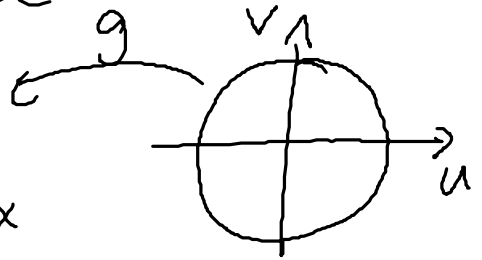
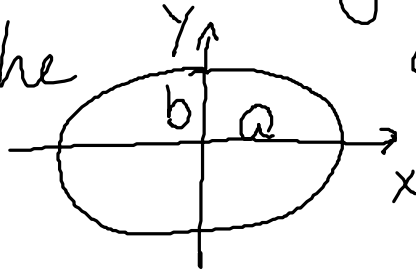


Gestern: Integration über 2-dim. Bereichen

Transformationsregel

Bsp.: Ellipsenfläche

$$E: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$



$$g(u, v) = \begin{pmatrix} a \cdot u \\ b \cdot v \end{pmatrix}$$

$$K: u^2 + v^2 \leq 1$$

$$F(K) = \pi \cdot r^2 = \pi$$

$$|\det(J_g(u, v))| = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = a \cdot b$$

$$F(E) = \iint_E 1 \, dx \, dy = \iint_K 1 \cdot |\det J_g(u, v)| \, du \, dv =$$

$$= \iint_K ab \, du \, dv = ab \iint_K 1 \, du \, dv = ab \pi //$$

Heute: Integration über 3-dim. Bereiche

Transformationsformel mit $n=3$

Def: Sei $I = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid a_1 \leq x \leq b_1, a_2 \leq y \leq b_2, a_3 \leq z \leq b_3 \right\}$
 $= [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

$$f: I \rightarrow \mathbb{R} \text{ stetig}$$

$$Z = a_1 = x_0 < \dots < x_n = b_1 \text{ eine Zerlegung von } I$$

$$a_2 = y_0 < \dots < y_m = b_2$$

$$a_3 = z_0 < \dots < z_p = b_3$$

Konvergieren die Folgen Riemannsches Summen
von f ⁿ bzgl. z_p

$$S_z(f) = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^l f(\xi_i, \eta_j, \mu_k) (x_i - x_{i-1}) (y_j - y_{j-1}) (z_k - z_{k-1})$$

$\in [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$

für beliebig kleine Zerlegungen von I und für
bel. Zwischenstellen (ξ_i, η_j, μ_k) gegen einen
Grenzwert S , so heißt f auf I integrierbar
und S das bestimmte Integral von f auf I .

$$S = \iiint_I f \, dV = \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x, y, z) \, dx \, dy \, dz$$

Volumen-
element

Bsp.: "Masse eines Quaders"
 $Q = [0, 2] \times [0, 1] \times [1, 2]$ mit Dichtefunktion

$$\rho(x, y, z) = x^2 y + z$$

($V = \iiint_Q 1 \, dV$ "Volumen"), Dichte = $\frac{\text{Masse}}{\text{Volumen}}$

$$M = \iiint_Q \rho \, dV$$

$$M = \int_1^2 \int_0^1 \int_0^2 x^2 y + z \, dx \, dy \, dz$$

$$= \int_1^2 \int_0^1 \left(\left[\frac{1}{3} x^3 y + z x \right]_0^2 \right) dy \, dz$$

$$= \int_1^2 \int_0^1 \left[\frac{8}{3} y + 2z \right] dy \, dz = \int_1^2 \left[\frac{8}{6} y^2 + 2zy \right]_0^1 dz$$

