Lecture IV

Bivariate subdivision schemes for quad meshes with arbitrary connectivity

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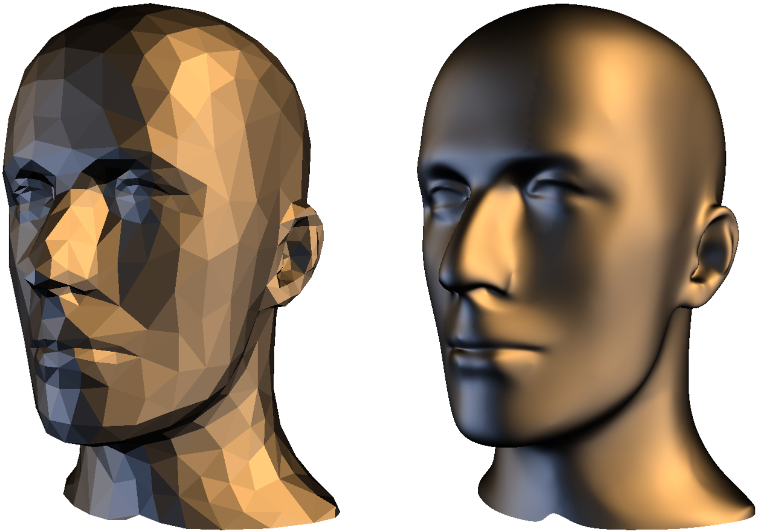
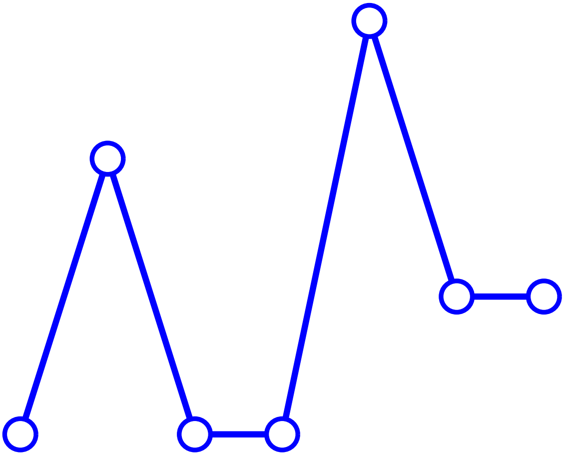
Bertinoro, May 19, 2010

The reason for success in Computer Graphics:



Subdivision for univariate cubic B-splines

Subdivision for cubic B-splines uses two different rules:



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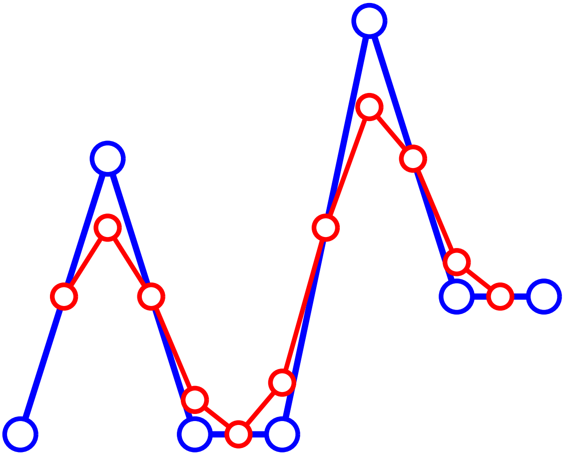

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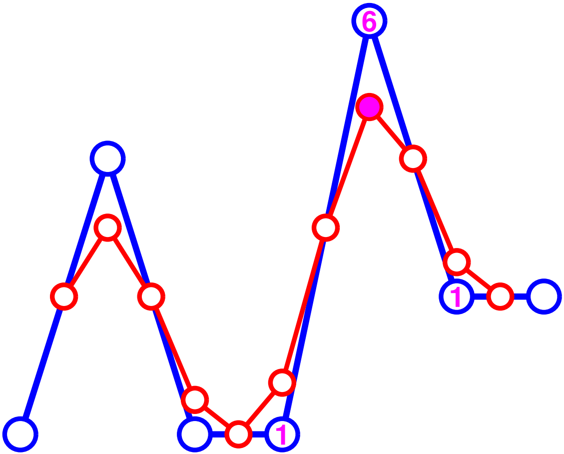

