal, hence violating the maximum principle. Thus we need to locate an appropriate "sheet" of M, "crossing" the torus.

By the Theorem of Sard (which we will not explain here) we may alter R slightly, if necessary, so that M is nowhere tangent to ∂T. The embeddedness of M then implies that $M \cap \partial T$ consists of disjoint simple loops.

Claim (i): For R large enough there is a simple loop $\delta \subset M \setminus T$ which bounds a disk $\Delta \subset M$ with $\operatorname{link}(\delta, c) = 1$ but $\operatorname{link}(\delta, \mu) = 0$, where c is the core curve and μ a meridian of the torus ∂T (e.g., $\mu \subset \partial T$ is a circle in a vertical plane).

Since both halfspaces contain noncompact components of M, we can increase R to a value such that there are curves $\delta_\pm \subset H_\pm \cap M$, running from ∂B_R to ∞. Let $d_\pm := F^{-1}(\delta_\pm) \subset D$ be the preimages, each of them approaching 0. There are compact subsets of $d_\pm$, again denoted by $d_\pm$, which can be completed to a loop

$$d = d_+ \cup d_0 \cup d_- \cup d_\infty \subset D \quad \text{with} \quad F(d_0) \subset B_R, \;\; F(d_\infty) \subset \mathbb{R}^3 \setminus B_{3R},$$

such that d is simple and null homotopic in $D \setminus \{0\}$. Then d is the boundary of a disk in $D \setminus \{0\}$, and by the embeddedness of M, the image loop $\delta = F(d)$ bounds a disk $\Delta \subset M$.

For large R, since δ is disjoint from ∂T, the meridian curve μ bounds a disk disjoint from δ, and so $\operatorname{link}(\delta, \mu) = 0$. On the other hand, to see $\operatorname{link}(\delta, c) = 1$ note that $F(d_0)$ intersects the horizontal disk bounded by c from below to the top. This proves the claim.

Claim (ii): There is a domain $\Delta^0 \subset \Delta \cap T$ such that $\partial\Delta^0 \subset \partial T$ contains one boundary component homotopic to the meridian $\pm\mu$.

To show the claim, decompose $\Delta \cap \partial T$ into its components $\gamma_1 \cup \ldots \cup \gamma_k$, all of which are topological circles. We first show that within the set ∂T either γ_i is null homotopic or homotopic to $\pm\mu$. To see this note that the torus ∂T has fundamental group $\mathbb{Z} \oplus \mathbb{Z}$, generated by the meridian μ and a latitude λ. So $\pi_1(\gamma_i) = (m, l) \in \mathbb{Z} \oplus \mathbb{Z}$. But then it is not hard to see that the linking number satisfies

$$\operatorname{link}(\gamma_i, \delta) = m \operatorname{link}(\mu, \delta) + l \operatorname{link}(\lambda, \delta) \overset{(i)}{=} l \operatorname{link}(\lambda, \delta).$$

On the other hand, γ_i and δ are not linked since γ_i spans a disk disjoint to δ. So indeed $l = 0$.

By (i) also $\operatorname{link}(c, \delta) = 1$, and moreover δ is homotopic to $\bigcup \gamma_i$ within $\Delta \setminus T$. Thus at least one of the γ_i is homotopic to $\pm\mu$.

Now consider $\Delta \cap T$. The preimage of this set $F^{-1}(\Delta \cap T)$ decomposes into a finite union of domains, bounded by $\bigcup \gamma_i$. From one of these domains pick an innermost one, bounded by γ_i with $\gamma_i \sim \pm\mu$. We let Δ^0 be the component bounded by γ_i. This establishes claim (ii).

We now lift T to its universal cover $\tilde{T}$. Since the fundamental group $\pi_1(\Delta^0)$ has trivial image in $\pi_1(T)$ we can also choose a compact lift $\tilde{\Delta}^0$ in $\tilde{T}$. We claim that $\tilde{\Delta}^0$ separates the two ends of T. Indeed, if $\alpha \subset T$ is an arc running from one end to the other, that is, α is homotopic to the lift of the core curve c, then α meets $\tilde{\Delta}^0$, because the number of signed intersections of α with $\tilde{\Delta}^0$ agrees with the number of signed intersections of $\tilde{c}$ with $\tilde{\Delta}^0$, which is

$$\operatorname{link}(\tilde{c}, \partial\tilde{\Delta}^0) = \operatorname{link}(c, \partial\Delta^0) = \operatorname{link}(c, \pm\mu) = \pm 1.$$