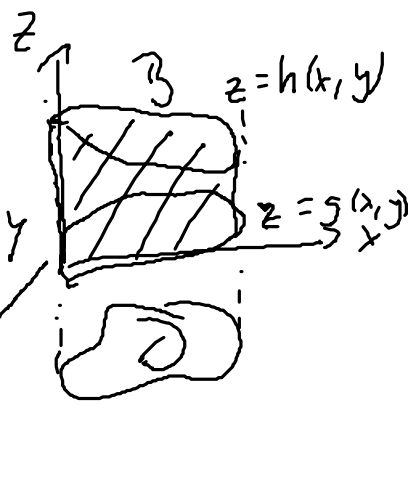
$$= \int_1^2 \left[\frac{4}{3} + 2z \right] dz = \left[\frac{4}{3} z + z^2 \right]_1^2 = \frac{4}{3} (2-1) + (2^2 - 1^2) = \frac{4}{3} + 3$$

Integration über 3-dim. Normalbereiche (falls Integrationsbereich kein Quader)

Is $D \subseteq \mathbb{R}^2$ ein Normalbereich (vom Typ I oder II) in der x - y -Ebene, (d.h. $D = \{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq h(x)\}$) und $g(x, y)$ und $h(x, y)$ Funktionen auf D mit $g(x, y) \leq h(x, y)$ für $(x, y) \in D$, so heißt $B = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D, g(x, y) \leq z \leq h(x, y)\}$ 3-dim. Normalbereich vom Typ I.

Satz: Sei $f: B \rightarrow \mathbb{R}$, B Normalbereich vom Typ I

$$\begin{aligned} \text{So gilt } \iiint_B f \, dV &= \iint_D \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z) \, dz \right) dx dy \\ &= \int_a^b \int_{u(x)}^{v(x)} \int_{g(x,y)}^{h(x,y)} f(x,y,z) \, dz \, dy \, dx \end{aligned}$$



$D = \{(x, y) \mid a \leq x \leq b, u(x) \leq y \leq v(x)\}$ Normalbereich vom Typ I.

Bsp.:

$$B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x \geq 0, y \geq 0, z \geq 1, x + y + z \leq 2 \right\}$$

$$f(x, y, z) = x$$

ges. $g(x, y), h(x, y)$ mit $g(x, y) \leq z \leq h(x, y)$

$$1 \leq z \leq 2 - x - y$$

$$1 \leq 2 - x - y \Rightarrow x + y \leq 1 \Rightarrow y \leq 1 - x$$

$$\Rightarrow 0 \leq y \leq 1 - x$$

$$\Rightarrow 0 \leq x \leq 1$$

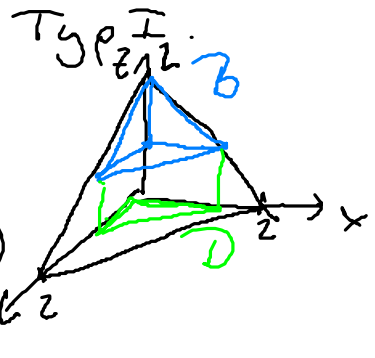
$$a = 0, b = 1$$

$$u(x) = 0, v(x) = 1 - x$$

$$g(x, y) = 1, h(x, y) = 2 - x - y$$

$$B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 1 \leq z \leq 2 - x - y \right\}$$

$$\iiint_B f \, dV = \int_0^1 \int_0^{1-x} \int_1^{2-x-y} x \, dz \, dy \, dx$$



$$\begin{aligned}
&= \int_0^1 \int_0^{1-x} (2-x-y) \cdot x - x \, dy \, dx = \int_0^1 \int_0^{1-x} (2-x-y) \cdot x - x \, dy \, dx \\
&= \int_0^1 \int_0^{1-x} (x - x^2 - xy) \, dy \, dx = \int_0^1 \left(xy - x^2 y - \frac{xy^2}{2} \right) \Big|_0^{1-x} \, dx \\
&= \int_0^1 x(1-x) - x^2(1-x) - \frac{1}{2} x(1+x^2-2x) \, dx = \int_0^1 \frac{x}{2} - x^2 + \frac{x^3}{2} \, dx \\
&= \left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} \right]_0^1 = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} = \frac{6-8+3}{24} = \frac{1}{24} //
\end{aligned}$$

