An Introduction to Topology

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Introduction

In the basic Analysis courses one encounters metric spaces as an extremely useful abstract concept which is used to prove central theorems and to formulate basic principles. A typical example is the Banach Contraction Principle, an abstract theorem on contractions on complete metric spaces, that is used to prove the Inverse Function Theorem and the existence of local solutions of ordinary differential equations with a Lipschitz continuous right hand side.

For the formulation of the contraction principle, the metric is indispensable, but for many other concepts, such as convergence and limits, the specific metric is not needed at all, it suffices to use the concepts of open subsets and neighborhoods. Therefore the context of topological spaces, where everything is formulated in terms of open subsets alone, is certainly more natural in many situations. In particular, the appropriate version of the *Maximal Value Theorem* is that every real-valued function on a compact topological space has a maximal value. No metric is needed to formulate and prove it. Another basic result on real-valued functions is the *Intermediate Value Theorem*, which finds its optimal formulation in the topological context, where it reduces to the simple observation that intervals of real numbers are connected and continuous images of connected spaces are connected. All these concepts, and many others, will be developed in this course.

As will soon become apparent, topology goes far beyond metric spaces in that it can also be used to find a natural home for non-uniform convergence of sequences of functions, such as pointwise convergence, which cannot be formulated as convergence in a metric space (Chapter 3).

A new issue showing up in the context of topological spaces is that in some topological spaces where points have "too many" neighborhoods, the concept of a convergent sequence is no longer sufficient to characterize continuity of a function, and this leads us to the convergence of filters and nets (Chapter 3).

In the second half of this course we shall be concerned with developing tools that are needed in other fields, such as Functional Analysis, Spectral Theory and Differential Geometry. Typical tools of this kind are Tychonov's Theorem on the compactness of product spaces (Chapter 4), the Stone–Weierstrass Theorem and Ascoli's Theorem on compact sets in spaces of continuous functions (Chapter 5). We finally take a closer look at coverings of topological spaces, the fundamental group and the existence of simply connected covering spaces. These are concepts that arise already in analysis on open subsets of $\mathbb{R}^n$ or the complex plane, where the contractibility of loops is an important property (Chapter 6).

Chapter 1

Basic Concepts

In this first chapter, we introduce some of the most basic concepts in topology. We start with the axiomatics of topological spaces, discuss continuous maps and the concept of connectedness.

1.1 Topological Spaces

1.1.1 Open Sets

We start with the definition of a topological space and then discuss some typical classes of examples.

Definition 1.1.1. Let X be a set. A *topology on* X is a subset $\tau \subseteq \mathbb{P}(X)$, i.e., a set of subsets of X, satisfying the following axioms:

(O1) The union of any family of sets in τ belongs to τ. Applying this to the empty family, we obtain in particular $\emptyset \in \tau$.

(O2) The intersection of any finite family of sets in τ belongs to τ. Applying this to the empty family, we obtain in particular $X \in \tau$.

If τ is a topology on X, then the pair (X, τ) is called a *topological space* and the elements of τ are called *open subsets*. Often it will be clear from the context what τ is. By abuse of language, we then call X a topological space, not writing τ explicitly. Elements of topological spaces are called *points*. [1]

Examples 1.1.2. (a) For each set X, $\tau = \{X, \emptyset\}$ defines a topology on X. It is called the *indiscrete* or *chaotic topology*. Since $\emptyset$ and X are contained in every topology on X, this is the minimal topology on X.

(b) Similarly, there is a maximal topology on X. It is given by $\tau := \mathbb{P}(X)$. It is called the *discrete topology*. For this topology all subsets are open.

[1] Metric spaces were introduced by Maurice Fréchet in 1906 and topological spaces were introduced in 1914 by Felix Hausdorff (1868–1942). As so often in the history of mathematics, the more abstract and powerful concept was introduced later, when it was clear what the essential features of the theory are.

To see more examples of topological spaces, we recall the concept of a metric space.

Definition 1.1.3. (a) Let X be a set. A function $d\colon X \times X \to \mathbb{R}_+ := [0, \infty[$ is called a *metric* if

(M1) $d(x, y) = 0 \Leftrightarrow x = y$ for $x, y \in X$.

(M2) $d(x, y) = d(y, x)$ for $x, y \in X$ (Symmetry).

(M3) $d(x, z) \leq d(x, y) + d(y, z)$ for $x, y, z \in X$ (Triangle Inequality).

If, instead of (M1), only the weaker condition

(M1') $d(x, x) = 0$ for $x \in X$,

holds, then d is called a *semimetric*. If d is a (semi-)metric on X, then the pair (X, d) is called a *(semi-)metric space*.

(b) Let (X, d) be a semimetric space. For $r \geq 0$, the set

$$B_r(p) := \{q \in X\colon d(p, q) < r\}$$

is called the *(open) ball of radius r around p*. A subset $O \subseteq X$ is said to be *open* if for each $x \in O$ there exists an $\varepsilon > 0$ with $B_\varepsilon(x) \subseteq O$.

Lemma 1.1.4. *If (X, d) is a semimetric space, then the set τ_d of open subsets of X is a topology.*

Proof. To verify (O1), let $(O_i)_{i \in I}$ be a family of open subsets of X and $x \in O := \bigcup_{i \in I} O_i$. Then there exists a $j \in I$ with $x \in O_j$ and, since O_j is open, there exists an $\varepsilon > 0$ with $B_\varepsilon(x) \subseteq O_j$. Then $B_\varepsilon(x) \subseteq O$, and since $x \in O$ was arbitrary, O is open. This proves (O1).

To verify (O2), we first note that X itself is open because it contains all balls. Now let I be a finite non-empty set and $x \in U := \bigcap_{i \in I} O_i$, then $x \in O_i$ for each i, so that there exists an $\varepsilon_i > 0$ with $B_{\varepsilon_i}(x) \subseteq O_i$. Then $\varepsilon := \min\{\varepsilon_i\colon i \in I\}$ is positive, $B_\varepsilon(x) \subseteq U$, and since $x \in U$ was arbitrary, U is open. This proves (O2), and therefore (X, τ_d) is a topological space. □

Clearly, the topology τ_d depends on the semimetric. The following examples illustrate this point.

Examples 1.1.5. (a) On a set X, the metric defined by

$$d(x, y) := \begin{cases} 1 & \text{for } x = y \\ 0 & \text{otherwise,} \end{cases}$$

is called the *discrete metric*. Each subset of the metric space (X, d) is open, so that $\tau_d = \mathbb{P}(X)$ is the discrete topology (Exercise 1.1.10).

(b) If X has two different elements, then the indiscrete topology on X is not of the form τ_d for some metric d on X. Why? In particular, not every topology comes from a metric.

(c) On $X = \mathbb{R}$ the discrete metric and the standard metric $d(x, y) := |x - y|$ define different topologies coming from a metric. In fact, the one-point subset $\{0\}$ is not open for the standard metric, so that τ_d is not the discrete topology.

Examples 1.1.6. (of metric spaces) (a) $X = \mathbb{C}^n$ with the metrics

$$d_1(x, y) := \sum_{i=1}^n |x_i - y_i|, \quad d_2(x, y) := \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{1/2}$$

or

$$d_\infty(x, y) := \max\{|x_i - y_i| \colon i = 1, \ldots, n\}.$$

(b) The space $X := C([a, b], \mathbb{R})$ of continuous real-valued functions on a bounded closed interval $[a, b] \subseteq \mathbb{R}$ with the metrics

$$d_\infty(f, g) := \sup\{|f(x) - g(x)| \colon x \in [a, b]\}$$

or

$$d_1(f, g) := \int_0^1 |f(x) - g(x)|\, x$$

(Exercise 1.1.6).

(c) Typical examples of semimetric spaces which are not metric are

(1) $X = \mathbb{R}^2$ with $d(x, y) = |x_1 - y_1|$,

(2) $X = C([0, 2], \mathbb{R})$ with $d(f, h) = \int_0^1 |f(x) - h(x)|\, dx$. In this case $d(f, h) = 0$ is equivalent to $f(x) = h(x)$ for $x \in [0, 1]$ (Exercise 1.1.6).

We know already that all metric spaces carry a natural topology. It is also clear that every subset of a metric space inherits a natural metric, hence a topology. This is also true for subsets of topological spaces:

Definition 1.1.7. If (X, τ) is a topological space and $Y \subseteq X$ a subset, then

$$\tau_Y := \{U \cap Y \colon U \in \tau\}$$

is a topology on Y, called the *subspace topology* (cf. Exercise 1.1.5).

1.1.2 Closed Sets and Neighborhoods

Definition 1.1.8. Let (X, τ) be a topological space.

(a) A subset $A \subseteq X$ is called *closed* if its complement $A^c := X \setminus A$ is open.

(b) A subset $U \subseteq X$ containing $x \in X$ is called a *neighborhood of* x if there exists an open subset O with $x \in O \subseteq U$. We write $\mathfrak{U}(x)$ for the set of neighborhoods of x.

(c) (X, τ) is called a *Hausdorff space* or *hausdorff* or *separated*, if for $x \neq y \in X$ there exist disjoint open subsets O_x and O_y with $x \in O_x$ and $y \in O_y$.

Remark 1.1.9. (a) The discrete topology on a set X is always hausdorff. However, the indiscrete topology is hausdorff if and only if $|X| \leq 1$.

(b) Neighborhoods of a point are not necessarily open. All supersets of any neighborhood are neighborhoods.

