Lecture II

Univariate subdivision schemes and their analysis by the matrix approach

Ulrich Reif

Technische Universität Darmstadt

Bertinoro, May 18, 2010

Overview

- Univariate subdivision as a binary tree of products of square matrices.
- Contractivity of matrices $\leftrightarrow$ continuity
- Joint spectral radius matrices $\leftrightarrow$ Hölder continuity
- Strategies for computing the JSR.
- Examples

Approach 1: Laurent Series

- Sequence of data points at level $k \in \mathbb{N}_0$,

$$\mathbf{f}^k = \{f_i^k\}_{i\in\mathbb{Z}}.$$

- Identify with Laurent series

$$\mathbf{f}^k(z) := \sum_{i\in\mathbb{Z}} f_i^k z^i.$$

- Subdivision represented by symbol a,

$$\mathbf{f}^{k+1}(z) = a(z)\mathbf{f}^k(z^2).$$

- Study properties of product function

$$a(z)a(z^2)a(z^4)\cdots a(z^{2^k}) \quad \longrightarrow \quad \textit{exponential growth in } k.$$

Approach 2: Matrices (global)

- Sequence of data points at level $k \in \mathbb{N}_0$,

$$\mathbf{f}^k = \{f_i^k\}_{i\in\mathbb{Z}}$$

- Subdivision represented by *infinte* matrix $A \in \mathbb{R}^{\mathbb{Z}\times\mathbb{Z}}$,

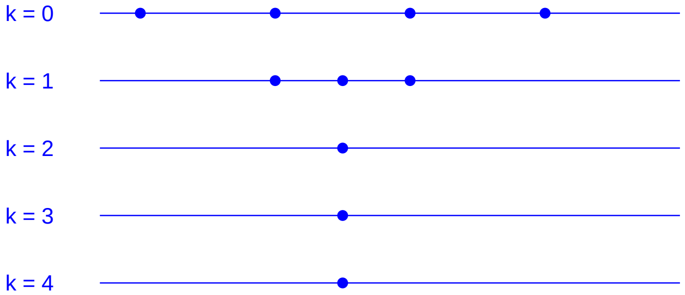
$$\mathbf{f}^k = A\mathbf{f}^{k-1} = A^2\mathbf{f}^{k-2} = \cdots = A^k\mathbf{f}^0$$

- *Problem:* How to study properties of A^k?

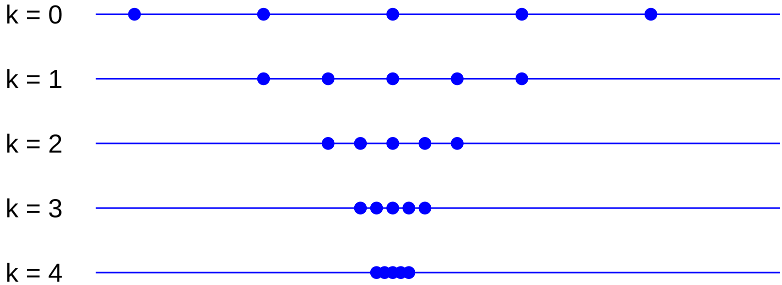
Approach 2: Matrices (local)

- *Idea:* Reduce infinite sequence $\mathbf{f}^0$ of initial data points to the vector F^0 defining the limit function f on $[0, 1]$,

- *Idea:* Reduce infinite sequence $\mathbf{f}^0$ of initial data points to the vector F^0 defining the limit function f on $[0, 1]$,

