Lecture V

C^1-Analysis of bivariate subdivision schemes near extraordinary vertices

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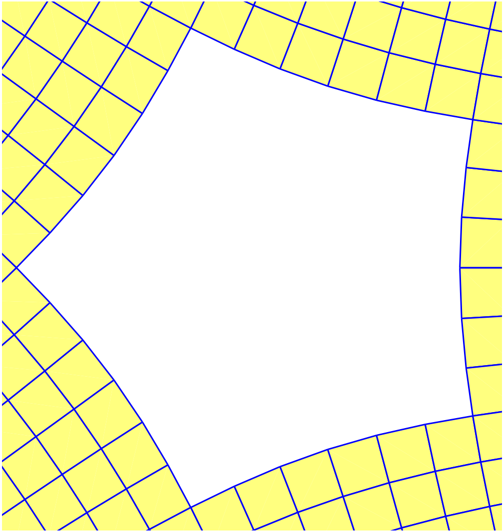
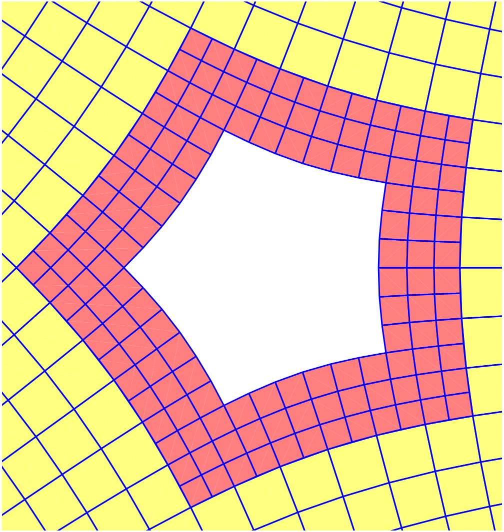
Bertinoro, May 20, 2010

Subdivision near an extraordinary vertex



Subdivision near an extraordinary vertex



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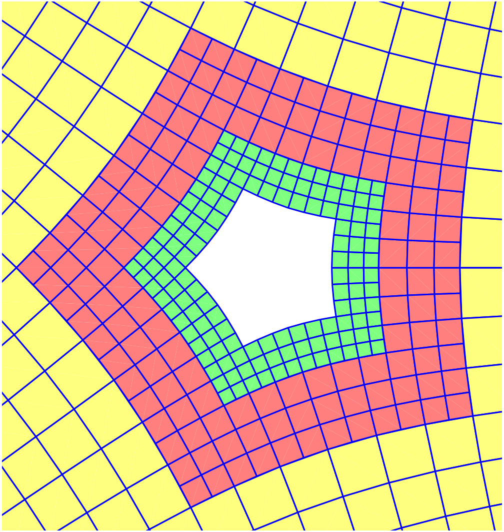

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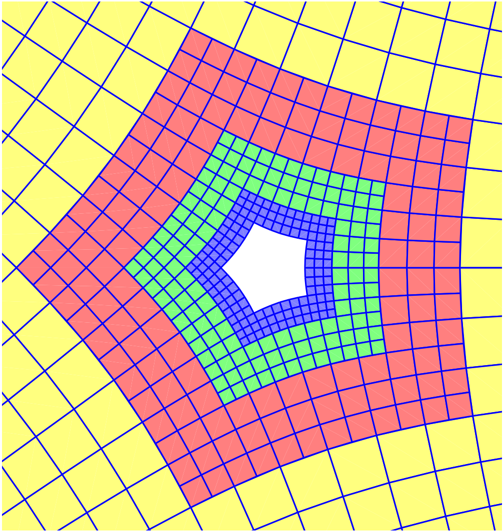

Setup

- In the vicinity of an extraordinary vertex of valence n, a subdivision surface $\mathbf{x}$ can be written as the union of *rings* $\mathbf{x}^m$,

$$\mathbf{x} = \bigcup_{m \in \mathbb{N}_0} \mathbf{x}^m, \quad \mathbf{x}^m : \Sigma_0 \times \{1, \dots, n\} \to \mathbb{R}^3.$$

- The space of rings $\mathbf{x}^m$ is spanned by a common *generating system*

$$G = [g_0, g_1, \dots, g_I], \quad \sum_i g_i = 1.$$

- The ring $\mathbf{x}^m$ is determined by *control points* $\mathbf{p}_i^m \in \mathbb{R}^3$,

$$\mathbf{x}^m = \sum_i g_i \, \mathbf{p}_i^m = G \, \mathbf{P}^m.$$

- The sequence of control points is obtained by repeated application of the *subdivision matrix A*,

$$\mathbf{P}^m = A^m \mathbf{P}^0,$$

where the rows of A sum up to 1.