Lemma 1.1.10. *The set of closed subsets of a topological space (X, τ) satisfies the following conditions:*

(C1) *The intersection of any family of closed subsets of X is closed. In particular, X is closed.*

(C2) *The union of any finite family of closed subsets of X is closed. In particular, $\emptyset$ is closed.*

Proof. This follows immediately from (O1) and (O2) by taking complements and using de Morgan's Rules: $(\bigcup_{i\in I} O_i)^c = \bigcap_{i\in I} O_i^c$ and $(\bigcap_{i\in I} O_i)^c = \bigcup_{i\in I} O_i^c$. □

Lemma 1.1.11. *In a semimetric space (X, d), we have:*

(i) *The sets $B_r(x)$, $x \in X$, $r > 0$, are open.*

(ii) *The sets $B_{\leq r}(x) := \{y \in X : d(x, y) \leq r\}$ are closed.*

(iii) *(X, d) is hausdorff if and only if d is a metric.*

Proof. (i) Let $y \in B_r(x)$, so that $d(x, y) < r$. We claim that, for $s := r - d(x, y)$, the ball $B_s(y)$ is contained in $B_r(x)$. In fact, this follows from the triangle inequality, which asserts for $z \in B_s(y)$ that

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + s = r.$$

Since y was arbitrary in $B_r(x)$, the set $B_r(x)$ is open.

(ii) Let $y \in B_{\leq r}(x)^c$, so that $d(x, y) > r$. We claim that, for $s := d(x, y) - r$, the ball $B_s(y)$ is contained in $B_{\leq r}(x)^c$. In fact, this follows from the triangle inequality, which asserts for $z \in B_s(y)$ that

$$d(x, z) \geq d(x, y) - d(y, z) > d(x, y) - s = r.$$

Since $y \in B_{\leq r}(x)^c$ was arbitrary, it follows that $B_{\leq r}(x)^c$ is an open set.

(iii) If d is not a metric, then there exist two different points $x \neq y \in X$ with $d(x, y) = 0$. Then $y \in B_r(x)$ for each $r > 0$ implies that every open subset containing x also contains y, and therefore X is not hausdorff.

If, conversely, d is a metric and $x \neq y$, then we pick a positive $r < \frac{1}{2}d(x, y)$. Then the triangle inequality implies that the two balls $B_r(x)$ and $B_r(y)$ are disjoint: For $z \in B_r(x) \cap B_r(y)$, we obtain a contradiction

$$d(x, y) \leq d(x, z) + d(z, y) < 2r < d(x, y).$$

Since the balls $B_r(x)$ and $B_r(y)$ are open by (i), X is hausdorff. □

Definition 1.1.12. Let (X, τ) be a topological space. For $E \subseteq X$ the set

$$\overline{E} := \bigcap \{F \subseteq X : E \subseteq F \text{ and } F \text{ closed}\}$$

is called the *closure of* E and the subset E is called *dense in* $\overline{E}$. Clearly, E is closed if and only if $\overline{E} = E$.

This is the smallest closed subset of X containing E. The subset

$$E^0 := \bigcup \{U \subseteq X : U \subseteq E \text{ and } U \text{ open}\}$$

is called the *interior of* E. This is the largest open subset of X contained in E. Elements of E^0 are called *interior points of* E. The set

$$\partial E := \overline{E} \setminus E^0$$

is called the *boundary of* E.

Lemma 1.1.13. *For a topological space* (X, τ)*, a subset* $E \subseteq X$ *and* $x \in X$*, the following assertions hold:*

(i) $x \in E^0 \Leftrightarrow \big[(\exists U \in \mathfrak{U}(x))\ U \subseteq E\big] \Leftrightarrow E \in \mathfrak{U}(x)$.

(ii) $x \in \overline{E} \Leftrightarrow (\forall U \in \mathfrak{U}(x))\ U \cap E \neq \emptyset \Leftrightarrow E^c \notin \mathfrak{U}(x)$.

(iii) $x \in \partial E \Leftrightarrow (\forall U \in \mathfrak{U}(x))\ U \cap E \neq \emptyset$ *and* $U \cap E^c \neq \emptyset$.

Proof. (i) That $x \in E^0$ is equivalent to the existence of an open subset O of E containing x, which is equivalent to the existence of a neighborhood of x contained in E and also equivalent to E being a neighborhood of x.

(ii) That $x \in \overline{E}$ means that each closed subset containing E also contains x, which is equivalent to the non-existence of an open subset O containing x and intersecting E trivially. This in turn is equivalent to the assertion that each neighborhood of x intersects E. Clearly, this also means that the complement E^c is not a neighborhood of x.

(iii) follows by combining (i) and (ii) because $x \in \partial E$ means that $x \in \overline{E}$, but not $x \in E^0$. □

Exercises for Section 1.1

Exercise 1.1.1. (a) Show that all metrics d on a finite set define the discrete topology.

(b) Show that all finite Hausdorff spaces are discrete.

Exercise 1.1.2. Find an example of a countable metric space (X, d) for which the topology τ_d is not discrete.

For the next exercise, we need the concept of a seminormed space.

Definition 1.1.14. Let V be a $\mathbb{K}$-vector space ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$). A function $p \colon V \to \mathbb{R}_+$ is called a *seminorm* if

(N1) $p(\lambda x) = |\lambda| p(x)$ for $\lambda \in \mathbb{K}$, $x \in V$, and

(N2) $p(x+y) \leq p(x) + p(y)$ for $x, y \in V$ (Subadditivity).
If, in addition,

(N3) $p(x) > 0$ for $0 \neq x \in V$,

then p is called a *norm*. If p is a (semi-)norm on V, then the pair (V, p) is called a *(semi-)normed space*.

Exercise 1.1.3. Show that if (V, p) is a (semi-)normed space, then $d(x, y) := p(x - y)$ is a (semi-)metric which is a metric if and only if p is a norm.

Exercise 1.1.4. Show that a subset M of a topological space X is open if and only if it is a neighborhood of all points $x \in M$.

Exercise 1.1.5. Let Y be a subset of a topological space (X, τ). Show that $\tau|_Y = \{O \cap Y : O \in \tau\}$ defines a topology on Y.

Exercise 1.1.6. Let $a < b < c$ be real numbers. Show that

$$d(f, g) := \int_a^b |f(x) - g(x)| \, dx$$

defines a semimetric on the space $C([a, c], \mathbb{R})$ of continuous real-valued functions on $[a, c]$. Show also that $d(f, g) = 0$ is equivalent to $f = g$ on $[a, b]$, and that d is a metric if and only if $b = c$.

Exercise 1.1.7. Let (X, d) be a metric space and $Y \subseteq X$ be a subset. Show that the subspace topology $\tau_d|_Y$ on Y coincides with the topology defined by the restricted metric $d_Y := d|_{Y \times Y}$.

Exercise 1.1.8. (Hausdorff's neighborhood axioms) Let (X, τ) be a topological space. Show that the collected $\mathfrak{U}(x)$ of neighborhoods of a point $x \in X$ satisfies:

(N1) $\{x\} \in \mathfrak{U}(x)$ and $X \in \mathfrak{U}(x)$.

(N2) $U \in \mathfrak{U}(x)$ and $V \supseteq U$ implies $V \in \mathfrak{U}(x)$.

(N3) $U_1, U_2 \in \mathfrak{U}(x)$ implies $U_1 \cap U_2 \in \mathfrak{U}(x)$.

(N4) Each $U \in \mathfrak{U}(x)$ contains a $V \in \mathfrak{U}(x)$ with the property that $U \in \mathfrak{U}(y)$ for each $y \in V$.

Exercise 1.1.9. Let X be a set and suppose that we have for each $x \in X$ a subset $\mathcal{U}(x) \subseteq \mathbb{P}(X)$, such that the conditions (N1)-(N4) from Exercise 1.1.9 are satisfied. We then call a subset $O \subseteq X$ open if $O \in \mathcal{U}(x)$ holds for each $x \in O$. Show that the set τ of open subsets of X defines a topology on X for which $\mathcal{U}(x)$ is the set of all neighborhoods of x.

Exercise 1.1.10. Show that the following assertions are equivalent for a metric space (X, d):

(a) The topological space (X, τ_d) is discrete.

(b) The points in X form τ_d-open subsets.

(c) For each $p \in X$ there exists an $\varepsilon > 0$ with $B_\varepsilon(p) = \{p\}$.

Exercise 1.1.11. For each norm $\|\cdot\|$ on $\mathbb{R}^n$, the metric $d(x,y) := \|x-y\|$ defines the same topology. Hint: Use that each norm is equivalent to $\|x\|_\infty := \max\{|x_i| \colon i = 1, \ldots, n\}$ (cf. Analysis II).

Exercise 1.1.12. (The cofinite topology) Let X be a set and

$$\tau := \emptyset \cup \{A \subseteq X \colon |A^c| < \infty\}.$$

Show that τ defines a topology on X. When is this topology hausdorff?

Exercise 1.1.13. (p-adic metric) Let p be a prime number. For $q \in \mathbb{Q}^\times$ we define $|q|_p := p^{-n}$ if we can write $q = p^n \frac{a}{b}$, where $a \in \mathbb{Z}, 0 \neq b \in \mathbb{Z}$ are not multiples of p. Note that this determines a unique $n \in \mathbb{Z}$. We also put $|0|_p := 0$. Show that

$$d(x,y) := |x-y|_p$$

defines a metric on $\mathbb{Q}$ for which the sequence $(p^n)_{n\in\mathbb{N}}$ converges to 0.

Exercise 1.1.14. Show that for a subset E of the topological space X, we have

$$\overline{E}^c = (E^c)^0 \quad \text{and} \quad (E^0)^c = \overline{E^c}.$$

1.2 Continuous Maps

After introducing the concept of a topological space as a pair (X, τ) of a set X with a distinguished collection of subsets called open, we now explain what the corresponding structure preserving maps are. They are called continuous maps, resp., functions.

Definition 1.2.1. Let (X, τ_X) and (Y, τ_Y) be topological spaces.

(a) A map $f \colon X \to Y$ is called *continuous* if for each open subset $O \subseteq Y$ the inverse image $f^{-1}(O)$ is an open subset of X.

We write $C(X,Y)$ for the set of continuous maps $f \colon X \to Y$.