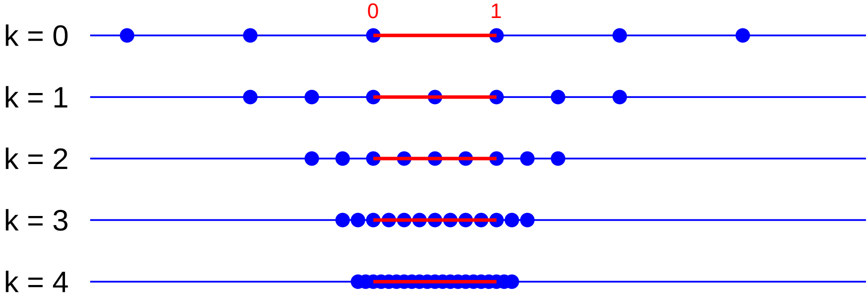


- *Idea:* Reduce infinite sequence $\mathbf{f}^0$ of initial data points to the vector F^0 defining the limit function f on $[0, 1]$,



Approach 2: Matrices (local)

- *Idea:* Reduce infinite sequence $\mathbf{f}^0$ of initial data points to the vector F^0 defining the limit function f on $[0, 1]$,



Approach 2: Matrices (local)

- *Idea:* Reduce infinite sequence $\mathbf{f}^0$ of initial data points to the vector F^0 defining the limit function f on $[0, 1]$,

$$
\begin{aligned}
F^0 &= \{f_i^0\}_{i=1:N_0}, & N_0 &= N \\
F^1 &= \{f_i^1\}_{i=1:N_1}, & N_1 &= N+1 \\
F^2 &= \{f_i^2\}_{i=1:N_2}, & N_2 &= N+3 \\
F^k &= \{f_i^k\}_{i=1:N_k}, & N_k &= N+2^k-1
\end{aligned}
$$

- Local subdivision represented by *finite* matrices A^k,

$$F^k = A^k F^{k-1} = A^k A^{k-1} F^{k-2} = \cdots = A^k A^{k-1} \cdots A^1 F^0$$

- *Problem:* How to study properties of the product matrix?

Approach 2: Matrices (local, square)

- *Idea:* Reduce infinite sequence $\mathbf{f}^0$ of initial data points to the vector F^0 defining the limit function f on $[0,1]$,

$$F^0 = \{f_i^0\}_{i=1:N_0}, \qquad N_0 = N$$
$$F^1 = \{f_i^1\}_{i=1:N_1}, \qquad N_1 = N+1$$

- *Idea:* Partition F^1 into two sub-vectors of length N,

$$F^1 = [f_1^1, f_2^1, \ldots, f_N^1, f_{N+1}^1]$$
$$F_\ell^1 = [f_1^1, f_2^1, \ldots, f_N^1]$$
$$F_r^1 = \quad [f_2^1, \ldots, f_N^1, f_{N+1}^1]$$

- Local subdivision represented by a pair (S_ℓ, S_r) of *square* matrices,

$$F_\ell^1 = S_\ell F^0 \quad \text{defining } f \text{ on } [0, 1/2]$$
$$F_r^1 = S_r F^0 \quad \text{defining } f \text{ on } [1/2, 1]$$

Approach 2: Matrices (local, square)

- *Idea:* Reduce infinite sequence $\mathbf{f}^0$ of initial data points to the vector F^0 defining the limit function f on $[0,1]$,

$$F^0 = \{f_i^0\}_{i=1:N_0}, \qquad N_0 = N$$
$$F^1 = \{f_i^1\}_{i=1:N_1}, \qquad N_1 = N+1$$

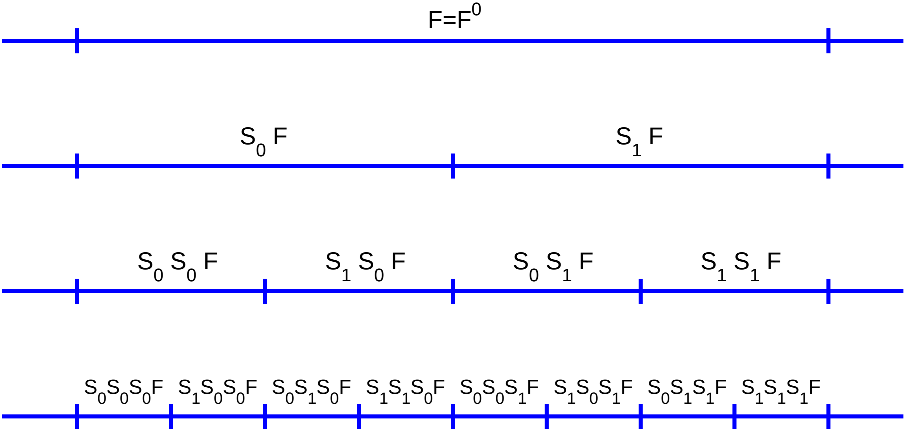
- *Idea:* Partition F^1 into two sub-vectors of length N,