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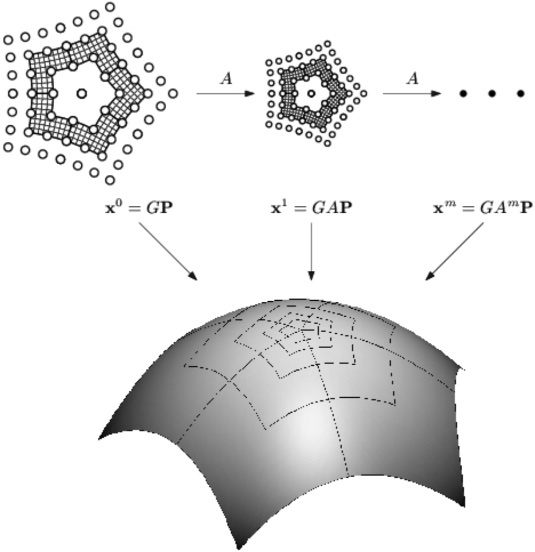
A
A
$\mathbf{x}^0 = G\mathbf{P}$
$\mathbf{x}^1 = GA\mathbf{P}$
$\mathbf{x}^m = GA^m\mathbf{P}$

Eigendecomposition

- The *eigenvalues* and *eigenvectors* of A are

$$Av_\ell = \lambda_\ell v_\ell, \quad \lambda_0 \geq \lambda_1 \geq \lambda_2 \cdots .$$

- The corresponding *eigenfunctions* are

$$f_\ell := G\, v_\ell.$$

- Decomposing the initial data $\mathbf{P}^0 = \sum_\ell v_\ell \mathbf{q}_\ell$ yields

$$\mathbf{x}^m = \sum_\ell GA^m v_\ell \mathbf{q}_\ell = \sum_\ell \lambda_\ell^m f_\ell\, \mathbf{q}_\ell = FD^m\mathbf{Q}.$$

- If the trivial eigenvalue $\lambda_0 = 1$ is dominant, i.e. $\lambda_0 = 1 > |\lambda_1|$, then the rings $\mathbf{x}^m$ form a continuous subdivision surface with central point

$$\mathbf{x}^c := \lim_{m\to\infty} \mathbf{x}^m = \mathbf{q}_0.$$

Subdominant eigenvalue and characteristic map

- Typically, for symmetric subdivision schemes, the *subdominant eigenvalue* is double and real,

$$1 > \lambda := \lambda_1 = \lambda_2 > \mu := |\lambda_3|.$$

- The second order expansion of the sequence of rings is

$$\mathbf{x}^m = \mathbf{q}_0 + \lambda^m(f_1\mathbf{q}_1 + f_2\mathbf{q}_2) + O(\mu^m).$$

- The *characteristic map* of the subdivision scheme is the planar ring

$$\boldsymbol{\Psi} := [f_1, f_2] = G[v_1, v_2].$$

- *The properties of the characteristic map are crucial for the regularity of the subdivision surface in a vicinity of an extraordinary point.*

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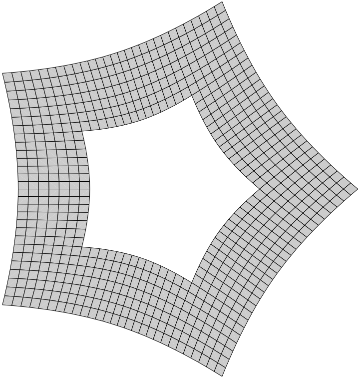
The characteristic map



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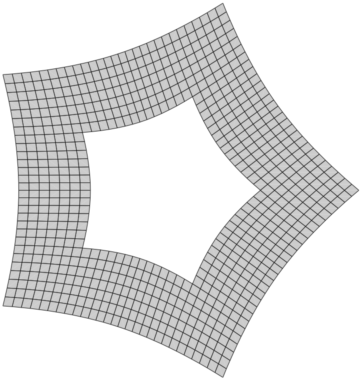