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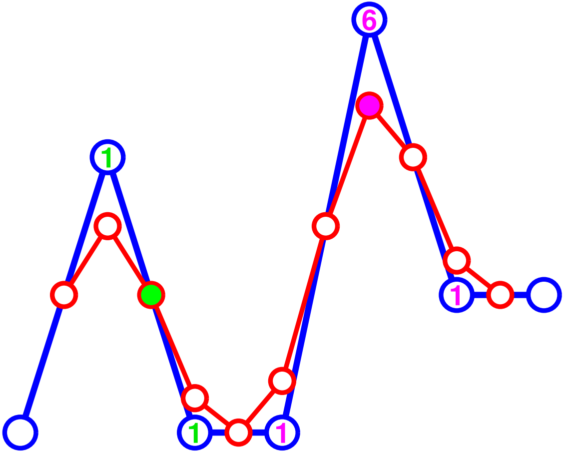

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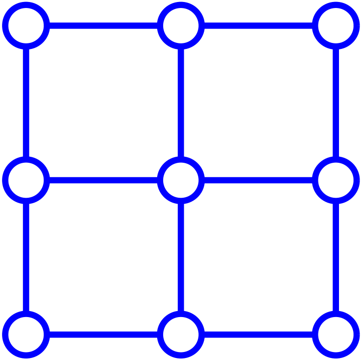

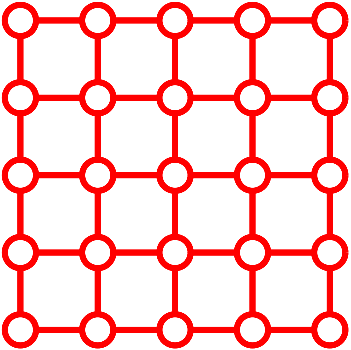

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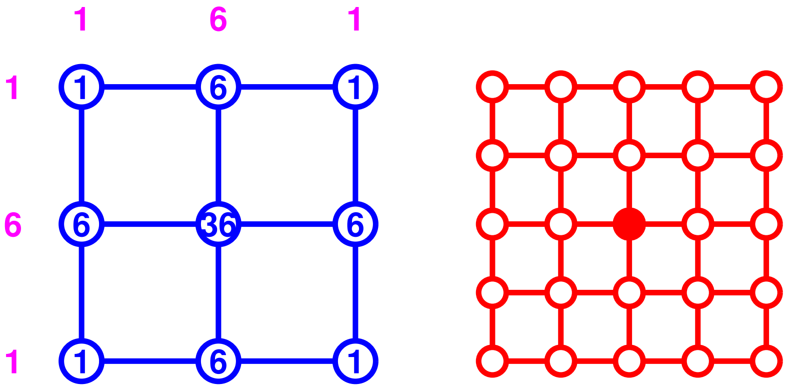

new vertex point

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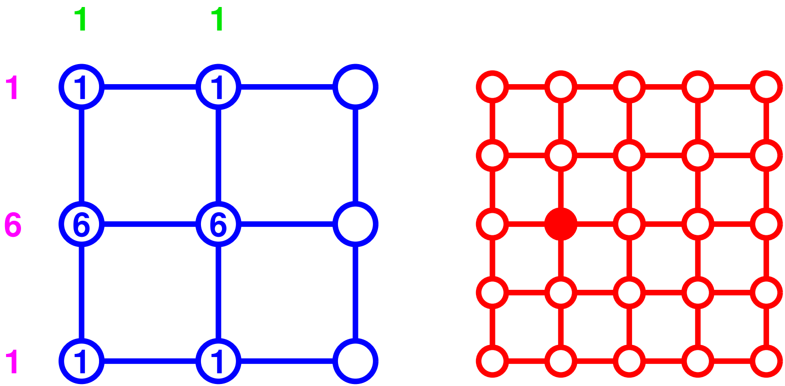