$$F^1 = [f_1^1, f_2^1, \ldots, f_N^1, f_{N+1}^1]$$
$$F_\ell^1 = [f_1^1, f_2^1, \ldots, f_N^1]$$
$$F_r^1 = \quad [f_2^1, \ldots, f_N^1, f_{N+1}^1]$$

- Local subdivision represented by a pair (S_0, S_1) of *square* matrices,

$$F_0^1 = S_0 F^0 \quad \text{defining } f \text{ on } [0, 1/2]$$
$$F_1^1 = S_1 F^0 \quad \text{defining } f \text{ on } [1/2, 1]$$

Approach 2: Matrices (local, square)



Approach 2: Matrices (local, square)

- At level k, there are 2^k sub-intervals, indexed by

$$I = [i_1, \ldots, i_k] \in \{0,1\}^k.$$

- The binary number $0.i_1 \cdots i_k$ is the left end-point of the sub-interval corresponding to I.
- The vector F_I^k of data defining the limit function f on the sub-interval with index I is given by

$$F_I^k = S_I F^0, \quad \text{where} \quad S_I := S_{i_k} \cdots S_{i_1}.$$

- Analyze binary tree of products of (S_0, S_1).

The difference scheme

- Let

$$\Delta := \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ & \ddots & & \ddots & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

denote the *difference matrix*.

- The matrices $D = (D_0, D_1)$ representing the *difference scheme*

$$\Delta F_i^1 = D_i \Delta F^0$$

must satisfy

$$\Delta S_i = D_i \Delta.$$

- A solution exists and is unique iff the rows of S_0, S_1 sum up to 1.

The difference scheme

- Let

$$\Delta := \begin{bmatrix} -1 & 1 & 0 & \cdots \\ & \ddots & \ddots & \\ \cdots & 0 & -1 & 1 \end{bmatrix}, \quad \Delta^{-1} := \begin{bmatrix} 0 & 1 & \cdots & 1 \\ & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 \end{bmatrix}^T$$

  denote the *difference matrix* and the *summation matrix*, resp.
- The matrices $D = (D_0, D_1)$ representing the *difference scheme*

$$\Delta F_i^1 = S_i \Delta F^0$$

  must satisfy

$$\Delta S_i = D_i \Delta.$$

- A solution exists and is unique iff the rows of S_0, S_1 sum up to 1,

$$D_i = \Delta S_i \Delta^{-1}.$$

The difference scheme

- The differences at level k are given by

$$\Delta F_I^k = D_I \, \Delta F^0, \quad D_I := D_{i_k} \cdots D_{i_1},$$

where $I = [i_1, \ldots, i_k] \in \{0,1\}^k$.

- The subdivision scheme $S = (S_0, S_1)$ generates a C^0-limit function iff the difference scheme $D = (D_0, D_1)$ is contractive, i.e., iff

$$D_{\mathcal{I}} = 0 \quad \text{for any infinite sequence } \mathcal{I} \in \{0,1\}^{\mathbb{N}}.$$

The difference scheme

- The differences at level k are given by

$$\Delta F_I^k = D_I\,\Delta F^0, \quad D_I := D_{i_k}\cdots D_{i_1},$$

where $I = [i_1, \dots, i_k] \in \{0,1\}^k$.

