Lecture VII

Shape analysis and higher regularity of bivariate subdivision schemes

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Assessment fo subdivision surfaces, today

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Subdivision surfaces are *sufficiently smooth, from afar.*

Designer 3 (Mercedes):

Subdivision surfaces are *far from sufficiently smooth.*

Setup

- A subdivision surface $\mathbf{x}$ is the union of *spline rings*,

$$\mathbf{x} = \bigcup_{m \in \mathbb{N}_0} \mathbf{x}^m.$$

- Each spline ring is a linear combination of *generating functions* and *control points*,

$$\mathbf{x}^m = \sum_i g_i \mathbf{p}_i^m = G\mathbf{P}^m.$$

- The sequence of control points is obtained by repeated application of the *subdivision matrix*,

$$\mathbf{P}^m = A^m \mathbf{P}^0.$$

Setup

- *Eigenvalues*

$$|\lambda_0| \geq |\lambda_1| \geq \cdots \geq |\lambda_L|,$$

- Left and right *eigenvectors*

$$Av_\ell = \lambda_\ell v_\ell, \quad w_\ell A = \lambda_\ell w_\ell,$$

- *Eigenfunctions* and *eigencoefficients*

$$f_\ell = G v_\ell, \quad \mathbf{q}_\ell = w_\ell \mathbf{P}.$$

- *Eigen-expansion*

$$\mathbf{x}^m = \sum_\ell \lambda^m f_\ell \mathbf{q}_\ell$$

Generic assumptions

- The *sub-dominant eigenvalue* is double

$$\lambda := \lambda_1 = \lambda_2 > |\lambda_3|$$

- The *characteristic map* is regular and injective,

$$\mathbf{\Psi} := [f_1, f_2] = G[v_1, v_2], \quad \det D\mathbf{\Psi} \neq 0.$$

- The *subsub-dominant eigenvalue* is denoted by μ,

$$1 > \underbrace{\lambda_1 = \lambda_2}_{\lambda} > \underbrace{\lambda_3 = \cdots = \lambda_N}_{\mu} > |\lambda_{N+1}|.$$

- Curvature near central point determined by third order expansion

$$\mathbf{x}^m \doteq \mathbf{q}_0 + \mathbf{\Psi}[\mathbf{q}_1; \mathbf{q}_2] + \mu^m \sum_{\ell=3}^{N} f_\ell \mathbf{q}_\ell.$$

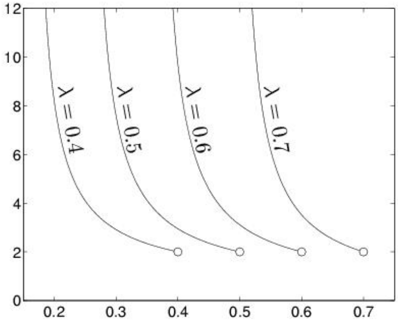
Curvature and the subsub-dominant eigenvalue

The principal curvatures

- converge to 0, if $\mu < \lambda^2$,
- are bounded, if $\mu = \lambda^2$,
- diverge, if $\mu > \lambda^2$.
- are in L^p for

$$p < \frac{2 \ln \lambda}{2 \ln \lambda - \ln \mu}.$$

C^1 always implies $H^{2,2}$.



C^2-conditions

- A subdivision schemes generates C^2-surfaces if and only if

$$\mu = \lambda^2$$

and if the subsub-dominant eigenfunctions satisfy

$$f_3, \ldots, f_N \in \mathrm{span}\{f_1^2,\ f_1 f_2,\ f_2^2\}.$$

- **Degree estimate:** If, on the regular part of the grid, the scheme generates polynomial patches of degree d joining C^k, then non-trivial curvature continuity is possible only if

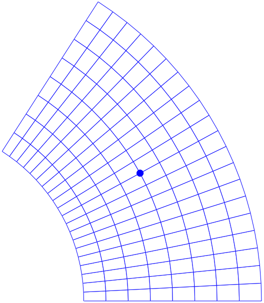
$$d \geq 2k + 2.$$

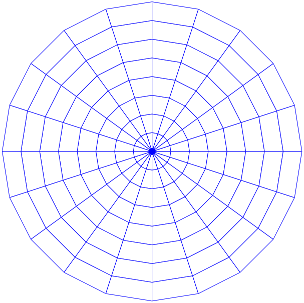
This rules out schemes generalizing uniform B-spline subdivision and box splines. The lowest order candidate is of bi-degree 6 with 4-fold knots.