new edge point

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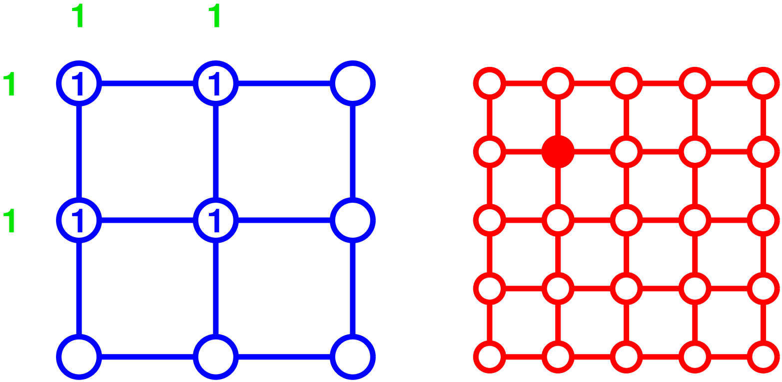
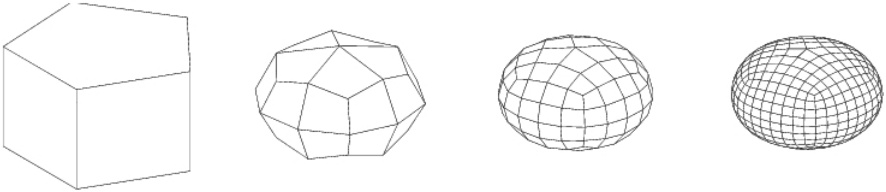


new face point

Catmull-Clark subdivision for general meshes



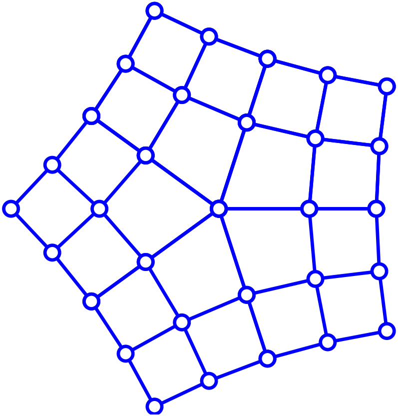
Topological rules:

- Each old *n-gon* is split into *n quadrilaterals.*
- After the first step, *all* faces are quadrilaterals.
- After a few steps, all extraordinary vertices are sufficiently well *separated*, and the vast majority of and vertices is regular.

Catmull-Clark subdivision for general meshes

Geometric rules:

- Use standard stencils, wherever it is possible.



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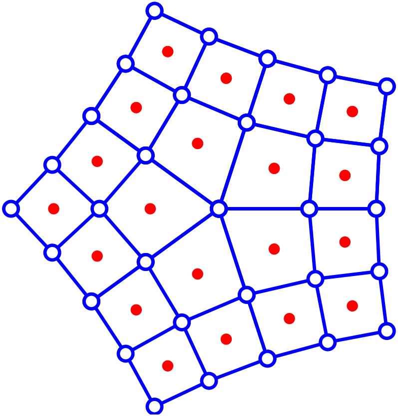

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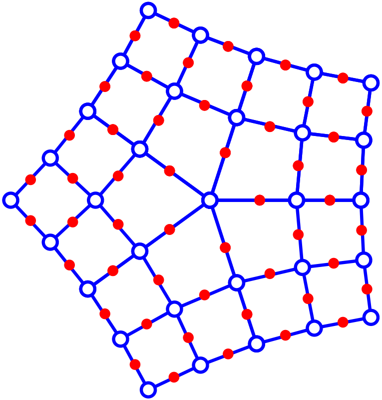

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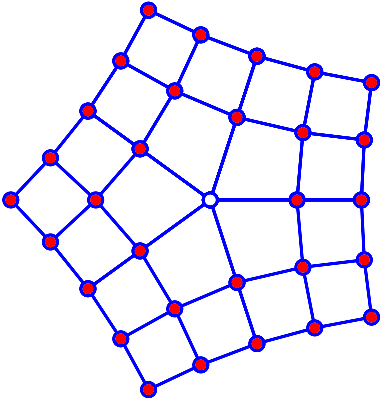