- The subdivision scheme $S = (S_0, S_1)$ generates a C^0-limit function iff the difference scheme $D = (D_0, D_1)$ is contractive, i.e., iff

$$D_{\mathcal{I}} = 0 \quad \text{for any infinite sequence } \mathcal{I} \in \{0,1\}^{\mathbb{N}}.$$

- The difference scheme is *not contractive* if

$$\varrho(D_I) \geq 1 \quad \text{for some index vector } I.$$

The difference scheme

- The differences at level k are given by

$$\Delta F_I^k = D_I \, \Delta F^0, \quad D_I := D_{i_k} \cdots D_{i_1},$$

where $I = [i_1, \ldots, i_k] \in \{0,1\}^k$.

- The subdivision scheme $S = (S_0, S_1)$ generates a C^0-limit function iff the difference scheme $D = (D_0, D_1)$ is contractive, i.e., iff

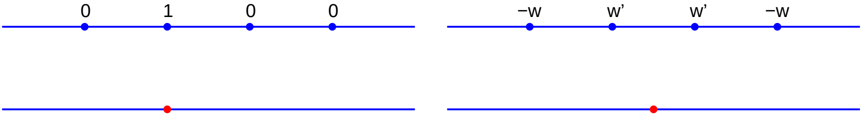
$$D_{\mathcal{I}} = 0 \quad \text{for any infinite sequence } \mathcal{I} \in \{0,1\}^{\mathbb{N}}.$$

- The difference scheme is *not contractive* if

$$\varrho(D_I) \geq 1 \quad \text{for some index vector } I.$$

- The difference scheme is *contractive* if there exists $N \in \mathbb{N}$ such that

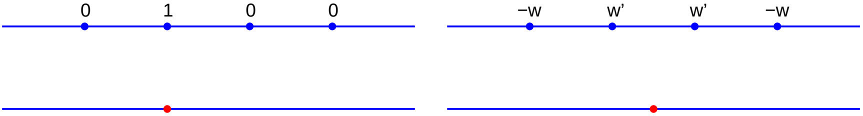
$$\|D_I\| < 1 \quad \text{for all index vectors } I \text{ of length } N.$$



Subdivision scheme S:

$$S_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega & \omega' & \omega' & -\omega & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\omega & \omega' & \omega' & -\omega & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega & \omega' & \omega' & -\omega \end{bmatrix}, \quad S_1 = \begin{bmatrix} -\omega & \omega' & \omega' & -\omega & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\omega & \omega' & \omega' & -\omega & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega & \omega' & \omega' & -\omega \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

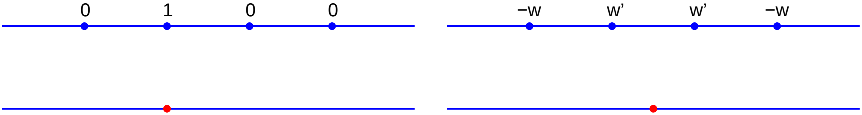
The four-point scheme I



Difference scheme D:

$$D_0 = \begin{bmatrix} -\omega & \frac{1}{2} & \omega & 0 & 0 \\ \omega & \frac{1}{2} & -\omega & 0 & 0 \\ 0 & -\omega & \frac{1}{2} & \omega & 0 \\ 0 & \omega & \frac{1}{2} & -\omega & 0 \\ 0 & 0 & -\omega & \frac{1}{2} & \omega \end{bmatrix}, \quad D_1 = \begin{bmatrix} \omega & \frac{1}{2} & -\omega & 0 & 0 \\ 0 & -\omega & \frac{1}{2} & \omega & 0 \\ 0 & \omega & \frac{1}{2} & -\omega & 0 \\ 0 & 0 & -\omega & \frac{1}{2} & \omega \\ 0 & 0 & \omega & \frac{1}{2} & -\omega \end{bmatrix}$$