Now place a unit sphere S with midpoint on $\tilde{c}$ in the component of $\tilde{T} \setminus \tilde{\Delta}^0$ which has positive mean curvature for the interior normal. When sliding towards $\tilde{\Delta}^0$, the first point of contact will occur at an interior point of $\tilde{\Delta}^0$. By the maximum principle, $\tilde{\Delta}^0$ and hence M will agree with S, which is a contradition since S is entirely contained in T, unlike M. $\square$

`24. Lecture, Thursday 8.7.10` ________________________________

7.2. **Height bounds.**

Lemma 55. *Suppose a graph $M = \{x^{n+1} = h(x^1, \dots, x^n)\}$ has constant mean curvature $H > 0$ and boundary values $h = 0$ over a compact domain in $\mathbb{R}^n$. Then $h(x) \le 1/H$.*

The estimate is sharp for a hemisphere of radius $1/H$.

For the proof, we need a Riemannian version of the Laplace operator. Given an immersion $f\colon U \to \mathbb{R}^m$ with first fundamental form g, the operator

$$L_g u := \sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \partial_i\big(g^{ij}\sqrt{g}\,\partial_j\big)u = \sum_{i,j=1}^{n} g^{ij}\partial_{ij}u + \sum_j \Big(\sum_i \frac{1}{\sqrt{g}}\partial_i\big(g^{ij}\sqrt{g}\big)\Big)\partial_j u \tag{94}$$

is a linear partial differential operator of second order, as in (66). The first fundamental form g is positive definite, and so its inverse g^{-1} is also positive definite: Indeed, the eigenvalues of g^{-1} are inverses of the positive eigenvalues of g. Hence the operator (94) is elliptic, and it is uniformly elliptic on each compact subset of U.

If we have two parameterizations f^α, f^β of a submanifold Σ, and a function $u \in C^\infty(\Sigma, \mathbb{R})$ then the operators (94) agree, $L_{g^\alpha}(u \circ f^\alpha) = L_{g^\beta}(u \circ f^\beta)$. See e.g., Forster, Analysis III, p.28ff, for a proof. That is, there is an operator

$$\Delta_\Sigma\colon C^2(\Sigma, \mathbb{R}) \to C^0(\Sigma, \mathbb{R}),$$

defined by the local representations (94). It is called the *Laplace-Beltrami operator*. Another way to introduce the Laplace-Beltrami operator, and to understand its specific form is in terms of integration by parts: When we integrate the Riemannian gradient by parts we get the Riemannian Laplacian, which is the Laplace-Beltrami operator, see the computation in the proof.

Proof. We need two equations:

$$\begin{aligned} \Delta_M h &= -nH\nu^{n+1}, \\ \Delta_M \nu^{n+1} &= -(\kappa_1^2 + \ldots + \kappa_n^2)\nu^{n+1}. \end{aligned} \tag{95}$$

It is equivalent, and more common, to write the second equation in terms of the squared L^2-norm of the second fundamental form $\|B\|^2$, see (36).

Although it is nicer to derive the first equation from first principles, using Riemannian terminology, it can also be checked explicitely for a graph: Then $\nu^{n+1} = 1/\sqrt{1 + |\nabla h|^2}$ and $g_{ij} = \delta_{ij} + \partial_i h \partial_j h$; in particular, for dimension $n = 2$, which we are ultimately interested in, the inverse is $g^{-1} = \frac{1}{\det g}\left(\begin{smallmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{smallmatrix}\right)$ and so the computation of the operator (94) is straightforward.

The second equation in (95) comes from the second variation formula (35) after integration by parts. To see that, consider a particular variation, given by vertical translation of our graph, i.e.

$$\big(x, h(x) + t\big) = (x, h(x)) + te_{n+1}.$$

We need two properties.
• The variation vector is e_{n+1}. If we rewrite it as a normal variation $u\nu$ we have $u = \nu^{n+1}$ (draw a triangle!).
• The second variation of J is constant,

$$0 = \delta_{u\nu} J \overset{(36)}{=} \int_U \|\nabla u\|^2 - u^2(\kappa_1^2 + \ldots + \kappa_n^2)\, dS.$$

Indeed, the area of the translated graphs stays constant, and the volume is affine linear in t, hence the second derivative vanishes.
Let us consider the first term; we let N be the interior normal to the domain U.

$$\begin{aligned}
\int_M \|\nabla u\|^2\, dS &= \int_U \sum_{ij} g^{ij}\partial_i u\, \partial_j u (\sqrt{g}\, dx) \\
&= \int_U \sum_{ij} \partial_i \Big(g^{ij} u\, \partial_j u \sqrt{g}\Big) - u\, \partial_i \Big(\sqrt{g} g^{ij} \partial_j u\Big) dx \\
&\overset{\text{Div. Thm.}}{=} \int_{\partial U} \sum_{ij} \Big(g^{ij} u\, \partial_j u \sqrt{g}\, N^i\Big)\, dA - \int_U u \frac{1}{\sqrt{g}} \sum_{ij} \partial_i \Big(\sqrt{g} g^{ij} \partial_j u\Big) dS \\
&= 0 - \int_U u \Delta_\Sigma u\, dS
\end{aligned}$$