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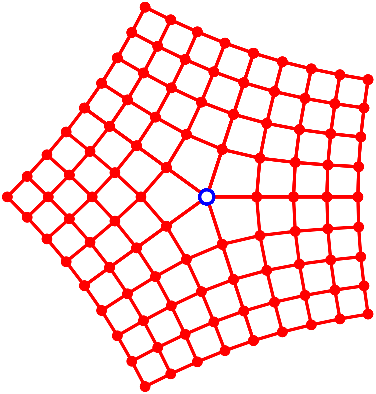
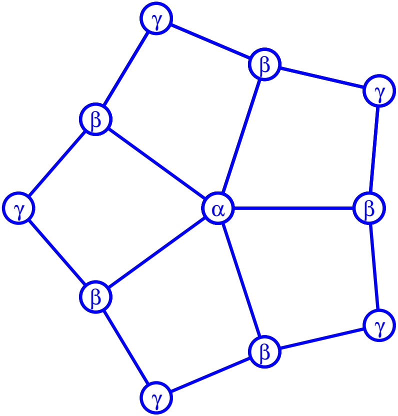


Catmull-Clark subdivision for general meshes

Geometric rules:

- Use standard stencils, wherever it is possible.
- Only at the extraordinary vertex itself, a new stencil is needed.



Catmull and Clark suggest:

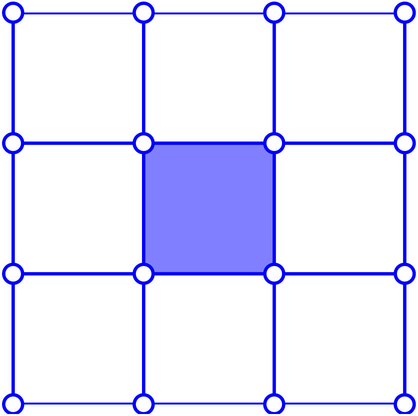
$$\alpha = 1 - \frac{7}{4n}$$

$$\beta = \frac{3}{2n^2}$$

$$\gamma = \frac{1}{4n^2}$$

Known parts of the limit surface

Always 4×4 control points define one patch.



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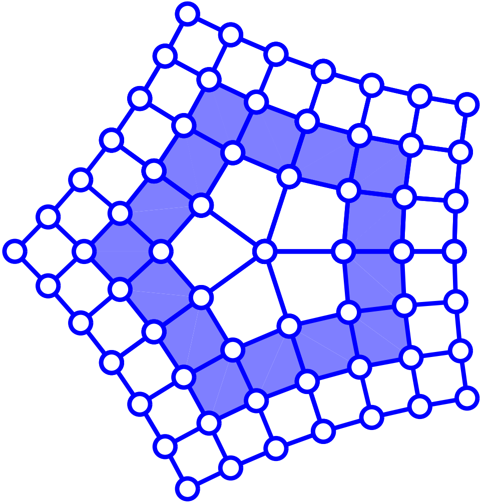

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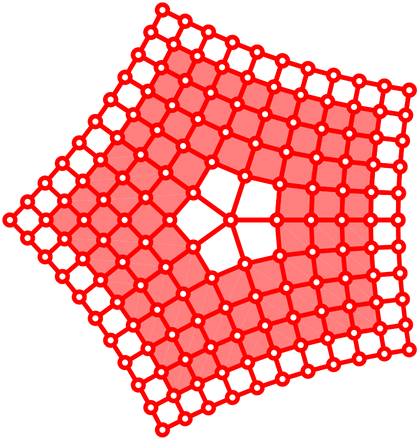

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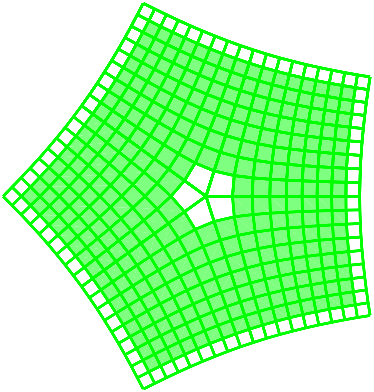