Rechenregeln

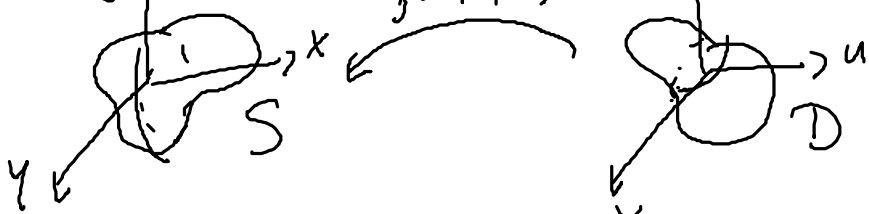
- ① Linearität: $\iiint_B (af + bg) \, dV = a \iiint_B f \, dV + b \cdot \iiint_B g \, dV$
- ② Monotonie: $f(x,y,z) \leq g(x,y,z) \Rightarrow \iiint_B f \, dV \leq \iiint_B g \, dV$
- ③ Additivität: $B = B_1 \cup B_2$

$$\iiint_B f \, dV = \iiint_{B_1} f \, dV + \iiint_{B_2} f \, dV$$

- ④ Vertauschbarkeit von Variablen / Satz von Fubini:

$$\begin{aligned}
B &= [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \\
\iiint_B f \, dV &= \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x,y,z) \, dz \, dx \, dy \\
&= \dots
\end{aligned}$$

⑤ Transformationsformel für $u=3$



$$g: D \rightarrow S, \quad g(u, v, w) = \begin{pmatrix} x(u, v, w) \\ y(u, v, w) \\ z(u, v, w) \end{pmatrix} \quad \begin{array}{l} \text{Koordinaten-} \\ \text{transformation} \\ \text{stetig partiell} \\ \text{diffbar} \end{array}$$

$f: S \rightarrow \mathbb{R}$ stetig

Dann ist

$$\iint_S f(x, y, z) \, d(x, y, z) = \iint_D f(g(u, v, w)) \cdot \left| \det g(u, v, w) \right| \, d(u, v, w)$$

Beispiele

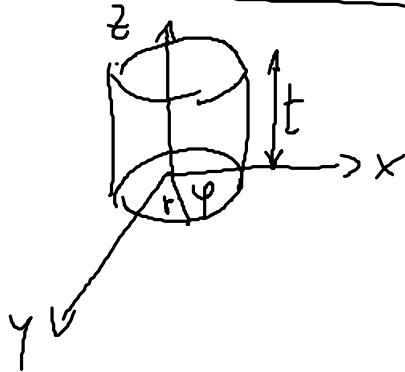
1) Zylinderkoordinaten

$$x = r \cos \varphi = x(r, \varphi, t)$$

$$y = r \sin \varphi = y(r, \varphi, t)$$

$$z = t = z(r, \varphi, t)$$

$$g(r, \varphi, t) = \begin{pmatrix} x(r, \varphi, t) \\ y(r, \varphi, t) \\ z(r, \varphi, t) \end{pmatrix}$$



$$\left| \det g(r, \varphi, t) \right| = \left| \begin{pmatrix} \cos \varphi & -r \sin \varphi & 0 \\ \sin \varphi & r \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \left| \begin{array}{cc} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{array} \right|$$

$$= r (\cos^2 \varphi + \sin^2 \varphi) = r$$

$$B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x, y \geq 0, x^2 + y^2 \leq 16, 0 \leq z \leq 10 \right\}$$

$$f(x, y, z) = \frac{z}{1 + x^2 + y^2}$$

$$\Rightarrow \iiint_B f d(x, y, z) = \int_0^4 \int_0^{\frac{\pi}{2}} \int_0^{10} f(g(r, \varphi, z)) \cdot r \cdot dz d\varphi dr$$

$$= \int_0^4 \int_0^{\frac{\pi}{2}} \int_0^{10} \frac{z}{1 + r^2} \cdot r \cdot dz d\varphi dr$$