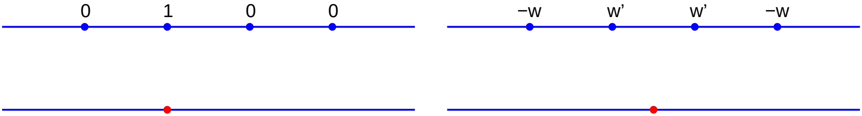
The four-point scheme I



Divided difference scheme $\bar{D} := 2D$:

$$\bar{D}_0 = 2 \begin{bmatrix} -\omega & \frac{1}{2} & \omega & 0 & 0 \\ \omega & \frac{1}{2} & -\omega & 0 & 0 \\ 0 & -\omega & \frac{1}{2} & \omega & 0 \\ 0 & \omega & \frac{1}{2} & -\omega & 0 \\ 0 & 0 & -\omega & \frac{1}{2} & \omega \end{bmatrix}, \quad \bar{D}_1 = 2 \begin{bmatrix} \omega & \frac{1}{2} & -\omega & 0 & 0 \\ 0 & -\omega & \frac{1}{2} & \omega & 0 \\ 0 & \omega & \frac{1}{2} & -\omega & 0 \\ 0 & 0 & -\omega & \frac{1}{2} & \omega \\ 0 & 0 & \omega & \frac{1}{2} & -\omega \end{bmatrix}$$

The four-point scheme I



Difference scheme D^2 of divided difference scheme:

$$D_0^2 = 2\begin{bmatrix} 2\omega & 2\omega & 0 & 0 \\ -\omega & \omega' - \omega & -\omega & 0 \\ 0 & 2\omega & 2\omega & 0 \\ 0 & -\omega & \omega' - \omega & -\omega \end{bmatrix}, \quad D_1^2 = 2\begin{bmatrix} -\omega & \omega' - \omega & -\omega & 0 \\ 0 & 2\omega & 2\omega & 0 \\ 0 & -\omega & \omega' - \omega & -\omega \\ 0 & 0 & 2\omega & 2\omega \end{bmatrix}$$

The four-point scheme I

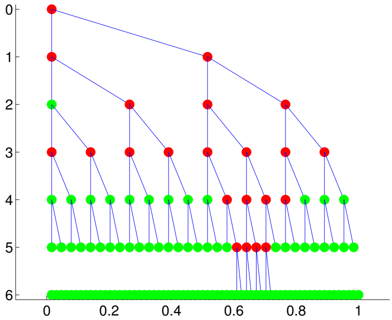
- The FPS generates C^1-limit functions iff the difference scheme D^2 of the divided difference scheme is contractive.
- Determining the maximal set $(0, \omega_{\text{sup}})$ providing C^1 is a challenge:
  - '87, based on level $n = 2$, $\omega_{\text{sup}} \geq \frac{1}{8} = .125$
  - '92, based on level $n = 3$, $\omega_{\text{sup}} \geq \frac{\sqrt{5}-1}{8} \approx .155$
  - '96, based on level $n = 22$, $\omega_{\text{sup}} \geq .188$

The four-point scheme I

- The FPS generates C^1-limit functions iff the difference scheme D^2 of the divided difference scheme is contractive.
- Determining the maximal set $(0, \omega_{\text{sup}})$ providing C^1 is a challenge:
  - '87, based on level $n = 2$, $\omega_{\text{sup}} \geq \frac{1}{8} = .125$
  - '92, based on level $n = 3$, $\omega_{\text{sup}} \geq \frac{\sqrt{5}-1}{8} \approx .155$
  - '96, based on level $n = 22$, $\omega_{\text{sup}} \geq .188$
  - '06, based on refined analytic approach,

$$\omega_{\text{sup}} := \frac{\left(27 + 3\sqrt{105}\right)^{2/3} - 6}{12\left(27 + 3\sqrt{105}\right)^{1/3}} \approx 0.192729.$$