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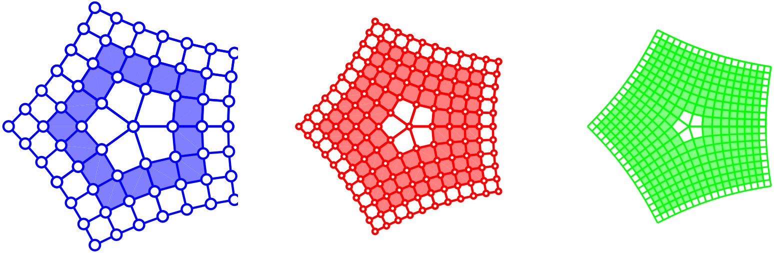
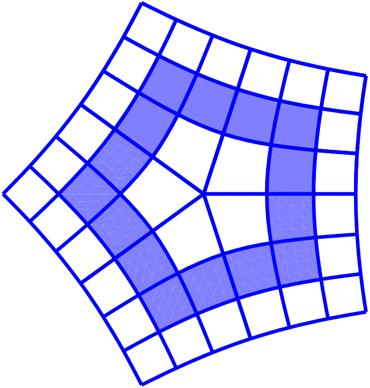


- At a given level, an *n-sided region* around the extraordinary vertex is unknown.
- As subdivision proceeds, the known parts grow and the unknown parts shrink.
- Eventually, only a single point at the center is not covered.
- Parts which are covered at different levels, coincide.

Subdivision surface as union of spline rings



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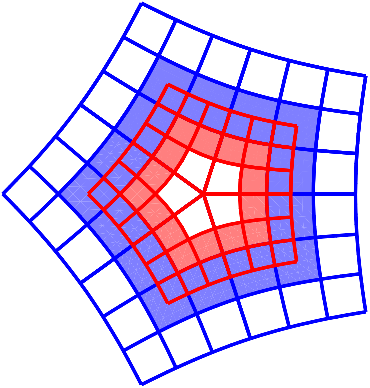

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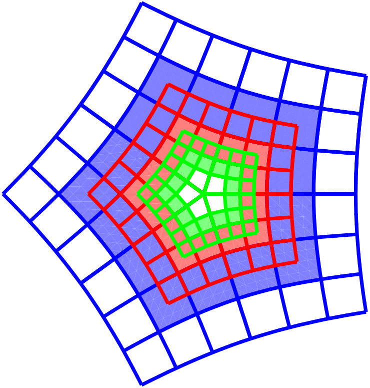

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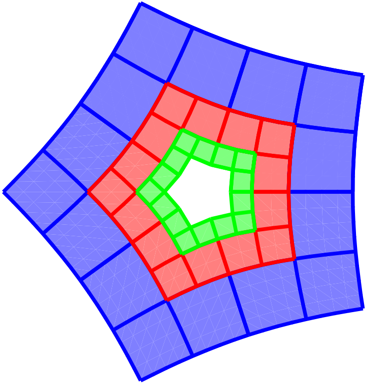

The subdivision surface can be regarded as the union of those ring-shaped parts which are newly added at every step of refinement.

General setup

- Locally, a subdivision surface can be represented as the union of *spline rings*, and a limit point, called the *central point*,

$$\mathbf{x} = \bigcup_{m\in\mathbb{N}} \mathbf{x}^m \cup \mathbf{x}^c, \qquad \mathbf{x}^m : \Sigma_0 \times \mathbb{Z}_n \ni (s,t,j) \mapsto \mathbb{R}^d.$$

- All spline rings have a similar structure. They consist of a fixed number n of *L-shaped patches*,