(b) A continuous map $f \colon X \to Y$ is called a *homeomorphism* or *topological isomorphism* if there exists a continuous map $g \colon Y \to X$ with

$$f \circ g = \mathrm{id}_Y \quad \text{and} \quad g \circ f = \mathrm{id}_X.$$

(c) A map $f \colon X \to Y$ is said to be *open* if for each open subset $O \subseteq X$, the image $f(O)$ is an open subset of Y. We similarly define *closed* maps $f \colon X \to Y$ as those mapping closed subsets of X to closed subsets of Y.

Proposition 1.2.2. *If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then their composition $g \circ f : X \to Z$ is continuous.*

Proof. For any open subset $O \subseteq Z$, the set $(g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O))$ is open in X because $g^{-1}(O)$ is open in Y. $\square$

Lemma 1.2.3. *(a) If $f\colon X \to Z$ is a continuous map and $Y \subseteq X$ a subset, then $f|_Y\colon Y \to Z$ is continuous with respect to the subspace topology on Y.*

(b) If $f\colon X \to Z$ is a map and $Y \subseteq Z$ is a subset containing $f(X)$, then f is continuous if and only if the corestriction $f^{|Y}\colon X \to Y$ is continuous with respect to the subspace topology on Y.

Proof. (a) If $O \subseteq Z$ is open, then $(f|_Y)^{-1}(O) = f^{-1}(O) \cap Y$ is open in the subspace topology. Therefore $f|_Y$ is continuous.

(b) For a subset $O \subseteq Z$, we have

$$f^{-1}(O) = f^{-1}(O \cap Y) = (f^{|Y})^{-1}(O \cap Y).$$

This implies that f is continuous if and only if the corestriction $f^{|Y}$ is continuous. $\square$

Presently, we only have a global concept of continuity. To define also what it means that a function is continuous in a point, we use the concept of a neighborhood.

Definition 1.2.4. Let X and Y be topological spaces and $x \in X$. A function $f\colon X \to Y$ is said to be *continuous in* x if for each neighborhood V of $f(x)$ there exists a neighborhood U of x with $f(U) \subseteq V$. Note that this condition is equivalent to $f^{-1}(V)$ being a neighborhood of x.

Remark 1.2.5. If (X, d_X) and (Y, d_Y) are metric spaces, then a map $f\colon X \to Y$ is continuous in $x \in X$ if and only if

$$(\forall \varepsilon > 0)(\exists \delta > 0) \quad f\big(B_\delta(x)\big) \subseteq B_\varepsilon\big(f(x)\big).$$

This follows easily from the observation that $V \subseteq Y$ is a neighborhood of $f(x)$ if and only if it contains some ball $B_\varepsilon(f(x))$ and $U \subseteq X$ is a neighborhood of x if and only if it contains some ball $B_\delta(x)$ (Exercise 1.2.1).

Lemma 1.2.6. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps between topological spaces. If f is continuous in x and g is continuous in $f(x)$, then the composition $g \circ f$ is continuous in x.*

Proof. Let V be a neighborhood of $g(f(x))$ in Z. Then the continuity of g in $f(x)$ implies the existence of a neighborhood V' of $f(x)$ with $g(V') \subseteq V$. Further, the continuity of f in x implies the existence of a neighborhood U of x in X with $f(U) \subseteq V'$, and then $(g \circ f)(U) \subseteq g(V') \subseteq V$. Therefore $g \circ f$ is continuous in x. $\square$

Proposition 1.2.7. *For a map $f\colon X \to Y$ between topological spaces, the following are equivalent:*

(1) *f is continuous.*

(2) *f is continuous in each $x \in X$.*

(3) *Inverse images of closed subsets of Y under f are closed.*

(4) *For each subset $M \subseteq X$, we have $f(\overline{M}) \subseteq \overline{f(M)}$.*

Proof. (1) $\Rightarrow$ (2): Let $V \subseteq Y$ be a neighborhood of $f(x)$. Then the continuity of f implies that $U := f^{-1}(V^0)$ is an open subset of X containing x, hence a neighborhood of x with $f(U) \subseteq V$.

(2) $\Rightarrow$ (1): Let $O \subseteq Y$ be open and $x \in f^{-1}(O)$. Since f is continuous in x, $f^{-1}(O)$ is a neighborhood of x, and since x is arbitrary, the set $f^{-1}(O)$ is open.

(1) $\Leftrightarrow$ (3): If $A \subseteq Y$ is closed, then $f^{-1}(A) = f^{-1}(A^c)^c$ implies that all these subsets of X are closed if and only if all sets $f^{-1}(A^c)$ are open, which is equivalent to the continuity of f.

(3) $\Rightarrow$ (4): The inverse image $f^{-1}(\,\overline{f(M)}\,)$ is a closed subset of X containing M, hence also $\overline{M}$.

(4) $\Rightarrow$ (3): If $A \subseteq Y$ is closed and $M := f^{-1}(A)$, then $f(\overline{M}) \subseteq \overline{f(M)} \subseteq A$ implies that $\overline{M} \subseteq M$, i.e., M is closed. □

Proposition 1.2.8. *For a continuous map $f\colon X \to Y$, the following are equivalent:*

(1) *f is a homeomorphism.*

(2) *f is bijective and $f^{-1}\colon Y \to X$ is continuous.*

(3) *f is bijective and open.*

(4) *f is bijective and closed.*

Proof. (1) $\Leftrightarrow$ (2): Let $g\colon Y \to X$ be continuous with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Then f is bijective, and $f^{-1} = g$ is continuous.

If, conversely, f is bijective and f^{-1} is continuous, then we see with $g := f^{-1}$ that f is a homeomorphism.

(2) $\Leftrightarrow$ (3): For $O \subseteq X$ we have $f(O) = (f^{-1})^{-1}(O)$. That this set if open for each open subset $O \subseteq X$ is equivalent to f being open and to f^{-1} being continuous.

(2) $\Leftrightarrow$ (4): For $A \subseteq X$ we have $f(A) = (f^{-1})^{-1}(A)$. That this set if closed for each closed subset $A \subseteq X$ is equivalent to f being closed and to f^{-1} being continuous (Proposition 1.2.7). □

Definition 1.2.9. (a) A sequence $(x_n)_{n\in\mathbb{N}}$ in the topological space X is said to *converge to p*, written,

$$\lim_{n\to\infty} x_n = p \quad \text{or} \quad x_n \to p,$$

if for each neighborhood $U \in \mathfrak{U}(p)$ there exists a number $n_U \in \mathbb{N}$ with $x_n \in U$ for $n > n_U$.

(b) A topological space X is said to be *first countable* if there exists for each $p \in X$ a sequence $(U_n)_{n\in\mathbb{N}}$ of neighborhoods of p such that each neighborhood V of p contains some U_n. Then the sequence $(U_n)_{n\in\mathbb{N}}$ is called a *basis of neighborhoods of* p and the U_n are called *basic neighborhoods*. Note that by replacing U_n by $U_n' := U_1 \cap \ldots \cap U_n$, we even obtain a decreasing sequence of basic neighborhoods of p.

Proposition 1.2.10. *(a) If $f\colon X \to Y$ is a continuous map between topological spaces, then*

$$x_n \to p \Rightarrow f(x_n) \to f(p) \quad \textit{for each } p \in X. \tag{1.1}$$

(b) If, conversely, X is first countable and for each sequence $x_n \to p$ we have $f(x_n) \to f(p)$, then f is continuous.

Proof. (a) Suppose first that f is continuous in p and that $x_n \to p$ in X. Let V be a neighborhood of $f(p)$ in Y. Then there exists a neighborhood U of p in X with $f(U) \subseteq V$. Pick $n_U \in \mathbb{N}$ with $x_n \in U$ for $n > n_U$. Then $f(x_n) \in V$ for $n > n_U$ implies that $f(x_n) \to f(p)$.

(b) Now assume that X is first countable and for each sequence $x_n \to p$ we have $f(x_n) \to f(p)$. To show that f is continuous in p, pick a neighborhood V of $f(p)$ and a decreasing sequence $(U_n)_{n\in\mathbb{N}}$ of basic neighborhoods of p.

If f is not continuous in p, then $f(U_n) \not\subseteq V$ for each n, so that we find $x_n \in U_n$ with $f(x_n) \notin V$. Then $x_n \to p$ follows from the fact that for each neighborhood U of p there exists an n with $U_n \subseteq U$, and then $x_m \in U_m \subseteq U_n \subseteq U$ holds for $m \geq n$. On the other hand $f(x_n) \not\to f(p)$, by construction. □

Example 1.2.11. We shall see later that there exists a topology (the topology of pointwise convergence) on the set X of measurable functions $f\colon [0,1] \to [0,1]$ for which $f_n \to f$ for a sequence in X if and only if $f_n(x) \to f(x)$ for each $x \in [0,1]$. In view of Lebesgue's Theorem of Dominated Convergence, the map

$$I\colon X \to \mathbb{R}, \quad f \mapsto \int_0^1 f(x)\,dx$$

is sequentially continuous, i.e., $f_n \to f$ pointwise implies $I(f_n) \to I(f)$ (here we use that $|f_n| \leq 1$). However, I is not a continuous map because $I(f) = 0$ holds for each function f which is non-zero at most in finitely many places, but the constant function 1 is contained in the closure of this set (Exercise 2.2.10). This shows that I is not continuous because the continuity of I would imply $I(\overline{M}) \subseteq \overline{I(M)}$ for any subset $M \subseteq X$.

Exercises for Section 1.2

Exercise 1.2.1. Verify the assertion of Remark 1.2.5 about the continuity of a function between metric spaces.

Exercise 1.2.2. Show that, if $f\colon X \to Y$ is a continuous function into a discrete space Y, then the sets $f^{-1}(y)$, $y \in Y$, form a partition of X by open closed subsets.