Shape analysis

To achieve curvature continuity, convergence of the

- principal curvatures is not sufficient.
- principal directions is not necessary.
- Weingarten map is necessary and sufficient, but ...





The Weingarten map revisited

- The Weingarten map W is a linear map in the tangent space $T\mathbf{x}$, defined by

$$\nabla\mathbf{n} = -W\nabla\mathbf{x}.$$

- Its eigenvalues and eigenvectors are the principal curvatures and directions, respectively.
- With respect to basis $\mathbf{x}_u, \mathbf{x}_v$ of $T\mathbf{x}$,

$$D\mathbf{n} = -W\,D\mathbf{x} \;\Rightarrow\; -D\mathbf{n}D\mathbf{x}^{\mathrm{t}} = W\,D\mathbf{x}D\mathbf{x}^{\mathrm{t}} \;\Rightarrow\; W = H\,G^{-1},$$

where

$$D := \begin{bmatrix}\partial_u \\ \partial_v\end{bmatrix}, \quad G := D\mathbf{x}\,D\mathbf{x}^{\mathrm{t}}, \quad H := -D\mathbf{n}\,D\mathbf{x}^{\mathrm{t}}.$$

Problem: For spline surfaces, $D\mathbf{x}$ and hence W is *discontinuous*.

The Weingarten map revisited

- *Trick:* Instead of $D\mathbf{n} = -WD\mathbf{x}$, consider the dual equation,

$$D\mathbf{n}^{\mathrm{t}} = -E\, D\mathbf{x}^{\mathrm{t}}.$$

The Weingarten map revisited

- *Trick:* Instead of $D\mathbf{n} = -WD\mathbf{x}$, consider the *extended* dual equation,

$$[D\mathbf{n}^{\mathrm{t}},\ 0] = -E\,[D\mathbf{x}^{\mathrm{t}},\ \mathbf{n}^{\mathrm{t}}].$$

The Weingarten map revisited

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$$[D\mathbf{n}^{\mathrm{t}},\ 0] = -E\,[D\mathbf{x}^{\mathrm{t}},\ \mathbf{n}^{\mathrm{t}}].$$

- With

$$D\mathbf{x}^{+} = D\mathbf{x}^{\mathrm{t}}\,G^{-1}$$

denoting the pseudo-inverse of $D\mathbf{x}$,

$$E = -D\mathbf{x}^{+}\,D\mathbf{n} = D\mathbf{x}^{+}\,H\,(D\mathbf{x}^{+})^{\mathrm{t}}$$

is a symmetric map acting on $\mathbb{R}^3$. By duality,

$$E_{|T\mathbf{x}} = W \quad \text{and} \quad E\mathbf{n}^{\mathrm{t}} = 0.$$

- We call E the *embedded Weingarten map* of $\mathbf{x}$.

The Weingarten map revisited

Properties:

- E is a second order geometric invariant.
- The principal directions are eigenvectors with respect to the principal curvatures.
- E refers to coordinates of the embedding space.
- Continuity of E is *necessary and sufficient* for $\mathbf{x}$ to be a C^2-manifold, i.e., in the subdivision setup, the limit

$$E^c := \lim_{m\to\infty} E^m, \quad E^m : \Sigma_0 \times \{1, \ldots, n\} \to \mathbb{R}^{3\times 3}$$

has to exist and to be constant.

The Weingarten map revisited

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  has to exist and to be constant.
- The integrability conditions are simple,

$$\mathbf{n}_u E_v^+ = \mathbf{n}_v E_u^+ \quad \Rightarrow \quad D\mathbf{x} = D\mathbf{n}\, E^+.$$

The central surface

- For simplicity, let

$$\begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = L \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix}.$$

- The third order asymptotic expansion of the rings is

$$\mathbf{x}^m \doteq \mathbf{q}_0 + [\lambda^m \mathbf{\Psi} L, \mu^m \varphi] , \quad \varphi := \sum_{i=3}^{N} f_i \langle \mathbf{q}_i, \mathbf{n}^c \rangle.$$