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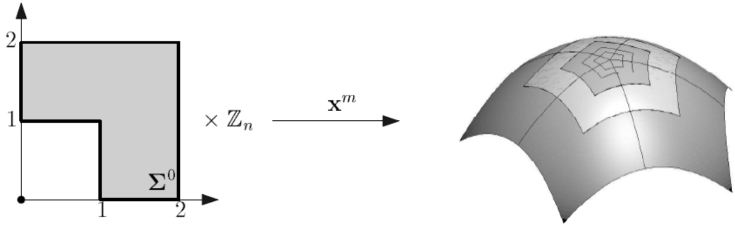

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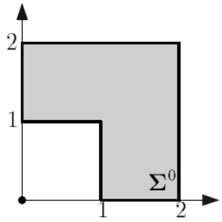

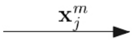

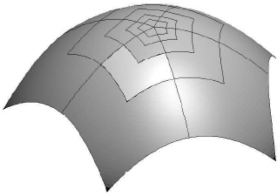

General setup

- Being part of a *regular* subdivision surface, the spline rings can be parametrized with the help of the basic limit functions of the *regular subdivision scheme* and the control points at level m,

$$\mathbf{x}^m(s,t,j) = \sum_{\ell=0}^{L} f_\ell(s,t,j)\mathbf{p}_\ell^m = F(s,t,j)\mathbf{P}^m,$$

where

$$F = [f_0, f_1, \ldots, f_L], \quad \mathbf{P}^m = \begin{bmatrix} \mathbf{p}_0^m \\ \vdots \\ \mathbf{p}_L^m \end{bmatrix}.$$

- The control points at level m are obtained from the previous level by application of square *subdivision matrix A*,

$$\mathbf{P}^m = A\mathbf{P}^{m-1}, \quad \mathbf{P}^m = A^m\mathbf{P},$$

where the control points $\mathbf{P} = \mathbf{P}^0$ at level 0 are the *initial data*.

General setup

- The sequence of spline rings to be analyzed is

$$\mathbf{x}^m = F\mathbf{P}^m = FA^m\mathbf{P}.$$

- F is built from the basic limit functions of the regular rules.
  - F is mapping control points to the corresponding spline ring.
  - F forms a partition of unity, $\sum_i f_i = 1$.
  - F is assumed to be at least C^1.
  - F is linearly independent.
- A represents the special rules.
  - A is mapping control points from one level to the next finer one.
  - The rows of A (the stencils) sum to 1.
- $\mathbf{P}$ contains the user-given initial set of control points.

Eigendecomposition

- Defining equation

$$Av_i = \lambda_i v_i.$$

- *Eigenvalues*, ordered by modulus,

$$|\lambda_0| \geq |\lambda_1| \geq \cdots \geq |\lambda_L|, \quad D = \begin{bmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_L \end{bmatrix}.$$

- *Right eigenvectors*, existence assumed,

$$V = [v_0, \ldots, v_L], \quad AV = VD.$$

- *Left eigenvectors*

$$W = V^{-1} = \begin{bmatrix} w_0 \\ \vdots \\ w_L \end{bmatrix}, \quad WA = DW.$$

Eigendecomposition

- With $A^m = VD^mV^{-1} = VD^mW$, the spline rings are

$$\mathbf{x}^m = FA^m\mathbf{P} = FVD^mW\mathbf{P}.$$

Eigendecomposition

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$$\mathbf{x}^m = FA^m\mathbf{P} = (FV)\,D^m\,(W\mathbf{P}) = GD^m\mathbf{Q}.$$

- The row-vector $G = FV$ contains the *eigen-functions* g_ℓ,

$$G = [g_0, \ldots, g_L], \quad g_i = Fv_i.$$

- The column-vector $\mathbf{Q} = W\mathbf{P}$ contains the *eigen-coefficients* $\mathbf{q}_\ell$,

$$\mathbf{Q} = \begin{bmatrix} \mathbf{q}_0 \\ \vdots \\ \mathbf{q}_L \end{bmatrix}, \quad \mathbf{q}_\ell = w_\ell\mathbf{P}.$$

- Finally,

$$\mathbf{x}^m = GD^m\mathbf{Q} = \sum_\ell \lambda_\ell^m g_\ell \mathbf{q}_\ell.$$

Convergence and the dominant eigenvalue

- If $|\lambda_0| > 1$, then the sequence

$$\mathbf{x}^m = \sum_i \lambda_i^m g_i \mathbf{q}_i$$

  is typically divergent. This case is excluded.