[Sorry, I realize it takes two more arguments to finish the argument: 1. The boundary integral vanishes so that the last equality holds. 2. We have $0 = \int_U u(\Delta_\Sigma u + |B|^2 u)\, dS$. We need to show that this implies $\Delta_\Sigma u + |B|^2 u = 0$. This comes from the fact a graph is strictly stable, or $\Delta_\Sigma + |B|^2$ is positive.]

Summing (95) we obtain

$$\Delta_M \big(Hh - \nu^{n+1}\big) = (\kappa_1^2 + \ldots + \kappa_n^2 - nH^2)\nu^{n+1} \geq 0,$$

where the inequality follows from the Cauchy-Schwarz as follows:

$$n \sum \kappa_i^2 = \|(1,\ldots,1)\|^2 \|(\kappa_1,\ldots,\kappa_n)\|^2 \geq \big\langle (1,\ldots,1), (\kappa_1,\ldots,\kappa_n)\big\rangle^2 = \Big(\sum \kappa_i\Big)^2 = (nH)^2$$

Now we apply the weak maximum principle Lemma 28 to the Laplace-Beltrami operator $L := \Delta_\Sigma$. As $Hh - \nu^{n+1} \leq 0$ on ∂M it gives $Hh - \nu^{n+1} \leq 0$ in the interior of the graph domain. In fact, since L is uniformly elliptic only on compact subsets (the gradient could explode at the boundary!) we employ this argument on compact subdomains where $h \geq \varepsilon > 0$, and so obtain the above result with $H\varepsilon$ in place of 0; letting $\varepsilon \to 0$ then gives the conclusion we stated. Since ν is a unit vector we conclude $h \leq \frac{1}{H}\nu^{n+1} \leq \frac{1}{H}$, as desired. □

We can now bound the height of any embedded compact surface with $H \equiv 1$:

Lemma 56 (Height Estimate)**.** *Let $M \subset \mathbb{R}^{n+1}$ be a bounded embedded surface with constant mean curvature $H > 0$ and $\partial M \subset \{x^{n+1} = 0\}$. Then $M \subset \{|x^{n+1}| \leq 2n/H\}$.*

A sphere, punctured at the south pole, shows that the estimate is sharp. In the immersed case, no such estimate can hold. Note that the limiting case $H \to 0$ is also correct as a direct application of the maximum principle shows.

Proof. We apply Alexandrov reflection w.r.t. horizontal planes $\{x_{n+1} = C\}$ twice, once starting from the top, once from the bottom.

To carry out Alexandrov reflection at the top, consider the component of $\mathbb{R}^n_+ \setminus M$ which has compact closure; denote its closure with V. We let all other notation agree with the proof of Thm. 36, in particular let $a \geq 0$ again be the inf of the real numbers $s > 0$ such that the reflected image of M^s_+ is contained in V^s_-.

As a result of Alexandrov reflection, the reflected image of M^a_+ and the remaining portion of M in the lower half space must agree, $\sigma_a(M^a_+) = M^a_- \cap \mathbb{R}^n_+$. Note that by Cor. 39 each of these two portions are graph, and so their height is bounded by the previous lemma. For M^a_- to be graph we must have $a \leq n/H$. Moreover, for M^a_+ to be graph the height of M^a_+ can be n/H at most, that is, the x^{n+1}-component of M^a_+ is at most $2n/H$.

Working similarly from below, wo find that the x^{n+1} component is also bounded from below by $2n/H$. ☐

7.3. **Cylindrical boundedness of ends.** Let M be a properly immersed annular end. We call $a \in \mathbb{S}^2$ an *axis vector* of M if there exist $p_i \in M$ with $|p_i| \to \infty$ and $p_i/|p_i| \to a$. We also need a notation for solid cylinders and half-cylinders ($a \in \mathbb{S}^2$, $R > 0$, $p \in \mathbb{R}^3$):

$$C_{a,R}(p) := \big\{x \in \mathbb{R}^3 : \mathrm{dist}(x, \{p + \mathbb{R}a\}) \leq R\big\}, \qquad C^+_{a,R}(p) := C_{a,R}(p) \cap \{\langle x - p, a\rangle \geq 0\}$$