- *Definition:* The *central surface* is a spatial ring defined by

$$\tilde{\mathbf{x}} := (\mathbf{\Psi} L, \varphi).$$

Asymptotic expansions

- With $J := D\boldsymbol{\Psi} L$, the *first fundamental* form of $\mathbf{x}^m$ is

$$G^m \doteq \lambda^{2m}\, G, \quad G := J\, J^T.$$

- With $\tilde{G}$ and $\tilde{H}$ the fundamental forms of the central surface $\tilde{\mathbf{x}}$, the *second fundamental form* of $\mathbf{x}^m$ is

$$H^m \doteq \mu^m\, H, \quad H := \sqrt{\frac{\det \tilde{G}}{\det G}}\, \tilde{H}, \quad .$$

- The *embedded Weingarten map* of $\mathbf{x}^m$ is

$$E^m \doteq \varrho^m \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad E := L^{\mathrm{t}}\, J^{-\mathrm{t}}\, H\, J^{-1} L, \quad \varrho := \frac{\mu}{\lambda^2}.$$

Asymptotic expansions

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- The Gausian curvature of $\mathbf{x}^m$ is

$$\kappa_{\mathrm{G}}^m \doteq \varrho^{2m} \det E.$$

- The mean curvature of $\mathbf{x}^m$ is

$$\kappa_{\mathrm{M}}^m \doteq \varrho^m \operatorname{trace} E.$$

- The principal directions of $\mathbf{x}^m$ are

$$\mathbf{R}^m \doteq [\mathbf{R},\ 0], \quad \mathbf{R}\, E = K \mathbf{R}.$$

Consequences

- The deviation of E from a constant is a reliable indicator for the quality of a subdivision algorithm.
- An algorithm cannot generate *elliptic shape* unless

$$0 \in \mathcal{F}(\mu).$$

- An algorithm cannot generate *hyperbolic shape* unless

$$1, n-1 \in \mathcal{F}(\mu).$$

- Optimal spectrum

$$\begin{array}{ll} \text{simple } 1, & \mathcal{F}(1) = \{0\} \\ \text{double } \lambda \approx 1/2, & \mathcal{F}(\lambda) = \{1, n-1\} \\ \text{triple } \mu = \lambda^2, & \mathcal{F}(\mu) = \{0, 1, n-1\} \end{array}$$

C^2-schemes

- TURBS (R.' 95)
- Freeform splines (Prautzsch '96)
- Guided subdivision (Peters, Karciauskas '06)

General framework for C^2-subdivision

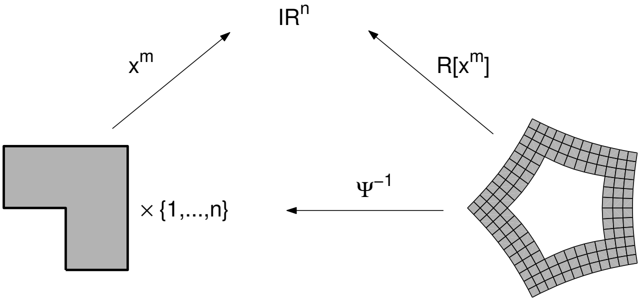
- Denote by $C^2_d(\mathbb{R}^n)$ the space of all C^2-rings in $\mathbb{R}^n$ composed of patches of coordinate degree d.
- A ring $\mathbf{\Psi} \in C^2_3(\mathbb{R}^2)$ is called a *concentric tesselation map* with scale factor $\lambda \in (0,1)$, if it is injective and regular, i.e., $\det D\mathbf{\Psi} \neq 0$, and if $\mathbf{\Psi}$ and $\lambda\mathbf{\Psi}$ join C^2 when regarded as consecutive rings.
- The image of $\mathbf{\Psi}$ and its *extension* are denoted

$$\Omega := \mathbf{\Psi}(\Sigma), \quad \Omega_e := \Omega \cup \lambda\Omega.$$

General framework for C^2-subdivision

The *reparametrization operator R* is mapping rings $\mathbf{x}^m \in C^2_6(\mathbb{R}^n)$ to functions on $\Omega \subset \mathbb{R}^n$ by

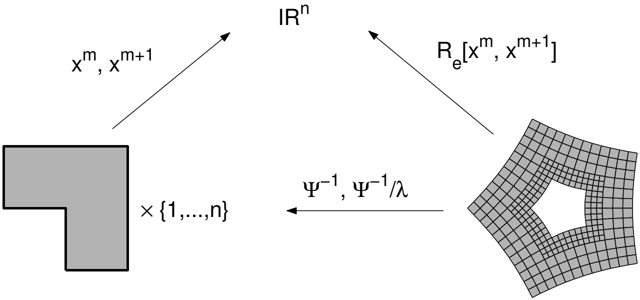
$$R[\mathbf{x}^m] : \Omega \ni \boldsymbol{\xi} \mapsto \mathbf{x}^m(\boldsymbol{\Psi}^{-1}(\boldsymbol{\xi})).$$