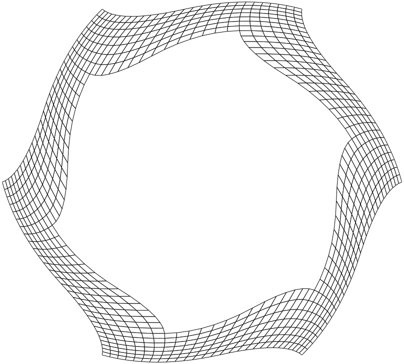

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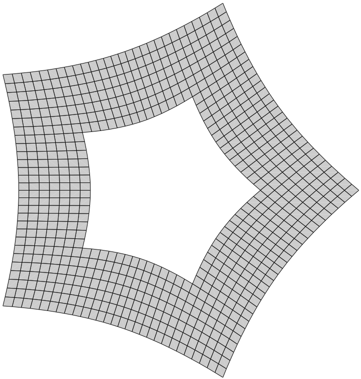
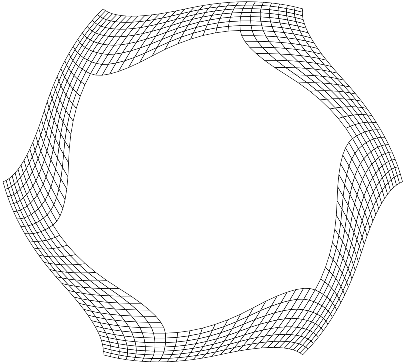


λ real



λ complex

Normal continuity

- Sequence of rings

$$\mathbf{x}^m = \mathbf{q}_0 + \lambda^m \boldsymbol{\Psi}[\mathbf{q}_1; \mathbf{q}_2] + O(\mu^m).$$

- Sequence of partial derivatives

$$D\mathbf{x}^m = [\mathbf{x}_u^m, \mathbf{x}_v^m] = \lambda^m \, D\boldsymbol{\Psi}\,[\mathbf{q}_1;\ \mathbf{q}_2] + O(\mu^m).$$

- Sequence of normals

$$\begin{aligned}\mathbf{n}^m &= \frac{\mathbf{x}_u^m \times \mathbf{x}_v^m}{\|\mathbf{x}_u^m \times \mathbf{x}_v^m\|} = \frac{\boldsymbol{\Psi}_u[\mathbf{q}_1;\ \mathbf{q}_2] \times \boldsymbol{\Psi}_v[\mathbf{q}_1;\ \mathbf{q}_2] + o(1)}{\|\boldsymbol{\Psi}_u[\mathbf{q}_1;\ \mathbf{q}_2] \times \boldsymbol{\Psi}_v[\mathbf{q}_1;\ \mathbf{q}_2] + o(1)\|} \\ &= \frac{\det D\boldsymbol{\Psi}(\mathbf{q}_1 \times \mathbf{q}_2) + o(1)}{\|\det D\boldsymbol{\Psi}(\mathbf{q}_1 \times \mathbf{q}_2) + o(1)\|}.\end{aligned}$$

Normal continuity

- The sequence of normals

$$\mathbf{n}^m = \frac{\det D\mathbf{\Psi}(\mathbf{q}_1 \times \mathbf{q}_2) + o(1)}{\|\det D\mathbf{\Psi}(\mathbf{q}_1 \times \mathbf{q}_2) + o(1)\|}$$

converges to the constant limit

$$\mathbf{n}^c := \lim_{m\to\infty} \mathbf{n}^m = \operatorname{sign}(\det D\mathbf{\Psi}) \frac{\mathbf{q}_1 \times \mathbf{q}_2}{\|\mathbf{q}_1 \times \mathbf{q}_2\|}$$

if

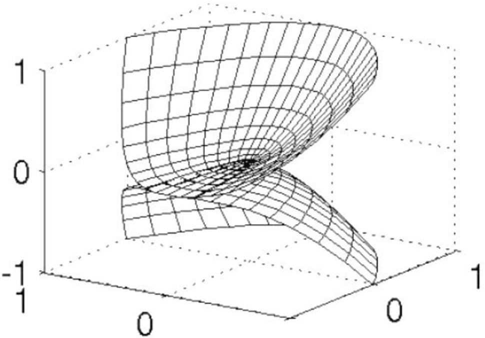
  - $\mathbf{q}_1$ and $\mathbf{q}_2$ are linearly independent
  - the characteristic map is regular, i.e.,

$$\det D\mathbf{\Psi} \neq 0.$$

- The sequence of normals does *not converge* if $\det D\mathbf{\Psi}$ changes sign.