`25. Lecture, Tuesday 13.7.10` ______

The embedded surfaces of revolution with nonzero constant mean curvature, the unduloids, have annular ends which are contained in a half-cylinder, whose radius could be chosen to be $R = 1/H$. We can make a similar statement for any embedded annular end:

Proposition 57. *Any properly embedded annular end $M \subset \mathbb{R}^3$ with constant mean curvature $H \equiv 1$ is contained in some solid half-cylinder $C^+_{a,R}(p)$.*

Remarks. 1. Embedded annular ends of minimal surfaces can be shown to be asymptotic to a (half-)catenoid or a plane. So they are naturally contained in a half-space. On the other hand, our statement also holds for any $H \neq 0$ and gives a radius $R \sim 1/H \to \infty$ as

$H \to 0$. Hence in the limit, the half-cylinders of our Proposition approach the halfspaces needed for the minimal case.

2. The proposition does not hold for the immersed case. An example for immersed annular ends is given by the Smyth surfaces which are immersed planes with constant mean curvature. As we pointed out before, they generalize Enneper's minimal surfaces and also have an intrinsic rotation. In terms of this property, B. Smyth established existence of the surfaces by solving an ODE in 1993. Their end is not contained in cylinder (I would need to have a closer look at them to come to a more precise statement).

Proof. Suppose $\partial M \subset B_{R-2}$ for $R > 4$, and pick an axis vector $a \in \mathbb{S}^2$ of M. Let Π be a plane containing a, and let N denote its unit normal. For simplicity, let us first consider the case that the halfspace $\{x : \langle x, N\rangle \leq 0\}$ contains a noncompact part of M.

Let $\Pi_{R-2} = \Pi + N(R-2)$ be a plane tangent to B_{R-2}, and H_{R-2} be the closed halfspace with boundary Π_{R-2} disjoint to B_{R-2}. The plane separation lemma gives that H_{R-2} will contain only compact pieces of M, and since ∂M is disjoint to H_{R-2} these compact pieces have boundary in Π_{R-2}. Lemma 56 then says that the similarly defined halfspace H_R cannot contain any points of M.

Let us now remove our additional assumption that the halfspace $\{\langle x, N\rangle \leq 0\}$ contains noncompact components of M. To do so, consider tilted planes $\Pi(\varepsilon)$ through the origin, normal to the vector $N(\varepsilon) := N\cos\varepsilon - a\sin\varepsilon$ where $0 < \varepsilon < \pi$. For all these angles, the halfspace of $\Pi(\varepsilon)$ to the side of $N(\varepsilon)$ contains the vector a, and hence contains a noncompact component of M. So the Plane Separation Lemma, applied as before, shows that the tilted halfspaces $H_R(\varepsilon)$ will be disjoint from M. In particular,

$$\bigcup_{0<\varepsilon\leq\pi/2} H_R(\varepsilon) \supset \operatorname{int} H_R(0) = \operatorname{int} H_R$$

does not contain any point of M, as claimed.

Now note that our argument works with a circle worth of normals $\{N \perp a\}$. Intersecting the corresponding halfspaces gives the desired statement

$$M \subset B_R \cup C^+_{a,R}(0) \subset C^+_{a,R}(-Ra).$$

□

The simplest generalization of compact surfaces are surfaces with finitely many ends:

Definition. (i) We call a submanifold $\Sigma \subset \mathbb{R}^3$ a surface of *finite topology* if Σ has finite genus g and $\overline{\Sigma} = \Sigma \cup \{p_1, \dots, p_k\}$ is a compact submanifold.
(ii) We call a proper immersion of a surface with finite topology a *surface of genus g with k ends.*

The terminology here is consistent: An *end* of a surface of finite topology is a neighbourhood of a puncture, homeomorphic to $\overline{D} \setminus \{0\}$. In this sense, a surface of genus g with k ends indeed has k properly immersed ends.

In contrast, a common example for surfaces with *infinite topology* are periodic surfaces. Examples go back to the 19th century (the triply periodic Schwarz minimal surfaces), while constant mean curvature examples are more recent (starting with Lawson 1970).

The following theorem gives an idea of how embedded constant mean curvature surfaces can look like:

Theorem 58 (Meeks 1988)**.** *A surface of finite topology which is properly embedded to* $\mathbb{R}^3$ *with constant mean curvature* $H > 0$ *has cylindrically bounded ends whose axis vectors cannot lie in an open hemisphere. In particular,*
$k = 1:$ *a properly embedded plane is impossible,*
$k = 2:$ *a properly embedded annulus is contained in a solid cylinder*
$k = 3:$ *a properly embedded surface with three ends is contained in a slab.*

Proof. As indicated above, if f_i^α parameterizes a neighbourhood of a singular point p_i in Σ by $\overline{D} \setminus \{0\}$, then the composition with the embedding φ gives a properly embedded annular end $F_i = \varphi \circ f_i^\alpha$. So Prop. 57 applies to show that F_i has image contained in a solid half-cylinder $C_i^+ := C_{a_i,r_i}^+(p_i)$.