Breadth-first vs. depth-first search

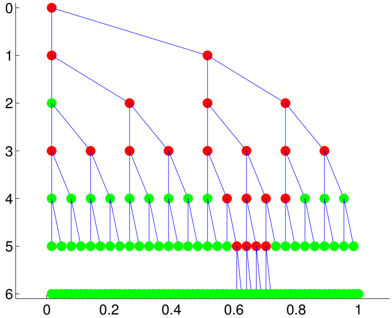


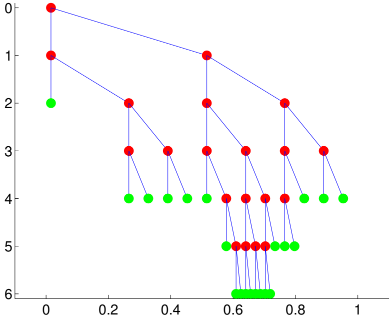
D is contractive if there exists a level N with contractive nodes,

$$\|D_I\| < 1 \quad \text{for all } I \text{ of length } N,$$

i.e., if a *breadth first search* terminates.

Breadth-first vs. depth-first search





D is contractive if there exists a proper subtree $\mathcal{T}$ with contractive nodes, i.e., if a *depth first search* terminates.

For $\omega = 0.188$, reduction from $4,000,000$ to 159 matrices.

Bad news I

The problem of checking a pair of matrices (D_0, D_1)
for contractivity is *undecidable*.

Hölder continuity and the joint spectral radius

- The rate of convergence of $\|D_I\| \to 0$ as $\#I \to \infty$ determines the Hölder continuity of the limit function f.
- The *joint spectral radius* of (D_0, D_1) is defined by

$$\operatorname{jsr}(D_0, D_1) := \sup_{n\in\mathbb{N}} \sup_{I\in\{0,1\}^n} \sqrt[n]{\varrho(D_I)}.$$

- The limit function f is C^0 if

$$\operatorname{jsr}(D_0, D_1) < 1.$$

- The limit function f is $C^{0,\alpha}$ if

$$\operatorname{jsr}(D_0, D_1) < 2^{-\alpha}.$$

Basic observations

- For any $I \in \{0,1\}^n$,

$$\sqrt[n]{\varrho(D_I)} \leq \operatorname{jsr}(D_0, D_1).$$

- For any norm,

$$\operatorname{jsr}(D_0, D_1) \leq \max_{I \in \{0,1\}^n} \sqrt[n]{\|D_I\|}.$$

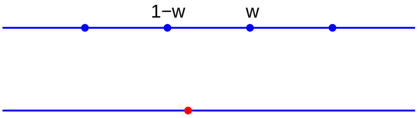
- There exists a norm $\|\cdot\|_*$ on $\mathbb{R}^d$ with

$$\operatorname{jsr}(D_0, D_1) = \max\{\|D_0\|_*, \|D_1\|_*\}.$$

- For symmetric subdivision schemes and (2×2)-matrices D_0, D_1,

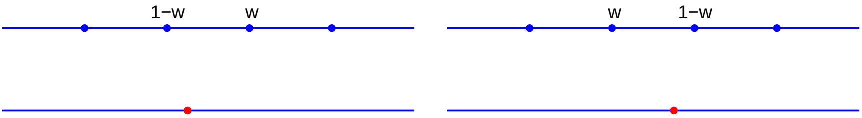
$$\operatorname{jsr}(D_0, D_1) = \max\{\varrho(D_0), \sqrt{\varrho(D_0 D_1)}\}.$$

Corner cutting



$$S_0 = \begin{bmatrix} 1-\omega & \omega & 0 \\ \omega & 1-\omega & 0 \\ 0 & 1-\omega & \omega \end{bmatrix}, \quad S_1 = \begin{bmatrix} \omega & 1-\omega & 0 \\ 0 & 1-\omega & \omega \\ 0 & \omega & 1-\omega \end{bmatrix}$$

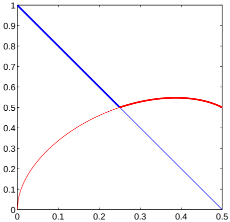
Corner cutting



$$D_0 = \begin{bmatrix} 1-2\omega & 0 \\ \omega & \omega \end{bmatrix}, \quad D_1 = \begin{bmatrix} \omega & \omega \\ 0 & 1-2\omega \end{bmatrix}$$