Normal continuity vs. regularity

Caution: Continuity of the normal vector does not imply C^1-regularity in the sense of manifolds.



$$\mathbf{x}(u, v) = [u^2 - v^2, uv, u^3]$$

Main theorem

Theorem (C^1-subdivision schemes)

A subdivision scheme with a double subdominant eigenvalue

- *generates C^1-limit surfaces for almost all initial data if*
  - $\boldsymbol{\Psi}$ *is regular and*
  - $\boldsymbol{\Psi}$ *is injective.*
- *does not generate C^1-limit surfaces for almost all initial data if*
  - $\det D\boldsymbol{\Psi}$ *changes sign or*
  - $\boldsymbol{\Psi}$ *is not injective on the interior of its domain.*

A necessary condition

- Discrete Fourier transform

$$A = \begin{bmatrix} A_0 & A_{n-1} & \cdots & A_1 \\ A_1 & A_0 & \cdots & A_2 \\ \dots & \vdots & \ddots & \vdots \\ A_{n-1} & A_{n-2} & \cdots & A_0 \end{bmatrix} \quad \Rightarrow \quad \hat{A} = \begin{bmatrix} \hat{A}_0 & 0 & \cdots & 0 \\ 0 & \hat{A}_1 & \cdots & 0 \\ \dots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{A}_{n-1} \end{bmatrix}$$

- The *Fourier index* of the eigenvalue λ_i is defined by

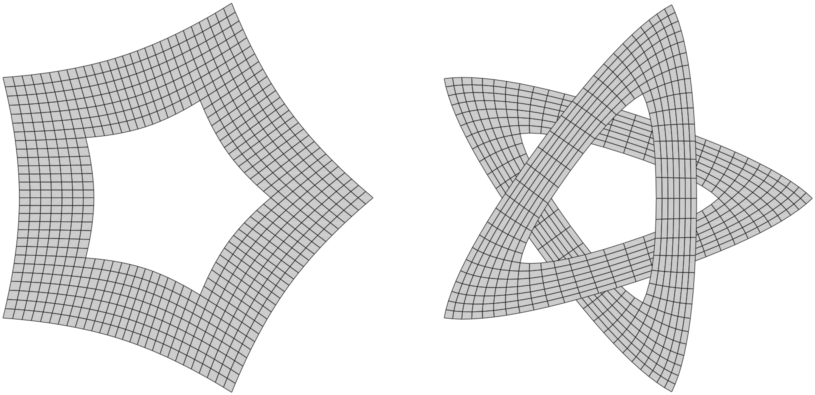
$$\mathcal{F}(\lambda_i) := \left\{ k \in \mathbb{Z}_n : \lambda_i \text{ is EV of } \hat{A}_k \right\}$$

- The characteristic map is not injective unless

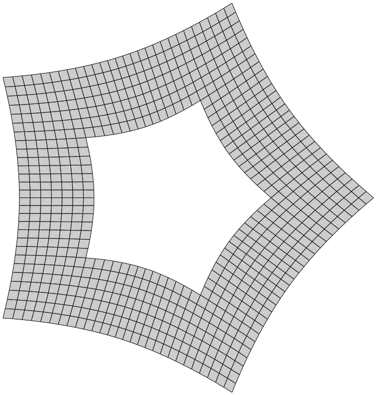
$$\mathcal{F}(\lambda) = \{1,\ n-1\}.$$

Proof based on the concept of winding number.