If we remove k disk-type open neighbourhoods of the p_i from our model surface Σ, we are left with a compact subset $\Sigma_0 \subset \Sigma$. So $M_0 := \varphi(\Sigma_0)$ is compact and hence contained in some ball B_R. Altogether this shows the containment of our surface in a set

$$M \subset \overline{B}_R \cup \bigcup_{i=1}^{k} C_i^+. \tag{96}$$

To prove that the axis vectors a_i cannot all be contained in an open hemisphere, let us suppose on the contrary that for some $e \in \mathbb{S}^2$ we have $\langle a_i, e\rangle < 0$ for all i. Then all half-cylinders C_i^+ point in the direction of e, and so for any $c \in \mathbb{R}$, the intersection of the halfspace $H^c := \{x : \langle x, e\rangle \geq c\}$ with the right hand of (96) is a compact set. By properness, $M \cap H^c$ is compact. Then the Height Estimate Lemma 56 gives that M is disjoint from H^{c+2}. But c is arbitrary and so M can only be empty, contradiction.

Let us now use elementary geometry to prove the statements for $k = 1, 2$ and 3:
$k = 1$: One axis vector is always contained in an open halfspace, making this case impossible.
$k = 2$: Two axis vectors must be opposite, $a_1 = -a_2$. Therefore the ends are contained in two solid half-cylinders, $C_{a_1,R_1}^+(p_1)$ and $C_{a_2,R_2}^+(p_2)$, while the remaining compact subset

of M is contained in a ball. These three sets are contained in a solid cylinder with axis $a_1 = -a_2$, and suitably large radius.
$k = 3$: If the first two axis vectors are linearly dependent then together with the third vector they span a 2-dimensional space W. The corresponding three solid half-cylinders, as well as a ball containing the remaining compact portion of M, are then contained in a sufficiently large slab parallel to W. If the first two axis vectors are not linearly dependent, and the third does not linearly dependent on them then the three vectors lie in an open halfspace. So again the surface is contained in slab. □

7.4. **Alexandrov reflection.** We now want to show that a properly embedded annulus with $H \equiv 1$ is an unduloid. For this end, Korevaar, Kusner, Solomon adapt the Alexandrov reflection technique to the non-compact setting. There are some points in their exposition which I am not certain about and so I use some workarounds here.

Let us first restate Alexandrov reflection. Consider a properly embedded complete surface $M \subset \mathbb{R}^3$. Then $\mathbb{R}^3 \setminus M$ has two components; suppose M has constant positive mean curvature with respect to a component with closure V (which may or may not be compact).

Let Π be a plane with unit normal e and denote planes parallel to Π by $\Pi^s := \{p + se : p \in \Pi\}$. Let $\pi\colon \mathbb{R}^3 \to \Pi$ be orthogonal projection, and L_p be the line through $p \in \Pi$ perpendicular to Π.

We assume that M is contained in a lower halfspace w.r.t. Π; actually it is sufficient to assume the points $p + te$ on L_p with t large are disjoint from V. Then we can define three functions, measuring certain heights of M, which for non-compact M can be unbounded:

$$t_\pm, \alpha\colon \pi(V) \subset \Pi \to \mathbb{R} \cup \{-\infty\},$$

$$t_+(p) := \sup\{t \in \mathbb{R} : p + te \in V\} \in \mathbb{R}, \qquad t_-(p) := \inf\{t \le t_+(p) : p + [t, t_+(p)]e \subset V\},$$

$$\alpha(p) := \frac{t_+ + t_-}{2}$$

That is, in $L_p \cap V$ the segment $\{p + te : t \in [t_-, t_+]\}$ is the topmost component. We also write $P_\pm(p) := p + t_\pm e$ for the endpoints in $\mathbb{R}^3$ of this line segment; it may happen that $P_+(p)$ and $P_-(p)$ agree. The function $\alpha(p)$ measures the height which leads to a first contact of M and its reflection, restricted to the line L_p.

`26. Lecture, Thursday 15.7.10` ____

Recall that a function $f\colon S \subset \mathbb{R}^n \to \mathbb{R}$ is *upper semicontinuous [oberhalbstetig]* if

$$\limsup_{x \to p} f(x) := \lim_{\varepsilon \searrow 0} \{\sup f(x) : x \in S \text{ with } |x - p| \le \varepsilon\} \;\le\; f(p) \qquad \text{for all } p \in S.$$

(Why is it equivalent to replace "$\le f(p)$" by "$= f(p)$"?) An upper semicontinuous function takes a maximum over each compact set of its domain (see problems).