Exercise 1.2.3. Let d_1 and d_2 be two metrics on the set X and write $B_r^j(x)$ for the balls with respect to d_j, $j = 1, 2$. Show that d_1 and d_2 define the same topology on X if and only if for each $p \in X$ and $\varepsilon > 0$ there exists a $\delta > 0$ with

$$B_\delta^1(p) \subseteq B_\varepsilon^2(p)$$

and for each $p \in X$ and $\varepsilon > 0$ there exists a $\delta > 0$ with

$$B_\delta^2(p) \subseteq B_\varepsilon^1(p).$$

Exercise 1.2.4. (Equivalent bounded metrics) Let (X, d) be a metric space. Show that:

(a) The function $f \colon \mathbb{R}_+ \to [0, 1[, f(t) := \frac{t}{1+t}$ is continuous with continuous inverse $g(t) := \frac{t}{1-t}$. Moreover, f is subadditive, i.e., $f(x+y) \leq f(x)+f(y)$ for $x, y \in \mathbb{R}_+$.

(b) $d'(x, y) := \frac{d(x,y)}{1+d(x,y)}$ is a metric on X with $\sup_{x,y \in X} d'(x, y) \leq 1$.

(c) d' and d induce the same topology on X.

Exercise 1.2.5. (Stereographic projection) We consider the n-dimensional sphere

$$\mathbb{S}^n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \colon x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}.$$

We call the unit vector $e_0 := (1, 0, \ldots, 0)$ the *north pole* of the sphere and $-e_0$ the *south pole*. We then have the corresponding *stereographic projection maps*

$$\varphi_+ \colon U_+ := \mathbb{S}^n \setminus \{e_0\} \to \mathbb{R}^n, \quad (y_0, y) \mapsto \frac{1}{1 - y_0} y$$

and

$$\varphi_- \colon U_- := \mathbb{S}^n \setminus \{-e_0\} \to \mathbb{R}^n, \quad (y_0, y) \mapsto \frac{1}{1 + y_0} y.$$

Show that these maps are homeomorphisms with inverse maps

$$\varphi_\pm^{-1}(x) = \Big(\pm \frac{\|x\|_2^2 - 1}{\|x\|_2^2 + 1}, \frac{2x}{1 + \|x\|_2^2} \Big).$$

1.3 Connectedness

Definition 1.3.1. (a) A topological space X is said to be *connected* if for each decomposition $X = O_1 \dot\cup O_2$ into two disjoint open subsets O_1 and O_2, one of the sets O_i is empty.

(b) A continuous map $\gamma : [0, 1] \to X$ is called a *path* and its image $\gamma([0, 1])$ an *arc*. The space X is called *arcwise connected*, if for $x, y \in X$ there exists a path $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. We call γ a *path* from x to y.

Lemma 1.3.2. *A topological space X is connected if and only if all continuous functions $f \colon X \to \{0, 1\}$ are constant, where $\{0, 1\}$ carries the discrete topology.*

From the Intermediate Value Theorem we thus immediately get:

Proposition 1.3.3. *A subset $I \subseteq \mathbb{R}$ is connected if and only if it is an interval, i.e., $x, z \in I$ and $x < y < z$ implies $y \in I$.*

Proof. According to the Intermediate Value Theorem, continuous functions $f : I \to \{0, 1\}$ on an interval are constant, so that intervals are connected. If, conversely, $I \subseteq \mathbb{R}$ is not an interval, then there exist $x < y < z \in \mathbb{R}$ with $x, z \in I$ and $y \notin I$. Then $I_1 := I \cap]-\infty, y[$ and $I_2 := I \cap]y, \infty[$ are non-empty disjoint open subsets of I with $I = I_1 \cup I_2$, and therefore I is not connected. □

Proposition 1.3.4. *If $f : X \to Y$ is a continuous map and X is (arcwise) connected, then $f(X)$ is (arcwise) connected.*

Proof. (a) First we assume that X is connected. Let $h : f(X) \to \{0, 1\}$ be a continuous function. Then $h \circ f : X \to \{0, 1\}$ is a continuous function, hence constant, and therefore h is also constant. This proves that $f(X)$ is connected.

(b) If X is arcwise connected and $a, b \in f(X)$, then there exist points $x, y \in X$ with $f(x) = a$ and $f(y) = b$. Since X is arcwise connected, there exists a path $\alpha : [0, 1] \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$. Then $f \circ \alpha : [0, 1] \to f(X)$ is a path from a to b. □

Proposition 1.3.5. *Arcwise connected spaces are connected.*

Proof. Let $x, y \in X$ and $f : X \to \{0, 1\}$ be a continuous function. Since X is arcwise connected, there exists a path $\alpha : [0, 1] \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$. Then $f \circ \alpha : [0, 1] \to \{0, 1\}$ is continuous, hence constant by the Intermediate Value Theorem. Therefore $f(x) = f(y)$, and since x and y were arbitrary elements of X, the function f is constant. □

Proposition 1.3.6. *Let $(Y_i)_{i \in I}$ be a family of (arcwise) connected subsets of the topological space X with $\bigcap_{i \in I} Y_i \neq \emptyset$. Then $\bigcup_{i \in I} Y_i$ is (arcwise) connected.*

Proof. Pick $x \in \bigcap_{i \in I} Y_i$ and let $Y := \bigcup_{i \in I} Y_i$.

(a) First we assume that each Y_i is connected. Let $f : Y \to \{0, 1\}$ be a continuous function and assume w.l.o.g. that $f(x) = 0$. Then the restriction to each Y_i is also continuous, so that the connectedness of Y_i implies that $f|_{Y_i} = 0$. This implies that $f = 0$, and hence that Y is connected (Lemma 1.3.2).

(b) Let $y, z \in Y$ and choose $i, j \in I$ with $y \in Y_i$ and $z \in Y_j$. Then there exists a path $\alpha : [0, 1] \to Y_i$ with $\alpha(0) = y$ and $\alpha(1) = x$ and a path $\beta : [0, 1] \to Y_j$ with $\beta(0) = x$ and $\beta(1) = z$. Then

$$(\alpha * \beta)(t) := \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \beta(2t - 1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

defines a path $\alpha * \beta : [0, 1] \to Y$ connecting y to z. Therefore Y is arcwise connected. □

Definition 1.3.7. Let X be a topological space.

(a) For $x \in X$, the *connected component* C_x of x is X is the union of all connected subsets of X containing x. Its connectedness follows from Proposition 1.3.6.

(b) For $x \in X$, the *arc-component* A_x of x is X is the union of all arcwise connected subsets of X containing x. Its arcwise connectedness follows from Proposition 1.3.6.

Remark 1.3.8. (a) Clearly, $A_x \subseteq C_x$ follows from Proposition 1.3.5.

(b) The connected components of a topological space X form a partition of X.

In fact, if C_x and C_y are two connected components which intersect nontrivially, then Proposition 1.3.4 implies that $C_x \cup C_y$ is connected, so that the maximality of C_x, resp., C_y yields $C_x = C_y$.

(c) A similar argument shows that the arc-components of a topological space X form a partition of X.

Proposition 1.3.9. *Let X be a topological space in which every point x has an arcwise connected neighborhood U_x. Then the arc-components of X are open and coincide with the connected components.*

Proof. Clearly, $U_x \subseteq A_x$ because U_x is arcwise connected. Therefore A_x is a neighborhood of x. For any other $y \in A_x$, we have $A_x = A_y$ (Remark 1.3.8), so that A_x also is a neighborhood of y, and hence A_x is open.

Since arcwise connected spaces are connected (Proposition 1.3.5), $A_x \subseteq C_x$. If $y \in C_x \setminus A_x$, then $A_y \subseteq C_x \setminus A_x$ follows from $A_x \cap A_y = \emptyset$, and therefore

$$C_x \setminus A_x = \bigcup_{y \in C_x \setminus A_x} A_y$$

is a union of open subsets, hence an open subset of C_x. Since C_x is connected and $A_x \neq \emptyset$, we obtain $C_x \setminus A_x = \emptyset$, i.e., $C_x = A_x$. $\square$

Definition 1.3.10. A topological space X is called an *n-dimensional manifold* if each $x \in X$ has an open neighborhood U homeomorphic to an open subset of $\mathbb{R}^n$.

Remark 1.3.11. (a) The preceding proposition applies in particular to each open subset $U \subseteq \mathbb{R}^n$.

(b) From (a) we immediately derive that Proposition 1.3.9 also applies to n-dimensional manifolds.

Exercises for Section 1.3

Exercise 1.3.1. Show that if Y is a connected subset of the topological space X, then its closure $\overline{Y}$ is also connected.

Exercise 1.3.2. For a continuous function $f\colon]0,1] \to \mathbb{R}$, we consider its graph

$$\Gamma(f) := \{(x, f(x))\colon 0 < x \leq 1\}.$$

Show that:

(a) $\Gamma(f)$ is an arcwise connected subset of $\mathbb{R}^2$.

(b) $\overline{\Gamma(f)} = \Gamma(f) \cup (\{0\} \times I_f)$, where $I_f \subseteq \mathbb{R}$ is the set of all those points y for which there exists a sequence $x_n \to 0$ in $]0,1]$ with $f(x_n) \to y$.

(c) $\overline{\Gamma(f)}$ is connected.

(d) For $f(x) := \sin(1/x)$, the set $\overline{\Gamma(f)}$ is not arcwise connected.

(e) $\overline{\Gamma(f)}$ is arcwise connected if and only if $|I_f| \leq 1$.

Exercise 1.3.3. Show that the connected components of a topological space are closed.

Exercise 1.3.4. Find an example of an arc-component of a topological space which is not closed.

Exercise 1.3.5. A topological space X is called *locally (arcwise) connected*, if each neighborhood U of a point x contains a connected (an arcwise connected) neighborhood V of x.

Show that in a locally connected space the connected components are open and in a locally arcwise connected space the arc-components are open and coincide with the connected components.