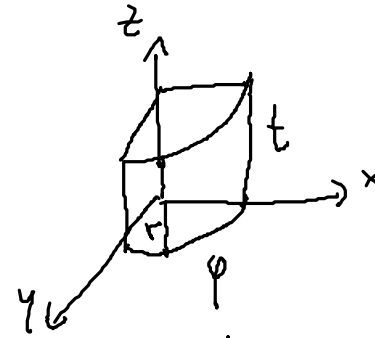
$$= \int_0^4 \int_0^{\frac{\pi}{2}} \left[\frac{z^2}{2(1+r^2)} r \right]_0^{10} d\varphi dr$$

$$= \int_0^4 \int_0^{\frac{\pi}{2}} \frac{50r}{1+r^2} d\varphi dr$$

$$= \int_0^4 \left[\frac{50r\varphi}{1+r^2} \right]_0^{\frac{\pi}{2}} dr = \int_0^4 \frac{50r\pi}{2(1+r^2)} dr$$

$$= 25\pi \int_0^4 \frac{r}{1+r^2} dr$$

$$= 25\pi \left[\frac{1}{2} \ln(1+r^2) \right]_0^4 = \frac{25\pi}{2} \cdot \ln(17)$$



2) Kugelkoordinaten

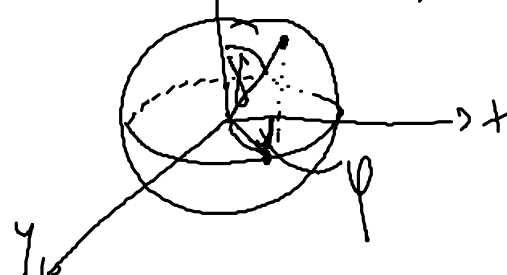
$$x = r \cos \varphi \cdot \sin \gamma = x(r, \varphi, \gamma)$$

$$y = r \sin \varphi \cdot \sin \gamma = y(r, \varphi, \gamma)$$

$$z = r \cos \gamma = z(r, \varphi, \gamma)$$

$$r \geq 0, \varphi, \gamma \in [0, 2\pi]$$

$$g(r, \varphi, \gamma) = \begin{pmatrix} x(r, \varphi, \gamma) \\ y(r, \varphi, \gamma) \\ z(r, \varphi, \gamma) \end{pmatrix}$$



$$\begin{aligned}
 |Jg(r, \varphi, \gamma)| &= \begin{vmatrix} \cos \varphi \sin \gamma & -r \sin \varphi \sin \gamma & r \cos \varphi \cos \gamma \\ \sin \varphi \sin \gamma & r \cos \varphi \sin \gamma & r \sin \varphi \cos \gamma \\ \cos \gamma & 0 & -r \sin \gamma \end{vmatrix} \\
 &= r^2 \sin \gamma \begin{vmatrix} \cos \varphi \sin \gamma & -\sin \varphi & \cos \varphi \cos \gamma \\ \sin \varphi \sin \gamma & \cos \varphi & \sin \varphi \cos \gamma \\ \cos \gamma & 0 & -\sin \gamma \end{vmatrix} \\
 &= \dots = r^2 \sin \gamma = r^2 \sin \gamma
 \end{aligned}$$

Anwendung

1) Volumen $V = \iiint_B 1 \, dV$

2) Masse $M = \iiint_B g(x, y, z) \, dV$

3) Schwerpunkt von B $x_s = \frac{1}{M} \iiint_B x \cdot g(x, y, z) \, dV$

$$y_s = \frac{1}{M} \iiint_B y \cdot g(x, y, z) \, dV$$

$$z_s = \frac{1}{M} \iiint_B z \cdot g(x, y, z) \, dV$$