Clearly, the function α is continuous at a point p where M meets L_p transversally, but it can be discontinuous if the tangent plane is vertical at one of these points. We would like α to take a maximum over each compact set. In order to do this, we replace α by

$$\overline{\alpha}\colon \pi(V) \to \mathbb{R} \cup \{-\infty\}, \qquad \overline{\alpha}(p) := \limsup_{x \to p} \alpha(x),$$

which we call the *Alexandrov function*. To see what this means geometrically, suppose the line L_p contains a point P in the interior of the interval $[P_2, P_1] \subset L_p$, such that L_p touches ∂M at P, but does not traverse it. Then the value of $\overline{\alpha}$ is the average height of P_1 and P, which is larger than the average height of P_1 and P_2.

We can now rephrase Alexandrov reflection:

Lemma 59. *If for some plane Π the function $\overline{\alpha}$ takes a local maximum m at $p \in \Pi$ then Π^m is a plane of symmetry for M (and so $\overline{\alpha} \equiv m$).*

Proof. We use the notation introduced in the proof of Alexandrov reflection. Reflect M_+^m in Π^m. By assumption the reflected points $\sigma^m(P_+)$ are contained in V. So

$$m - \big(t_+(q) - m\big) \geq t_-(q),$$

for all $q \in \pi(V)$ in a neighbourhood of p. Either equality holds at p and the two surfaces intersect in L_p. Or equality only holds in the limit of a sequence $q \to p$, so that by the closedness of M the surfaces will intersect in the limit as well. This touching must be tangential, with common tangent plane (else the tangent planes would intersect, and so m would not be a maximum). The maximum principle shows the two surfaces coincide locally; as before, we represent these surfaces as graphs over their common tangent plane to apply the Hopf boundary maximum principle in case $P_+ = P_-$, or the interior maximum principle otherwise. Global symmetry then follows from an openness and closedness argument, as in the proof of Alexandrov reflection. $\square$

We would like to apply this principle to a surface with ends. It could be, however, that the maximum of $\overline{\alpha}$ occurs at infinity so that a first point of contact would never arise. The following lemma serves to rule out this case.

Let $v \in \mathbb{S}^2$ and set $H_- := \{x : \langle x, v\rangle \leq 0\}$. Moreover suppose an annular end M has axis vector a with $\langle a, v\rangle > 0$, and $\partial M \subset H_-$. We can cap off M by a disk $\Delta \subset H_-$ and disjoint to M, such that $M \cup \Delta$ bounds the component to the side of the mean curvature normal, whose closure we define to be V. Then the Alexandrov function $\overline{\alpha}$ is defined w.r.t. any plane containing the vector v, and it does not depend on the choice of Δ, provided we restrict $\overline{\alpha}$ to points p with $\langle p, v\rangle \geq 0$. We now maximize $\overline{\alpha}$ perpendicularly to v:

$$\beta\colon [0,\infty) \to \mathbb{R}, \qquad \beta(x) := \max\{\overline{\alpha}(p) : p \in M, \langle p, v\rangle = x\} \tag{97}$$

Lemma 60. *Let M be a properly embedded annular end with $H \equiv 1$ and axis $a \in \mathbb{S}^2$, and $v \in \mathbb{S}^2$ such that $\langle a, v \rangle > 0$. Then for each plane Π containing a and v, either $x \mapsto \beta(x)$ is strictly decreasing or else M has a mirror plane parallel to Π.*

Proof. We claim that $\beta(0) \geq \beta(x)$ for all $x > 0$. Let $\sigma_t(M_+^t)$ be the reflected upper portion of M. Our claim is then equivalent to showing that

$$\sigma_t(M_+^t) \cap \{x > 0\} \ \subset \ V \qquad \text{for all } t > \beta(0). \tag{98}$$

Indeed, if $\beta(0) \geq \beta(x)$ for all x then by definition of β the reflection of all points with $x > 0$ stays inside V. Conversely, if the reflection stays inside V then the first point of contact is not traversed, and so $\beta(x) \leq \beta(0)$.