- Since all rows of A sum to 1,

$$A \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad \Rightarrow \quad \lambda_0 = 1, \quad v_0 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

- The eigen-function to $\lambda_0 = 0$ is

$$g_0 = F v_0 = \sum_i f_i = 1.$$

- The eigen-coefficient to $\lambda_0 = 0$ is

$$\mathbf{q}_0 = w_0 \mathbf{P}, \qquad \text{where } w_0 A = w_0.$$

Convergence and the dominant eigenvalue

- Asymptotic expansion:

$$\mathbf{x}^m = \lambda_0^m g_0 \mathbf{q}_0 + \sum_{i\geq 1} \lambda_i^m g_i \mathbf{q}_i = \mathbf{q}_0 + O(\lambda_1^m).$$

- If $1 = \lambda_0 > |\lambda_1|$, then the sequence of spline rings converges to the central point

$$\mathbf{x}^c := \lim_{m\to\infty} \mathbf{x}^m = \mathbf{q}_0.$$

In other words: If $\lambda_0 = 1$ *is the strictly dominant* eigenvalue of the subdivision matrix, then the subdivision surface $\mathbf{x}$ is *continuous.*

Ineffective eigenvectors

- What happens if the generating system G is not linearly independent?
- Convergence of

$$\mathbf{x}^m = GA^m\mathbf{P}$$

is possible even if $\varrho(A) > 1$.

- There might exist *ineffective eigenvectors* of A, i.e.,

$$Av = \lambda v, \quad \lambda \neq 0, \quad Gv = 0.$$

If so, spectral properties of A cannot be related to smoothness properties of the subdivision scheme.

Ineffective eigenvectors

- For any given matrix A there exists an *equivalent matrix A_* without ineffective eigenvectors*, i.e.,

$$GA^m\mathbf{P} = GA_*^m\mathbf{P} \quad \text{for all } m \text{ and } \mathbf{P}.$$

- The eigenfunctions of A_* corresponding to equal eigenvalues are linearly independent.
- *Construction:* For an ineffective eigenvector v of A, compute w such that

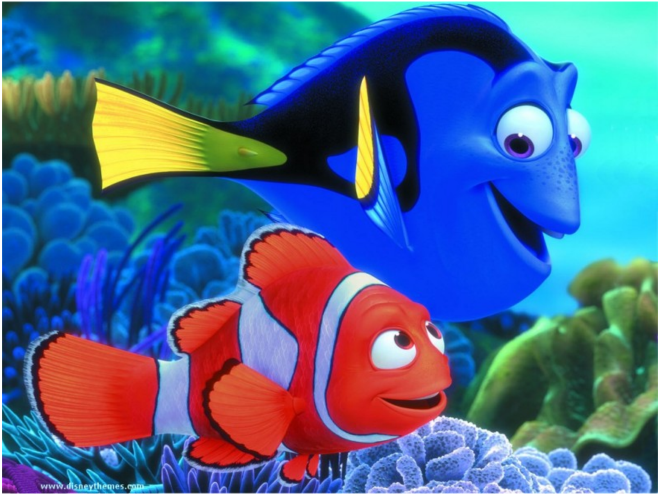
$$w^{\mathrm{t}}v = \lambda$$
$$w^{\mathrm{t}}v' = 0 \quad \text{for all other eigenvectors } v' \text{ of } A.$$

Set $\tilde{A} := A - vw^{\mathrm{t}}$ and repeat.

Popular schemes for quad meshes

- Catmull-Clark (generalizing cubic B-splines)
- Doo-Sabin (generalizing quadratic B-splines)
- Cashman's NURBS subdivision (generalizing B-splines of any order)
- Simplest subdivision (generalizing C^1 four-direction splines)
- Velho's 4-8 scheme (generalizing C^4 four-direction box splines)
- Kobbelt's interpolatory scheme (generalizing the four-point scheme)

Geri's game – An Oscar for subdivision



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