Exercise 1.3.6. In $\mathbb{R}^2$ we consider the set

$$X = ([0,1] \times \{1\}) \cup \Big(\Big\{\frac{1}{n}\colon n \in \mathbb{N}\Big\} \times [0,1]\Big) \cup (\{0\} \times [0,1]).$$

Show that X is arcwise connected but not locally arcwise connected.

Exercise 1.3.7. Show that the topological spaces

$$I = [0,1] \quad \text{and} \quad \mathbb{S}^1 := \{(x,y) \in \mathbb{R}^2\colon x^2 + y^2 = 1\}$$

are not homeomorphic. Hint: Consider the connectedness properties if one point is removed.

1.4 Separation Axioms

We have already seen that metric spaces are always hausdorff, but that in general topological spaces are not. Since the axioms of a topological space are very weak, they permit topologies, such as the indiscrete topology, which cannot distinguish the points in X. In almost all situations occurring in mathematical practice, the occurring topological spaces do have additional separation properties.

Definition 1.4.1. (The separation axioms) Let (X, τ) be a topological space. We distinguish the following separation properties T_n:

(T_0) X is called a T_0-*space* if for any two points $x \neq y$, there exists an open subset containing only one of them.

(T_1) X is called a T_1-*space* if for any two points $x \neq y$, there exists an open set O_x not containing y.

(T_2) X is called a T_2-*space*, or *hausdorff*, if for any two points $x \neq y$, there exist disjoint open subsets O_x containing x and O_y containing y. This is the same as the Hausdorff property.

(T_3) A T_1-space is called a T_3-*space*, or *regular*, if for any point x and a closed subset A not containing x there exist disjoint open subsets O_x containing x and O_A containing A.

(T_4) A T_1-space is called a T_4-*space*, or *normal*, if for any two disjoint closed subsets A_1, A_2 there exist disjoint open subsets O_j containing A_j, $j = 1, 2$.

Remark 1.4.2. (a) The T_0-axiom simply means that the points of X determined by the collection of open subsets in which they lie.

(b) The T_1-axion is equivalent to the condition that all points are closed. It states that the complement $\{y\}^c$ of each point is a neighborhood of each of its points, i.e., an open set.

Proposition 1.4.3. *A Hausdorff space X is regular if and only if each neighborhood U of a point contains a closed one.*

Proof. Suppose first that X is regular, let $x \in X$ and $V \in \mathfrak{U}(x)$ be an open neighborhood of x. Then $\{x\}$ and V^c are disjoint closed subsets, so that there exist disjoint open subsets U_1, U_2 of X with $x \in U_1$ and $V^c \subseteq U_2$. Then U_2^c is a closed neighborhood of x contained in V.

Suppose, conversely, that each neighborhood of a point x contains a closed one, and that $A \subseteq X$ is a closed subset not containing x. Then A^c is an open neighborhood of x, hence contains a closed neighborhood U. Then U^0 and U^c are disjoint open subsets with $x \in U^0$ and $A \subseteq U^c$. □

Exercises for Section 1.4

Exercise 1.4.1. For a non-empty subset A of the metric space (X, d), we consider the function

$$d_A(x) := \inf\{d(x, a) \colon a \in A\}.$$

Show that:

(i) $|d_A(x) - d_A(y)| \leq d(x, y)$ for $x, y \in X$. In particular, d_A is continuous.

(ii) $d_A(x) = 0$ if and only if $x \in \overline{A}$.

(iii) Every metric space (X, d) is normal. Hint: For two disjoint closed subsets $A, B \subseteq X$, consider the function $f := d_A - d_B$.

Chapter 2

Generating Topologies

In this chapter we discuss several methods to obtain new topologies from old ones. To this end, we start with a discussion of how topologies are generated by subsets of $\mathbb{P}(X)$ and then turn to the two key constructions: initial and final topologies. Two of the most important applications are product and quotient topologies.

2.1 Bases and Subbases of a Topology

Definition 2.1.1. Let X be a set. If τ and σ are topologies on X, we say that

(1) τ *is finer than* σ if $\sigma \subseteq \tau$ (as subsets of $\mathbb{P}(X)$).

(2) τ *is coarser than* σ if $\tau \subseteq \sigma$.

Lemma 2.1.2. *If $(\tau_i)_{i\in I}$ are topologies on X, then $\bigcap_{i\in I} \tau_i$ is a topology on X. It is the finest topology which is coarser that all the topologies τ_i.*

Proof. To see that $\tau := \cap_{i\in I}\tau_i$ is a topology, let $(O_j)_{j\in J}$ be a family of elements of τ. To verify (O1), put $O := \bigcup_{j\in J} O_j$. Since $O_j \in \tau_i$ for each i, the same holds for O, and therefore $O \in \tau$. To verify (O2), assume J is finite and put $O := \bigcap_{j\in J} O_j$. Since $O_j \in \tau_i$ for each i, the same holds for O, and therefore $O \in \tau$. □

Definition 2.1.3. If $\mathcal{A} \subseteq \mathbb{P}(X)$, then

$$\tau := \langle \mathcal{A} \rangle_{top} := \bigcap \{\sigma \colon \mathcal{A} \subseteq \sigma, \sigma \text{ is a topology}\}$$

is a topology on X. It is the coarsest topology on X containing $\mathcal{A}$. It is called the *topology generated by* $\mathcal{A}$.

Conversely, a set $\mathcal{A} \subseteq \mathbb{P}(X)$ is called a *subbasis* of a topology τ if τ is generated by $\mathcal{A}$. The set $\mathcal{A}$ is called a *basis of the topology* τ if each $O \in \tau$ is a union of elements of $\mathcal{A}$.

Lemma 2.1.4. *A subset $\mathcal{A} \subseteq \tau$ is a subbasis of the topology τ if and only if τ consists of all unions of finite intersections of elements of $\mathcal{A}$.*

Proof. Let σ be the set of all unions of finite intersections of elements of $\mathcal{A}$. We claim that σ is a topology. Clearly, σ is stable under arbitrary unions, so that we only have to show that it is also stable under finite intersections. The whole space X is contained in σ because X is the intersection of the empty family in $\mathcal{A}$. So let $O_1, \ldots, O_n$ be elements of σ. We write each O_i as $O_i = \bigcup_{j \in J_i} A_{i,j}$, where each $A_{i,j}$ is a finite intersection of elements of $\mathcal{A}$. Then

$$\bigcap_{i=1}^{n} O_i = \bigcap_{i=1}^{n} \bigcup_{j \in J_i} A_{i,j} = \bigcup_{j_i \in J_i} A_{1,j_1} \cap \cdots \cap A_{n,j_n}$$

is a union of finite intersections of elements of $\mathcal{A}$, hence in σ.

We conclude that $\mathcal{A}$ is subbasis of τ if and only if $\tau = \sigma$, which is the assertion of the lemma. □

If we know a subbasis for a topology, we can simplify the verification of continuity of a map:

Lemma 2.1.5. *If $f : (X, \tau_X) \to (Y, \tau_Y)$ is a map between topological spaces and $\mathcal{B}$ a subbasis of τ_Y, then f is continuous if and only if for each $B \in \mathcal{B}$, the inverse image $f^{-1}(B)$ is open.*

Proof. The set

$$f_*\tau_X := \{A \subseteq Y : f^{-1}(A) \in \tau_X\}$$

is easily seen to be a topology on Y (Exercise 2.1.2). Now f is continuous if and only if $f_*\tau_X \supseteq \tau_Y$, and since $\mathcal{B}$ generates τ_Y, this happens if and only if $\mathcal{B} \subseteq f_*\tau_X$. □

Exercises for Section 2.1

Exercise 2.1.1. A subset $\mathcal{A} \subseteq \mathbb{P}(X)$ is a basis for a topology on X if and only if

(1) $\bigcup \mathcal{A} = X$ and

(2) for each $x \in A \cap B$, $A, B \in \mathcal{A}$, there exists a $C \in \mathcal{A}$ with $x \in C \subseteq A \cap B$.

Exercise 2.1.2. (a) Let $f : X \to Y$ be a map and τ_X be a topology on X. Show that

$$f_*\tau_X := \{U \subseteq Y : f^{-1}(U) \in \tau_X\}$$

is a topology on Y.

(b) For a family of maps $f_i : X_i \to Y$ and topologies τ_i on X_i, show that the corresponding final topology is $\bigcap_{i \in I} (f_i)_*\tau_i = \{U \subseteq Y : (\forall i) f_i^{-1}(U) \in \tau_i\}$.

2.2 Initial and Final Topologies

If $f\colon X \to Y$ is a map and τ_X is a topology on X, then the topology

$$f_*\tau_X := \{A \subseteq Y \colon f^{-1}(A) \in \tau_X\}$$

is called the *pushforward of* τ_X *by* f (cf. Exercise 2.1.2). Similarly, we obtain for each topology τ_Y on Y a topology

$$f^*\tau_Y := f^{-1}\tau_Y := \langle f^{-1}(O) \colon O \in \tau_Y\rangle_{\rm top}$$

on X, called the *pullback of* τ_Y *by* f. The main point of initial and final topologies is to extend these concepts to families of maps.

Definition 2.2.1. Let X be a set and $(Y_i, \tau_i)_{i\in I}$ be topological spaces.

(a) Let $f_i\colon X \to Y_i$ be maps. Then the topology

$$\tau := \langle f_i^{-1}(\tau_i), i \in I\rangle_{top}$$

generated by all inverse images $f_i^{-1}(O_i)$, $O_i \in \tau_i$, is called the *initial topology* defined by the family $(f_i, Y_i)_{i\in I}$.

(b) Let $f_i\colon Y_i \to X$ be maps. Then the topology

$$\tau := \{U \subseteq X \colon (\forall i \in I) f_i^{-1}(U) \in \tau_i\} = \bigcap_{i\in I} f_{i,*}\tau_i$$

is called the *final topology on* X defined by the family $(f_i, Y_i)_{i\in I}$. That τ is indeed a topology is due to the fact that the assignment $U \mapsto f_i^{-1}(U)$ preserves arbitrary intersections and unions (Exercise 2.1.2).