To see that (98) holds we again resort to the idea of tilted planes. Let Π_ε be the plane through the origin whose normal is $N(\varepsilon) := e \cos\varepsilon - v \sin\varepsilon$ for $0 < \varepsilon < \pi/2$. The plane Π_ε decomposes M into two components, a compact one above Π_ε, and noncompact one below. So these two components cannot be mirror images of one another, and thus Alexandrov reflection with respect to planes parallel to Π_ε must give a first contact at the boundary. We conclude, with notation similar to (98),

$$\sigma_t^\varepsilon(M_+^{t,\varepsilon}) \cap \{x > 0\} \ \subset \ V \qquad \text{for all } t \geq \beta^\varepsilon(0). \tag{99}$$

Here, by translation of the end, we may assume that ∂M is disjoint from $x > 0$ for all small angles $\varepsilon \geq 0$, and $M^{t,\varepsilon}$ denotes the upper component of M when intersecting with $\mathbb{R}^3 \setminus \Pi_\varepsilon$.

For $\varepsilon \to 0$ the tilted version (99) implies (98), provided we know

$$\limsup_{\varepsilon \to 0} \beta^\varepsilon(0) \leq \beta(0).$$

Thus we need to know that β^ε is upper semicontinuous in ε:

$$\text{If } p(\varepsilon) \in \Pi(\varepsilon) \to p \in \Pi \text{ as } \varepsilon \to 0 \qquad \text{then} \qquad \limsup_{\varepsilon \to 0} \beta^\varepsilon(\varepsilon) \leq \beta(0).$$

To see this, consider a sequence of points $Q_1^\varepsilon, Q_2^\varepsilon$ whose common projection $q(\varepsilon)$ converges to $p \perp a$ as $\varepsilon \to 0$, such that $\frac{1}{2}(Q_1^\varepsilon + Q_2^\varepsilon)$ converges to $\limsup_{\varepsilon \to 0} \beta^\varepsilon(\varepsilon)$. Since M is closed, a subsequence of $Q_1^\varepsilon, Q_2^\varepsilon$ converges to (not necessarily distinct) points $P_1, P_2 \in M$ projecting to p. By the closedness of V the interval $[P_1, P_2]$ is contained in V, and so $\beta(0)$ cannot be smaller than the limit. Since the sequence was chosen to approach $\limsup_{\varepsilon \to 0} \beta^\varepsilon(\varepsilon)$, no other sequence can have a smaller limit.

Finally, the same argument can be applied for any x_0 replacing 0, which verifies $\beta(x_0) \geq \beta(x)$ for $x > x_0$. Hence β is decreasing and so by Lemma 59 there must be a symmetry plane.

To show that β is strictly decreasing note that we need to rule out intervals on which β is constant. But any point in the interior of such an interval gives rise to a local maximum of $\overline{\alpha}$, and so by the Alexandrov Reflection Lemma 59 this is impossible. $\square$

To state the next result in compact form, the following terminology is useful. It describes consequences of Alexandrov reflection, namely that a symmetry plane decomposes the surface into two graphs, compare Cor. 39.

Definition. We call a surface $M^n \subset \mathbb{R}^n$ *Alexandrov symmetric* if there is a hyperplane Π such that $M \setminus \Pi$ has two components M_+ and M_-, which are
- mirror images of one another, such that
- each half $M_\pm$ is a graph and
- $\nu(M_\pm) \subset \mathbb{S}^2_\pm$.

Example. A figure 8 is a mirror symmetric curve in $\mathbb{R}^2$ which is not Alexandrov symmetric.

Theorem 61. *Let M be an embedded surface of finite topology with k ends and genus g. Suppose M has constant mean curvature $H > 0$.*
(i) If $k = 2$, or if $k \geq 3$ and M is contained in a solid cylinder, then M is a Delaunay surface; in particular $k = 2$ and $g = 0$.
(ii) If $k = 3$, or $k \geq 4$ and M is contained in a halfspace, then M is Alexandrov symmetric and so contained in a slab.

Proof. (i) We can assume that M is contained in a solid cylinder in any case: If M has $k = 2$ ends this follows from Meeks' Theorem 58(ii). Let a be the axis of the cylinder, choose $e \perp a$, and consider the plane $\Pi = e^\perp$. By Lemma 60 the function β then is strictly monotonically decreasing for $x > 0$ and increasing for $x < 0$, and so has a maximum at $x = 0$. Thus also α takes a maximum, and by Lemma 59 this in turn implies a symmetry plane.

Now apply the same argument to all vectors $e \perp a$ to obtain a symmetry plane perpendicular to e. All symmetry planes must intersect in an axis parallel to a: Otherwise the reflection planes would generate a non-compact orbit, see the proof of Lemma 37.

(ii) We can assume M is contained in a halfspace in any case: If $k = 3$ then this property is implied by Meeks' Theorem 58(iii). Let $\Pi = e^\perp$ be a plane parallel to the halfspace boundary. By Thm. 58(iii) not all ends can be contained in an open halfspace. Thus there exists an end with horizontal axis $a_1 \in \Pi$, and there exists at least one more end with $a_2 \in \Pi$ such that $\langle a_1, a_2 \rangle < 0$. We now consider the function β as in (97), with respect to $v := a_1$. Since M has no boundary we can can extend the domain of β to all $x \in \mathbb{R}$.