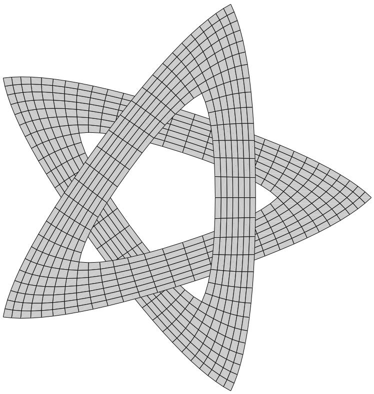
A necessary condition



A necessary condition



$\mathcal{F}(\lambda) = \{1, 4\}$



$\mathcal{F}(\lambda) = \{2, 5\}$

A sufficient condition

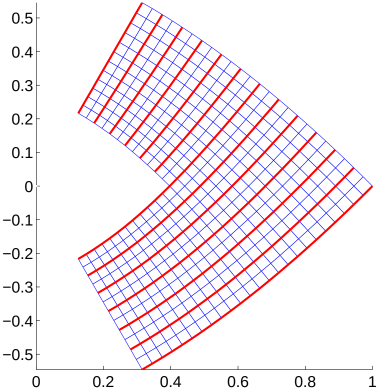
Consider the first segment

$$\mathbf{\Psi}^0 = [f_1^0, f_2^0] : \Sigma_0 \to \mathbb{R}^2$$

of the characteristic map. If

$$\partial_v \mathbf{\Psi}^0 > 0,$$

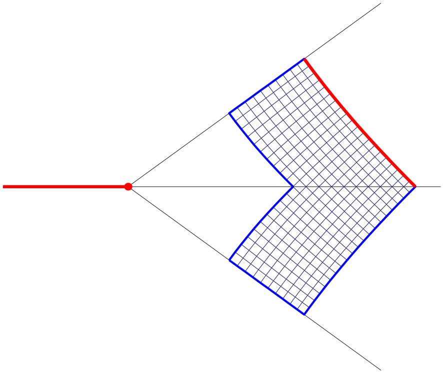
then the characteristic map is regular and injective.



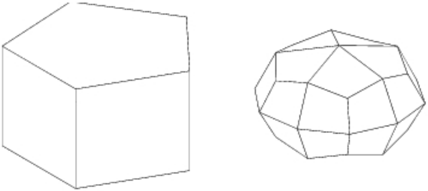
A necessary and sufficient condition

Let $\mathbf{\Psi}$ be regular. Then $\mathbf{\Psi}$ is injective if and only if

- $\mathcal{F}(\lambda) = \{1, n-1\}$ and
- the red lines do not intersect.



The Catmull-Clark algorithm



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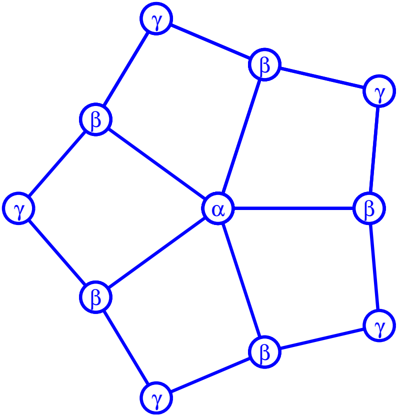


Catmull and Clark suggest:

$$\alpha = 1 - \frac{7}{4n}$$

$$\beta = \frac{3}{2n^2}$$

$$\gamma = \frac{1}{4n^2}$$

The Catmull-Clark algorithm

- For any reasonable choice of special weights, the spectrum of A is appropriate,

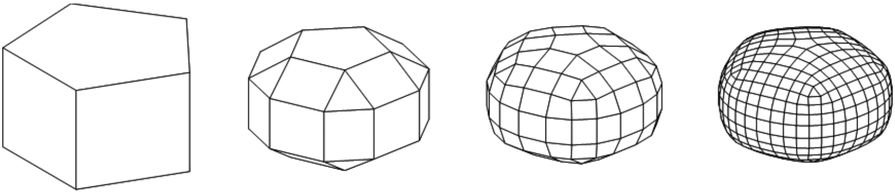
$$\lambda_0 = 1 > \lambda_1 = \lambda_2 > |\lambda_3|$$
$$\mathcal{F}(\lambda) = \{1, n-1\}.$$

- The subdominant eigenvalue is *independent* of the special choice of weights.
- The characteristic map is *independent* of the special choice of weights.

Theorem (Peters, R. '95)

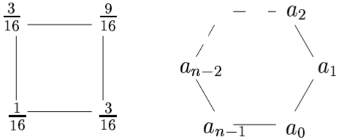
The Catmull-Clark algorithm is a C^1-scheme for all orders n.