Lemma 2.2.2. *The initial topology* τ *defined by the family* $f_i\colon X \to Y_i$*,* $i \in I$*, of maps is the coarsest topology for which all maps* f_i *are continuous. It has the following universal property: If* Z *is a topological space, then a map* $h\colon Z \to X$ *is continuous if and only if all maps* $f_i \circ h\colon Z \to Y_i$ *are continuous.*

Proof. Apply Lemma 2.1.5 to the subbasis $\{f_i^{-1}(O_i) \colon O_i \subseteq Y_i \text{ open}\}$ of τ. □

Lemma 2.2.3. *The final topology defined by the family* $f_i\colon Y_i \to X$*,* $i \in I$*, is the finest topology for which all maps* f_i *are continuous. It has the following universal property: If* Z *is a topological space, then a map* $h\colon X \to Z$ *is continuous if and only if all maps* $h \circ f_i$*,* $i \in I$*, are continuous.*

Proof. For an open subset $O \subseteq Z$, the inverse image $h^{-1}(O) \subseteq X$ is open if and only if for each $i \in I$, the set $f_i^{-1}(h^{-1}(O)) = (h \circ f_i)^{-1}(O)$ is open in Y_i. Therefore h is continuous if and only if each map $h \circ f_i$ is continuous. □

Example 2.2.4. Let (X, τ) be a topological space and $Y \subseteq X$ be a subset. We write $\iota_Y\colon Y \to X$ for the canonical embedding, mapping each $y \in Y$ to itself. Then the initial topology on Y with respect to ι_Y coincides with the subspace topology

$$\tau_Y = \{\iota_Y^{-1}(U) = U \cap Y \colon U \in \tau\}.$$

Definition 2.2.5. (Quotient topology) (a) Let $\sim$ be an equivalence relation on the topological space X, $[X] := X/\sim = \{[x] : x \in X\}$ be the set of equivalence classes, and $q : X \to [X], x \mapsto [x]$ the quotient map. Then the final topology on $[X]$ defined by $q : X \to [X]$ is called the *quotient topology.*

(b) According to Definition 2.2.1, a subset $U \subseteq [X]$ is open if and only its inverse image $q^{-1}(U)$ is an open subset of X and Lemma 2.2.3 implies that a map $h : [X] \to Z$ to a topological space Z is continuous if and only if $h \circ q\ X \to Z$ is continuous.

(c) An important special cases arises if $S \subseteq X$ is a subset and we define the equivalence relation $\sim$ in such a way that $S = [x]$ for each $x \in S$ and $[y] = \{y\}$ for each $y \in S^c$. Then the quotient space is also denoted $X/S := X/\sim$. It is obtained by collapsing the subset S to a point.

Example 2.2.6. We endow the set $S := (\{1\} \times \mathbb{R}) \cup (\{2\} \times \mathbb{R})$ with the subspace topology of $\mathbb{R}^2$ and define an equivalence relation on S by

$$(1, x) \sim (2, y) \quad \Longleftrightarrow \quad x = y \neq 0,$$

so that all classes except $[1, 0]$ and $[2, 0]$ contain 2 points. The topological quotient space

$$M := S/\sim = \{[1, x] : x \in \mathbb{R}\} \cup \{[2, 0]\} = \{[2, x] : x \in \mathbb{R}\} \cup \{[1, 0]\}$$

is the union of a real line with an extra point, but the two points $[1, 0]$ and $[2, 0]$ have no disjoint open neighborhoods.

The subsets $U_j := \{[j, x] : x \in \mathbb{R}\}$, $j = 1, 2$, of M are open, because their inverse images are the open subsets $X \setminus \{(1, 0)\}$, resp., $X \setminus \{(2, 0)\}$. Moreover, the maps

$$\varphi_j : U_j \to \mathbb{R}, \quad [j, x] \mapsto x,$$

are homeomorphisms. That φ_j is continuous follows from the continuity of the map $X \to \mathbb{R}, (j, x) \mapsto x$ and Definition 2.2.5(b). The continuity of the inverse follows from the continuity of the maps $\mathbb{R} \to X, x \mapsto (j, x)$ and the continuity of the quotient map $q : X \to X/\sim$.

Definition 2.2.7. (Product topology) Let $(X_i)_{i \in I}$ be a family of topological spaces and $X := \prod_{i \in I} X_i$ be their product set. We think of its elements as all tuples $(x_i)_{i \in I}$ with $x_i \in X_i$, or, equivalently, as the set of all maps $x : I \to \bigcup_{i \in I} X_i$ with $x_i := x(i) \in X_i$ for each $i \in I$.

We have for each $i \in I$ a projection map

$$p_i : X \to X_i, \ (x_j)_{j \in I} \mapsto x_i.$$

The initial topology on X with respect to this family $p_i : X \to X_i$ is called the *product topology* and X, endowed with this topology, it called the *topological product space.*

Remark 2.2.8. The sets $p_i^{-1}(O_i)$, $O_i \subseteq X_i$ open, clearly form a subbasis of the product topology, and therefore the sets $\prod_{i \in I} Q_i$, where $Q_i \subseteq X_i$ is open and only finitely many Q_i are different from X_i, form a basis (cf. Exercise 2.1.1).

Example 2.2.9. Typical examples of product spaces are $\mathbb{R}^n$ and $\mathbb{C}^n$. More generally, one can shows that for any finite collection $(X_1, d_1), \ldots, (X_n, d_n)$ of metric spaces, the metrics

$$d_1(x,y) := \sum_{j=1}^n d_j(x_j, y_j) \quad \text{and} \quad d_\infty(x,y) := \max\{d_j(x_j,y_j) \colon j = 1, \ldots, n\}$$

define the product topology on $X := \prod_{j=1}^n X_j$ (Exercise 2.2.4).

From Lemma 2.2.3, we immediately obtain:

Proposition 2.2.10. *A map $f = (f_i) : Y \to \prod_{i \in I} X_i$ to a product space is continuous if and only if all component maps $f_i = p_i \circ f : Y \to X_i$ are continuous.*

Example 2.2.11. (The topology of pointwise convergence) Let X be a set and Y be a topological space. We identify the set $\mathcal{F}(X,Y)$ of all maps $X \to Y$ with the product space $Y^X = \prod_{x \in X} Y$. Then the product topology on Y^X yields a topology on $\mathcal{F}(X,Y)$, called the *topology of pointwise convergence.* We shall see later, when we discuss convergence in topological spaces, why this makes sense.

It is the coarsest topology on $\mathcal{F}(X,Y)$ for which all evaluation maps

$$\mathrm{ev}_x \colon \mathcal{F}(X,Y) \to Y, \quad f \mapsto f(x)$$

are continuous because these maps correspond to the projections $Y^X \to Y$.

Example 2.2.12. (Coproducts) If $(X_i)_{i \in I}$ is a family of topological spaces, then their *coproduct* is defined as the disjoint union

$$\coprod_{i \in I} X_i := \dot{\bigcup}_{i \in I} X_i,$$

endowed with the final topology τ, defined by the inclusion maps $f_i \colon X_i \to X$. Then a subset $O \subseteq X$ is open if and only if $f_i^{-1}(O) = O \cap X_i$ is open for every i and a map $h \colon X \to Z$ is continuous if and only if all restrictions $h|_{X_i} = h \circ f_i \colon X_i \to Z$ are continuous.

Exercises for Section 2.2

Exercise 2.2.1. Let $X_1, \ldots, X_n$ be topological spaces. Show that the sets of the form

$$U_1 \times \ldots \times U_n, \quad U_i \subseteq X_i \ \text{ open},$$

form a basis for the product topology on $X_1 \times \ldots \times X_n$ and for $A_i \subseteq X_i$, $1 \leq i \leq n$, we have

$$\overline{\prod_{i=1}^n A_i} = \prod_{i=1}^n \overline{A_i} \quad \text{and} \quad \Big(\prod_{i=1}^n A_i\Big)^0 = \prod_{i=1}^n A_i^0.$$

Exercise 2.2.2. Let X and Y be topological spaces and $x \in X$. Show that the maps

$$j_x \colon Y \to X \times Y, \quad y \mapsto (x, y)$$

are continuous, and the corestriction

$$j_x^{|Y \times \{x\}} \colon Y \to Y \times \{x\}$$

is a homeomorphism.

Exercise 2.2.3. Let (X, d) be a semimetric space. We define an equivalence relation on X by

$$x \sim y :\Leftrightarrow d(x, y) = 0.$$

Then we obtain on $[X] = X/\sim$ a metric by $d([x], [y]) := d(x, y)$ (Why?). Show that the topology defined on $[X]$ by this metric coincides with the quotient topology obtained from the topology τ_d on X, defined by the semimetric d and the quotient map $q \colon X \to [X]$.

Exercise 2.2.4. Let (X_i, d_i), $i = 1, \ldots, n$, be metric spaces. Show that the metric s

$$d(x, y) := \sum_{i=1}^{n} d_i(x_i, y_i) \quad \text{and} \quad d_\infty(x, y) := \max\{d_i(x_i, y_i) \colon i = 1, \ldots, n\}$$

both induce the product topology on $X := \prod_{i=1}^{n} X_i$.

Exercise 2.2.5. Let $(X_i, d_i)_{i \in \mathbb{N}}$ be a sequence of metric spaces and $X := \prod_{i \in \mathbb{N}} X_i$ their topological product. Show that the product topology coincides with the topology on X induced by the metric

$$d(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$$

(cf. Exercise 1.2.4).

Show further that a sequence $(x^{(n)})_{n \in \mathbb{N}}$ in $X = \prod_{i \in \mathbb{N}} X_i$ converges if and only if all component sequences $(x_i^{(n)})_{n \in \mathbb{N}}$ converge.