If M does not admit a symmetry plane parallel to Π, the function β is decaying as $x \to \pm\infty$. So β must attain a maximum and Lemma 59 this implies, nevertheless, that M has a symmetry plane parallel to Π. $\square$

We stop here with the exposition of the results obtained by Korevaar, Kusner, Solomon. There are two more statements on properly embedded annular ends in their paper, whose proof requires further machinery:
• The ends converge to Delaunay unduloids in distance when we go to infinity,
• and the convergence is shown to be exponentially fast.

We remark that all results of this section hold not only for embedded surfaces, but for Alexandrov embedded immersions.

Let us comment on existence results for embedded surfaces of finite topology with constant mean curvature 1. It is hard to establish embeddedness for constructed examples, while Alexandrov embeddedness is usually much simpler to establish. Indeed, in continuous families of surfaces embeddedness can be lost, while Alexandrov embeddedness is preserved. That makes it a difficult quantitative problem to decide on embeddedness of a not explicitely given surface. The following results are in the Alexandrov embedded class:
1. Kapouleas constructed such surfaces for all $g \geq 0$ and $k \geq 3$ in 1990.
2. Grosse-Brauckmann, Kusner, Sullivan in 2003 classified all examples with $g = 0$ and $k = 3$, as well as all the Alexandrov symmetric examples with $g = 0$ and $k \geq 4$ (2007).

Let us finally mention two open problems related to our topic which I know from Harold Rosenberg:
1. Suppose M is an embedded surface of infinite topology and with constant mean curvature contained in a slab. Is M Alexandrov symmetric?
2. Suppose M is an embedded surface, contained in a slab, with one boundary loop on each of the two bounding planes. Does the genus of M necessarily vanish?

7.5. **Problems.**

Problem 38 – Properness:

a) Prove that the composition of proper maps is proper.
b) What does it mean for a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ to be proper?
c) Show that for a compact domain A, a continuous map $f\colon A \to \mathbb{R}^n$ is always proper.
d) Give an example for a homeomorphism $\varphi\colon B^n \to \mathbb{R}^n$; it is useful to construct a homeomorphism $\psi\colon (0,1) \to (0,\infty)$ first. Prove that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is proper, then $f \circ \varphi\colon B^n \to \mathbb{R}^n$ is proper. That is, it makes no difference if we define a properly embedded plane as the image of D or $\mathbb{R}^2$.

e) More advanced: Prove that the following two characterizations of the properness of a map $F\colon S \subset \mathbb{R}^m \to \mathbb{R}^n$ for an arbitrary domain S are equivalent.
   • Compact sets $K \subset \mathbb{R}^n$ have compact preimage $F^{-1}(K) \subset S$,
   • For all paths γ which leave any compact subset the image path $c = F \circ \gamma$ also leaves any compact subset.

Problem 39 – Laplace-Beltrami operator and mean curvature of a graph:

a) Write down the Laplace-Beltrami operator for polar coordinates $f\colon (0,\infty) \times \mathbb{R}$, $f(r,\varphi) = (r\cos\varphi, r\sin\varphi)$.
b) Let $u \in C^\infty(\Omega^n, \mathbb{R})$ and consider the graph $M = \{(x, u(x)) : x \in \Omega\}$. Check the equation $\Delta_M(x, u(x)) = -nH(x, u(x))$, where Δ_M is the Laplace-Beltrami operator for the graph.

Problem 40 – Upper semicontinuity:

a) Give a simple example of an upper semicontinuous function from $\mathbb{R}$ to $\mathbb{R}$, and one with many discontinuities.
b) Given a monotone function on $\mathbb{R}$, prove that the number of discontinuities is countable.
c) Prove that an upper semincontinuous function takes a maximum on each compact subset $K \subset \mathbb{R}$.

Problem 41 – Alexandrov reflection for minimal surfaces:

Prove that a properly embedded minimal surface with two ends $\varphi\colon \Sigma_g \setminus \{p_1, p_2\} \to \mathbb{R}^3$, such that the two ends are asymptotically catenoids must be a catenoid. You can assume convergence of the surface together with its normal when $|x| \to \infty$. In particular, Σ must have genus $g = 0$.

a) Use force balancing to show that the axes of the two catenoid ends are parallel (you might as well assume this property). By the way, what does this imply for the growth rates?
b) Apply Alexandrov reflection.

Remarks: 1. The asymptotics to catenoids is a consequence of the Weierstrass representation formulas.
2. The statement was first proved by Rick Schoen (1983).