The Doo-Sabin algorithm

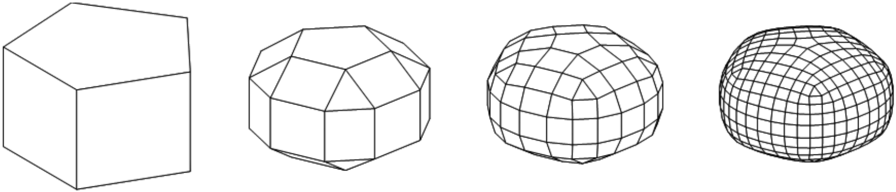


- Each old n-gon is mapped to a new, smaller n-gon.
- For quads, apply standard rules for biquadratic B-splines.
- For other n, use special weights $a = [a_0, \ldots, a_{n-1}]$.
- Doo and Sabin suggest

$$a_j = \frac{\delta_{j,0}}{4} + \frac{3 + 2\cos(2\pi j/n)}{4n}.$$



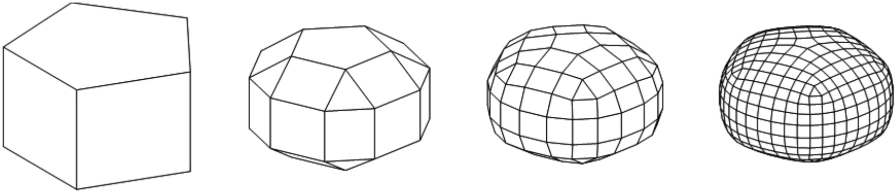
The Doo-Sabin algorithm



- Necessary condition for C^1: The discrete Fourier transform of the vector $a = [a_0, \ldots, a_{n-1}]$ is

$$\hat{a} = [1, \lambda, \hat{a}_2, \cdots, \hat{a}_{n-2}, \lambda], \quad 1 > \lambda > \max\{1/4, |\hat{a}_2|, \ldots, |\hat{a}_{n-2}|\}.$$

The Doo-Sabin algorithm

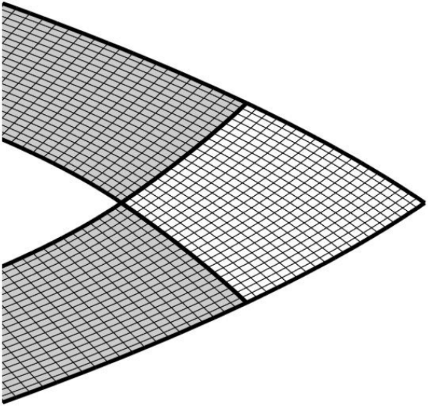


- Necessary and sufficient condition for C^1: The discrete Fourier transform of the vector $a = [a_0, \dots, a_{n-1}]$ is

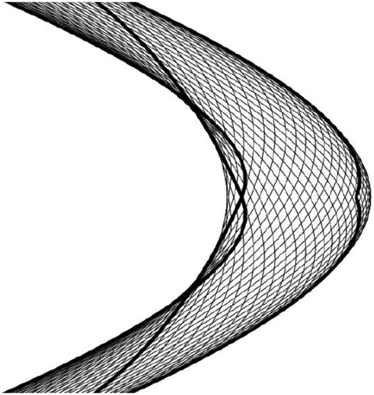
$$\hat{a} = [1, \lambda, *, \cdots, *, \lambda], \quad \lambda_n^* > \lambda > \max\{1/4, *\}$$

  for a certain upper bound λ_n^*.
- Loss of smoothness beyond the critical value $\lambda > \lambda_n^*$.