Exercise 2.2.6. Let (X, τ) be a topological space, $\sim$ be an equivalence relation on X, $q \colon X \to [X] := X/\sim$ be the quotient map, and endow $[X]$ with the quotient topology. Show that, if $f : X \to Y$ is a continuous map satisfying

$$x \sim y \quad \Rightarrow \quad f(x) = f(y) \qquad \forall x, y \in X,$$

then there exists a unique continuous map $\overline{f} \colon [X] \to Y$ with $f = \overline{f} \circ q$.

Exercise 2.2.7. Show that for each topological space X and $n \in \mathbb{N}$, the diagonal

$$\Delta_X \colon X \to X^n = \prod_{i=1}^{n} X, \quad x \mapsto (x, x, \ldots, x)$$

is continuous.

Exercise 2.2.8. Let X_1, X_2 and Y_1, Y_2 be non-empty topological spaces and $f_i\colon X_i \to Y_i$ be maps. Show that the product map

$$f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$$

is continuous if and only if both maps f_1 and f_2 are continuous.

Exercise 2.2.9. Let (X, d) be a metric space and $\mathcal{C}(X, d) \subseteq X^{\mathbb{N}}$ be the set of Cauchy sequences. Show that:

(a) $d(x, y) := \lim_{n\to\infty} d(x_n, y_n)$ defines a semimetric on $\mathcal{C}(X, d)$, where we write $x = (x_n)_{n\in\mathbb{N}}$ for a sequence in X. Let $\widehat{X} := \mathcal{C}(X, d)/\sim$ denote the corresponding quotient metric space (Exercise 2.2.3).

(b) The map $\eta\colon X \to \widehat{X}$, assigning to $x \in X$ the constant sequence $\eta(x) = (x, x, \ldots)$ is an isometric embedding, i.e., $d(\eta(x), \eta(y)) = d(x, y)$ for $x, y \in X$.

(c) $\eta(X)$ is dense in $\widehat{X}$.

(d) $(\widehat{X}, d)$ is complete. Hint: For a Cauchy sequence $(x_k)_{k\in\mathbb{N}}$ in $\widehat{X}$, pick $y_k \in X$ with $d(\eta(y_k), x_k) < \frac{1}{k}$ and show that $y := (y_k)_{k\in\mathbb{N}}$ is Cauchy.

Exercise 2.2.10. (a) On the set $X := [0, 1]^{[0,1]}$ of all functions $f\colon [0, 1] \to [0, 1]$, we consider the product topology. Show that each neighborhood U of the constant function 1 contains a function f_U which is non-zero at only finitely many places.

(b) Let $Y \subseteq X$ be the subset of all measurable functions, endowed with the subspace topology. Show that the integration map

$$I\colon Y \to \mathbb{R}, \quad f \mapsto \int_0^1 f(x)\,dx$$

is discontinuous. Recall from Example 1.2.11 that this map is sequentially continuous, so that Y is not first countable.

2.3 Topological Groups

Now that the product topology is available, we can define the concept of a topological group:

Definition 2.3.1. A *topological group* is a pair (G, τ) of a group G and a (Hausdorff) topology τ for which the group operations

$$m_G\colon G \times G \to G, \quad (x, y) \mapsto xy \quad \text{and} \quad \eta_G\colon G \to G, \quad x \mapsto x^{-1}$$

are continuous if $G \times G$ carries the product topology.

Remark 2.3.2. For a group G with a topology τ, the continuity of m_G and η_G already follows from the continuity of the single map

$$\varphi\colon G \times G \to G, \quad (g,h) \mapsto gh^{-1}.$$

In fact, if φ is continuous, then the inversion $\eta_G(g) = g^{-1} = \varphi(\mathbf{1}, g)$ is the composition of φ and the continuous map $G \to G \times G, g \mapsto (\mathbf{1}, g)$ (Proposition 2.2.10). The continuity of η_G further implies that the product map

$$\mathrm{id}_G \times \eta_G\colon G \times G \to G \times G, \quad (g,h) \mapsto (g, h^{-1})$$

is continuous (Exercise 2.2.8), and therefore $m_G = \varphi \circ (\mathrm{id}_G \times \eta_G)$ is continuous.

Example 2.3.3. (1) $G = (\mathbb{R}^n, +)$ is an abelian topological group.

(2) $(\mathbb{C}^\times, \cdot)$ is an abelian topological group and the circle group $\mathbb{T} := \{z \in \mathbb{C}^\times : |z| = 1\}$ is a compact subgroup.

(3) The group $\mathrm{GL}_n(\mathbb{R})$ of invertible $(n \times n)$-matrices is a topological group with respect to matrix multiplication. The continuity of the inversion follows from Cramer's Rule, which provides an explicit formula for the inverse in terms of determinants, resp., rational functions.

(4) All subgroups of topological groups are topological groups with respect to the subspace topology.

(5) Every group G is a topological group with respect to the discrete topology.

Exercises for Section 2.3

Exercise 2.3.1. If $(G_i)_{i\in I}$ is a family of topological groups, then the product group $G := \prod_{i\in I} G_i$ is a topological group with respect to the product topology.

Exercise 2.3.2. Show that the *n-dimensional torus*

$$\mathbb{T}^n := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon (\forall j)\, |z_j| = 1\}$$

is a topological group with respect to pointwise multiplication

$$(z_1, \ldots, z_n)(w_1, \ldots, w_n) := (z_1 w_1, \ldots, z_n w_n).$$

Chapter 3

Convergence in Topological Spaces

In Section 1.2 we defined continuity of functions between topological spaces without using the concept of convergence of sequences. For maps between metric spaces, it is often convenient to work with sequential continuity (which is actually equivalent) and which means that $x_n \to x$ implies $f(x_n) \to f(x)$. For topological spaces, simple examples, such as Example 1.2.11, demonstrate that the concept of a convergent sequence is not enough to catch all aspects of convergence and in particular not to test continuity. As we shall see below, this is due to the fact that the neighborhood filter $\mathfrak{U}(x)$ of a point need not have a countable basis.

We shall discuss two concepts of convergence in topological spaces. The most direct one, based on filters, rests on the characterization of continuity in terms of neighborhoods. It has many conceptual advantages because it only refers to subsets of the topological spaces under consideration.

There also is a generalization of the concept of a (convergent) sequence, called a (convergent) net. We shall only deal very briefly with this concept, based on replacing the domain $\mathbb{N}$ for sequences by a directed set $(I, \leq)$. This introduces subtle extra structure which creates many pitfalls and traps because it invites unjustified arguments, in particular, when it comes to subnets.

3.1 Filters

3.1.1 Convergence of Filters and Continuity

Definition 3.1.1. Let X be a set. A set $\mathcal{F} \subseteq \mathbb{P}(X)$ of subsets of X is called a *filter basis* if the following conditions are satisfied:

(FB1) $\mathcal{F} \neq \emptyset$.

(FB2) Each set $F \in \mathcal{F}$ is non-empty.

(FB3) $A, B \in \mathcal{F} \Rightarrow (\exists C \in \mathcal{F})\ C \subseteq A \cap B$.

Definition 3.1.2. (a) Let X be a set. A set $\mathcal{F} \subseteq \mathbb{P}(X)$ of subsets of X is called a *filter* if the following conditions are satisfied:

(F1) $\mathcal{F} \neq \emptyset$.

(F2) Each set $F \in \mathcal{F}$ is non-empty.

(F3) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$.

(F4) $A \in \mathcal{F}, B \supseteq A \Rightarrow B \in \mathcal{F}$.

(b) If $\mathcal{G}$, $\mathcal{F}$ are filters on X, then $\mathcal{G}$ is said to be *finer* than $\mathcal{F}$ if $\mathcal{F} \subseteq \mathcal{G}$. Then $\mathcal{F}$ is called *coarser* than $\mathcal{G}$.

A filter $\mathcal{U}$ is called an *ultrafilter* if there is no finer filter different from $\mathcal{U}$.

(c) If $\mathcal{F}$ is a filter basis, then

$$\widehat{\mathcal{F}} := \{A \subseteq X : (\exists B \in \mathcal{F})\ A \supseteq B\}$$

is a filter. It is called the *filter generated by* $\mathcal{F}$ and $\mathcal{F}$ is called a *basis for the filter* $\widehat{\mathcal{F}}$.

Remark 3.1.3. (F2) and (F3) imply in particular that all finite intersections of elements of a filter are non-empty and belong to $\mathcal{F}$.

Example 3.1.4. Let x be a point in the topological space X.

(a) The set $\mathfrak{U}(x)$ of all neighborhoods of X is a filter. It is called the *neighborhood filter of* x.

(b) The set $\mathfrak{U}_x$ of all subsets of X containing x is an ultrafilter which is finer than $\mathfrak{U}(x)$.

Definition 3.1.5. Let X be a topological space and $x \in X$. We say that a filter $\mathcal{F}$ on X *converges to* x if it is finer than the the neighborhood filter $\mathfrak{U}(x)$, i.e., $\mathcal{F}$ contains each neighborhood of x.

A filterbasis $\mathcal{F}$ is said to *converge to* x if the associated filter $\widehat{\mathcal{F}}$ converges to x, i.e., each neighborhood U of x contains an element of $\mathcal{F}$.

We then write

$$\mathcal{F} \to x \quad \text{or} \quad x \in \lim \mathcal{F}.$$

The notation $x = \lim \mathcal{F}$ means, in addition, that $\mathcal{F}$ converges only to x and not to some other point.

Remark 3.1.6. If X is separated, then each filter $\mathcal{F}$ on X converges at most to one point (Exercise 3.1.3).

Definition 3.1.7. Let $f : X \to Y$ be a map and $\mathcal{F}$ be a filter on X. Then we write

$$f(\mathcal{F}) := \{B \subseteq Y : (\exists A \in \mathcal{F})\ f(A) \subseteq B\}$$

for the filter of all supersets of images of elements of $\mathcal{F}$ (Exercise 3.1.1).