General framework for C^2-subdivision

The *extended reparametrization operator R_e* maps a pair $\mathbf{x}^m, \mathbf{x}^{m+1} \in C^2_6(\mathbb{R}^n)$ of consecutive rings to a single function acting on Ω_e according to

$$R_e[\mathbf{x}^m, \mathbf{x}^{m+1}] : \Omega_e \ni \boldsymbol{\xi} \mapsto \begin{cases} R[\mathbf{x}^m](\boldsymbol{\xi}) & \text{if } \boldsymbol{\xi} \in \Omega \\ R[\mathbf{x}^{m+1}](\boldsymbol{\xi}/\lambda) & \text{if } \boldsymbol{\xi} \in \lambda\Omega \end{cases}.$$



General framework for C^2-subdivision

- The subdivision matrix A has *quadratic precision*, if for consecutive rings $\mathbf{x}^m = B_6\mathbf{Q}^m$, $\mathbf{x}^{m+1} = B_6 A\mathbf{Q}^m$,

$$R[\mathbf{x}^m] \in \mathbb{P}_2(\Omega) \quad \text{implies} \quad R_e[\mathbf{x}^m, \mathbf{x}^{m+1}] \in \mathbb{P}_2(\Omega_e).$$

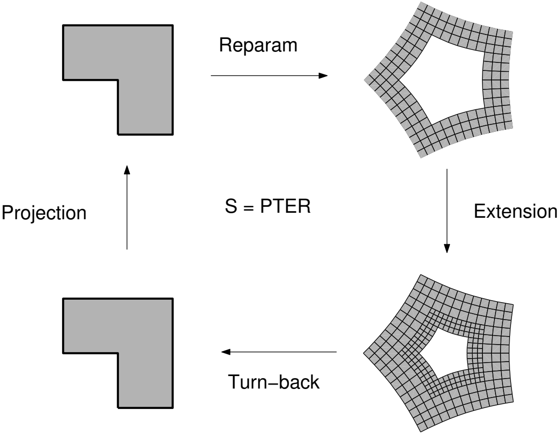
- If $\boldsymbol{\Psi}$ has scale factor λ and A has quadratic precision, then there exist eigenvalues λ_i, eigenvectors v_i and eigenfunctions $f_i = B_6 v_i$ satisfying

$$\lambda_0 = 1, \quad \lambda_1 = \lambda_2 = \lambda, \quad \lambda_3 = \lambda_4 = \lambda_5 = \lambda^2$$
$$f_0 = 1, \quad [f_1, f_2] = \boldsymbol{\Psi}, \quad f_3 = f_1^2,\ f_4 = f_1 f_2,\ f_5 = f_2^2.$$

- *Consequence:* Let A be a subdivision matrix with quadratic precision and eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{\bar{\ell}}$, where $\lambda_0, \ldots, \lambda_5$ are given above. If $\boldsymbol{\Psi}$ has scale factor λ and $|\lambda_i| < \lambda^2$ for all $i > 5$, then A defines a C^2-subdivision algorithm.

General framework for C^2-subdivision

Compute $\mathbf{x}^{m+1} = S(\mathbf{x}^m)$ in four steps:



General framework for C^2-subdivision

- *Extension:* Choose a linear *extension operator E* mapping the function $\mathbf{r} = R[\mathbf{x}^m]$ defined on Ω to the function $\mathbf{r}_e = E[\mathbf{r}]$ defined on Ω_e such that

$$\mathbf{r} \in \mathbb{P}_2 \quad \Rightarrow \quad \mathbf{r}_e \in \mathbb{P}_2.$$

- The subdivision matrix A corresponding to the scheme $S = PTER$ is given by

$$GA = PTER[G].$$

The algorithm is C^2, if

$$\lambda_i < \lambda^2, \quad i > 6.$$

A can be precomputed once and for all.

Summary

- Linear, univariate subdivision well understood.
- Linear, multivariate subdivision well understood in the regular setting.
- C^1-schemes for arbitrary topology available.
- C^2-schemes for arbitrary topology available, but not well established.
- Nonlinear schemes not well understood.
- Schemes for perfect shape sought.