The Doo-Sabin algorithm

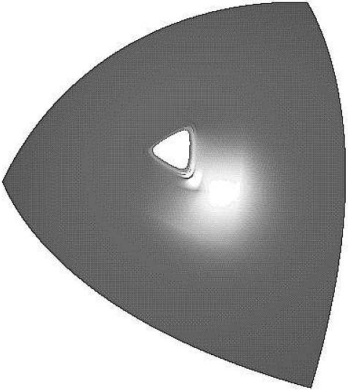


$\lambda = 0.5$



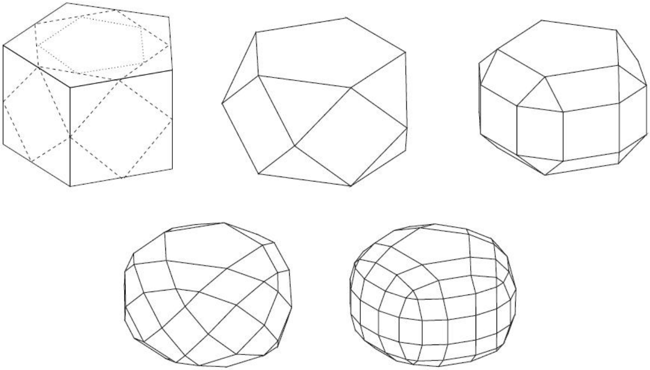
$\lambda = 0.95$

The Doo-Sabin algorithm

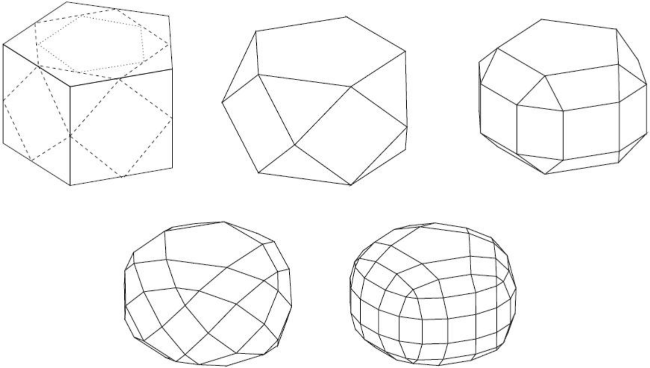
 

$$\lambda = 0.95 > \lambda_3^* = \frac{\sqrt{187}}{24} \cos\left(\frac{1}{3}\arctan\left(\frac{27\sqrt{5563}}{1576}\right)\right) + \frac{1}{3} \approx 0.8773.$$

Simplest subdivision and non-trivial Jordan blocks



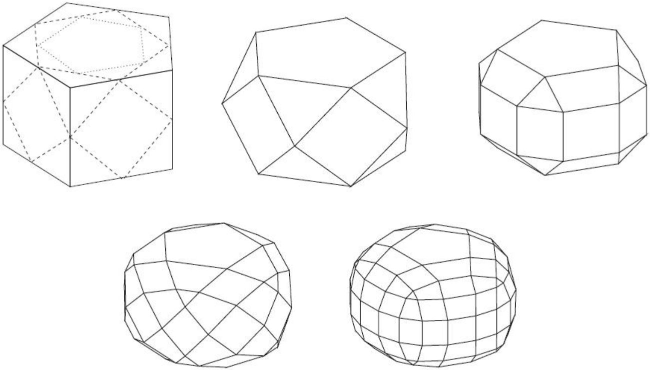
Simplest subdivision and non-trivial Jordan blocks



For $n = 3$, there exists an *eight-fold* subdominant eigenvalue $\lambda = 1/4$,

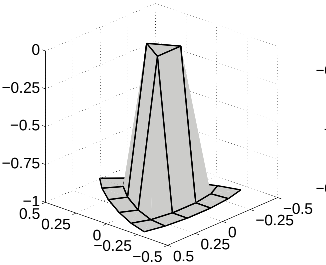
$$\begin{bmatrix} 1/4 & 1 \\ 0 & 1/4 \end{bmatrix}, \begin{bmatrix} 1/4 & 1 \\ 0 & 1/4 \end{bmatrix}, 1/4,\ 1/4,\ 1/4,\ 1/4.$$

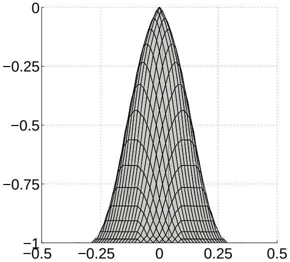
Simplest subdivision and non-trivial Jordan blocks



For $n = 3$, there exists an *eight-fold* subdominant eigenvalue $\lambda = 1/4$.
Nevertheless, *the scheme is C^1.*

Simplest subdivision and non-trivial Jordan blocks





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