The joint spectral radius is given by

$$\operatorname{jsr}(D_0, D_1) = \max\left\{1-2\omega,\ \omega/2+\sqrt{\omega - 7\omega^2/4}\right\}$$



The four-point scheme II

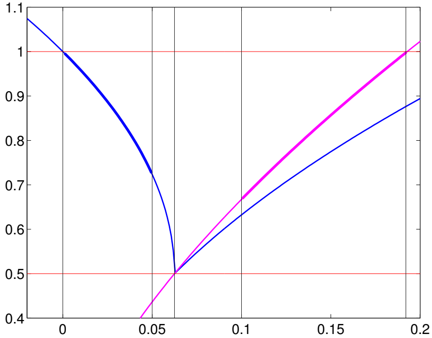
For the four-point scheme with parameter

- $\omega \in [0, 1/20]$, it is

$$\mathrm{jsr}(D_0^2, D_1^2) = \varrho(D_0^2).$$

- $\omega \in [1/10, 2/10]$, it is

$$\mathrm{jsr}(D_0^2, D_1^2) = \sqrt{\varrho(D_0^2 D_1^2)}.$$



The upper bound ω_{sup} is obtained from solving $\varrho(D_0^2 D_1^2) = 1$.

Bad news II

- The *finiteness conjecture*

$$\operatorname{jsr}(D_0, D_1) = \sqrt[n]{\varrho(D_I)} \quad \text{for some } I \in \{0,1\}^n$$

was disproven.

Bad news II

- The *finiteness conjecture*

$$\operatorname{jsr}(D_0, D_1) = \sqrt[n]{\varrho(D_I)} \quad \text{for some } I \in \{0,1\}^n$$

  was disproven.
- The jsr-problem is np-complete with respect to accuracy and dimension.
- In general, the numerical computation of $\operatorname{jsr}(D_0, D_1)$ with accuracy ε requires

$$O\big((\dim D_0)^{1/\varepsilon}\big)$$

  operations.

Good news

- So far, no counterexamples to the finiteness conjecture have been encountered in practice.
- Robust algorithm for verifying

$$\operatorname{jsr}(D_0, D_1) = \sqrt[n]{\varrho(D_I)}$$

for given I is available (implementation in progress).

Exact evaluation of the JSR

- Determine a candidate I for the finiteness conjecture,

$$\mathrm{jsr}(D_0, D_1) = \sqrt[n]{\varrho(D_I)}.$$

- Let

$$\tilde{D}_i = \frac{D_i}{\mathrm{jsr}(D_0, D_1)}.$$

- Verifying the conjecture is equivalent to showing

$$\mathrm{jsr}(\tilde{D}_0, \tilde{D}_1) = 1.$$

Exact evaluation of the JSR - Method 1

- Start with unit cube M^0.
- If the recursion

$$M^{k+1} = \operatorname{conv}(\tilde{D}_0 M^k, \tilde{D}_1 M^k)$$

  with stopping criterion

$$M^{k+1} \subseteq M^k$$

  terminates, then the conjecture is verified.
- The set M^k defines the unit ball wrt. the optimal norm $\|\cdot\|_*$.

Exact evaluation of the JSR - Method 2

- Conisder tree of matrix products with
  - edges of type

$$D_0,\ D_1,\ \{D_I^n : n \in \mathbb{N}_0\}$$

  - set-valued nodes of type

$$\mathcal{D} = \{D_J\, D_I^n D_K : n \in \mathbb{N}_0\}$$

- If a depth-first search with stopping criterion

$$\max_{D \in \mathcal{D}} \|D\| < 1$$

terminates, then the conjecture is verified.

Conclusion

- The matrix approach provides an alternative to the Laurent series formalism.
- From a theoretical point of view, both methods are equivalent.
- For special purposes, one approach may be more efficient than the other.
- In general, sharp results are *beyond* reach, and even good estimates may be *very* hard to determine.
- In practise, sharp results are *within* reach.