Note that for a filter basis $\mathcal{F}$ the set of all images of $f(A)$, $A \in \mathcal{F}$, is again a filter basis. In this sense $f(\mathcal{F})$ is the filter generated by the filter basis obtained by applying f to the elements of $\mathcal{F}$.

Proposition 3.1.8. *A map $f\colon X \to Y$ between topological spaces is continuous if and only if*

$$f\big(\mathfrak{U}(x)\big) \to f(x) \quad \text{for each} \quad x \in X.$$

Proof. Assume first that f is continuous and let $x \in X$. If $V \in \mathfrak{U}(f(x))$ is a neighborhood of $f(x)$, then the continuity implies the existence of a neighborhood U of x with $f(U) \subseteq V$. This implies that $V \in f(\mathfrak{U}(x))$. Since V was arbitrary, it follows that $f(\mathfrak{U}(x))$ is finer than $\mathfrak{U}(f(x))$, i.e., $f(\mathfrak{U}(x)) \to f(x)$.

Suppose, conversely, that $f(\mathfrak{U}(x)) \to f(x)$ holds for each $x \in X$. Then each neighborhood V of $f(x)$ is contained in $f(\mathfrak{U}(x))$, so that there exists some neighborhood U of x with $f(U) \subseteq V$, and this means that f is continuous in x. Since x was arbitrary, f is continuous (Proposition 1.2.7). □

3.1.2 Ultrafilters

We now turn to arguments leading to the existence of ultrafilters. The main point of ultrafilters in this notes is that they are a natural means to prove Tychonov's Theorem.

Definition 3.1.9. A relation $\leq$ on a set M is called a *partial ordered set* if:

(P1) $(\forall a \in M) \quad a \leq a \quad$ (Reflexivity).

(P2) $(\forall a, b, c \in M) \quad a \leq b, \quad b \leq c \quad \Rightarrow \quad a \leq c \quad$ (Transitivity).

(P3) $(\forall a, b \in M) \quad a \leq b, \quad b \leq a \quad \Rightarrow \quad a = a. \quad$ (Antisymmetry)

A pair $(M, \leq)$ of a set with a partial order is called a *partially order set.*

A subset K of a partially ordered set $(M, \leq)$ is called a *chain* if either $a \leq b$ or $b \leq a$ holds for $a, b \in K$. This means that all pairs of elements of K are comparable w.r.t. $\leq$.

An element $m \in M$ is called an *upper bound of the subset S* if $s \leq m$ holds for all $s \in S$. An element $m \in M$ is said to be *maximal* if $m \leq x$, $x \in M$, implies $x = m$.

Lemma 3.1.10. (Zorn's Lemma) *If each chain K in the partially orderer set $(M, \leq)$ possesses an upper bound, then there exists for each $a \in M$ a maximal element $b \in M$ with $a \leq b$.*

Proof. Since it is equivalent to the Axiom of Choice, which asserts the seemingly obvious fact that for a family $(X_i)_{i\in I}$ of non-empty sets the product set $\prod_{i\in I} X_i$ is non-empty, we may consider Zorn's Lemma as a set theoretic axiom. □

Proposition 3.1.11. *Each Filter $\mathcal{F}$ on X is contained in some ultrafilter.*

Proof. We order the set $\mathfrak{F}$ of all filters $\mathcal{F}$ on X by set inclusion as subsets of $\mathbb{P}(X)$. We claim that each chain $\mathfrak{K} \subseteq \mathfrak{F}$ has an upper bound. To verify this claim, we show that

$$\mathcal{M} := \bigcup \mathfrak{K} := \{A \subseteq X : (\exists \mathcal{F} \in \mathfrak{K})\ A \in \mathcal{F}\}$$

is an upper bound of $\mathfrak{K}$. To show that $\mathcal{M}$ is a filter, we note that the only non-trivial requirement is (F3). For F_1 and $F_2 \in \mathcal{M}$ there exist $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{K}$ with $F_i \in \mathcal{F}_i$. Since $\mathfrak{K}$ is a chain, we may w.l.o.g. assume that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Then we have $F_i \in \mathcal{F}_2$ for $i = 1, 2$ and thus $F_1 \cap F_2 \in \mathcal{F}_2 \subseteq \mathcal{M}$.

Now Zorn's Lemma applies to the ordered set of filters on X and this implies the proposition. □

Lemma 3.1.12. *Let $\mathcal{F}$ be a filter on X and $A \subseteq X$ with $A^c \notin \mathcal{F}$. Then there exists a filter $\mathcal{G}$ on X containing A which is finer than $\mathcal{F}$.*

Proof. We define $\mathcal{G}$ as the set of all supersets of the intersections of elements of $\mathcal{F}$ with A:

$$\mathcal{G} = \{B \subseteq X : (\exists F \in \mathcal{F})\ F \cap A \subseteq B\}.$$

To see that $\mathcal{G}$ is a filter, we show that

$$\mathcal{F} \cap A := \{F \cap A : F \in \mathcal{F}\}$$

is a filter basis and that $\mathcal{G}$ is the filter generated by $\mathcal{F} \cap A$. Clearly, $\mathcal{F} \cap A$ is non-empty. If $F \cap A = \emptyset$, then $F \subseteq A^c$ leads to $A^c \in \mathcal{F}$, contradicting our hypothesis. This proves (FB1) and (FB2). Finally, it is clear that $\mathcal{F} \cap A$ is stable under intersections, which implies (FB3). Therefore $\mathfrak{F} \cap A$ is a filter basis and hence $\mathcal{G} = (\mathcal{F} \cap A)\widehat{}$ is a filter. This filter contains A, and since each $F \in \mathcal{F}$ is a superset of $F \cap A$, it also contains $\mathcal{F}$. □

Proposition 3.1.13. *A filter $\mathcal{F}$ on X is an ultrafilter if and only if for each $A \subseteq X$ we either have $A \in \mathcal{F}$ of $A^c \in \mathcal{F}$.*

Proof. First assume that $\mathcal{F}$ is an ultrafilter. If $A^c \notin \mathcal{F}$, then Lemma 3.1.12 implies the existence of a finer filter $\mathcal{G} \supseteq \mathcal{F}$ containing A. Since $\mathcal{F}$ is an ultrafilter, $\mathcal{G} = \mathcal{F}$ implies that $A \in \mathcal{F}$.

If, conversely, $\mathcal{F}$ contains for each subset $A \subseteq X$ either A itself or A^c, then each finer filter $\mathcal{G} \supseteq \mathcal{F}$ which is properly larger contains for some subset $A \subseteq X$ both A and A^c, but then we obtain the contradiction $\emptyset = A \cap A^c \in \mathcal{G}$. Therefore $\mathcal{F}$ is maximal, i.e., an ultrafilter. □

Proposition 3.1.14. *If $\mathcal{F}$ is an ultrafilter on X and $f : X \to Y$ a map, then $f(\mathcal{F})$ is an ultrafilter on Y.*

Proof. In view of Proposition 3.1.13, we have to show that for each subset $B \subseteq Y$ we either have $B \in f(\mathcal{F})$ or $B^c \in f(\mathcal{F})$. Since $\mathcal{F}$ is an ultrafilter, it either contains $f^{-1}(B)$ or $f^{-1}(B)^c = f^{-1}(B^c)$. If $f^{-1}(B) \in \mathcal{F}$, then $B \supseteq f(f^{-1}(B))$ implies that $B \in f(\mathcal{F})$, and likewise we argue for the other case. □

Exercises for Section 3.1

Exercise 3.1.1. Show that for a map $f : X \to Y$ and a filter $\mathcal{F}$ on X, the subset

$$f(\mathcal{F}) := \{B \subseteq Y : (\exists A \in \mathcal{F})\ f(A) \subseteq B\}$$

of $\mathbb{P}(Y)$ is a filter.

Exercise 3.1.2. Consider the two element set $X = \{x, y\}$, endowed with the indiscrete topology. Show that $\mathcal{F} = \{\{x\}, \{x, y\}\}$ is a filter on X converging to x and y. This shows that limits of filters need not be unique.

Exercise 3.1.3. Show that if X is separated, then each filter $\mathcal{F}$ on X converges at most to one point.

Exercise 3.1.4. Let X be a finite set. Show that for each ultrafilter $\mathcal{U}$ on X there exists a point $x \in X$ with $\mathcal{U} = \{A \subseteq X : x \in A\}$.

Exercise 3.1.5. Let X be a topological space and $M \subseteq X$ a subset. Show that $x \in \overline{M}$ if and only if there exists a filter basis $\mathcal{F}$ in M with $\mathcal{F} \to x$.

3.2 Nets

Definition 3.2.1. (a) A partially ordered set $(I, \leq)$ is called *directed* if for $a, b \in I$ there exists an element $c \in I$ with $a, b \leq c$.

(b) A map $x\colon (I, \leq) \to X$ of a directed set $(I, \leq)$ to some set X is called a *net in* X. A net is mostly denoted $(x_i)_{i \in I}$, where the order on I is not explicitly mentioned.

(c) If X is a topological space, $(x_i)_{i \in I}$ a net in X and $p \in X$, then we say that (x_i) *converges to* p if for each neighborhood U of p there exists an index $i_U \in I$ with $x_j \in U$ for $j \geq i_U$. We then write

$$x_i \to p \quad \text{or} \quad \lim_I x_i = p.$$

Example 3.2.2. The ordered set $(\mathbb{N}, \leq)$ of natural numbers is directed, so that every sequence $(x_n)_{n \in \mathbb{N}}$ is in particular a net. Specializing the concept of convergence of nets, we find that for a sequence (x_n) in a topological space X, the relation

$$\lim_{n \to \infty} x_n = p$$

is equivalent to: For each neighborhood U of p there exists an $n_U \in \mathbb{N}$ with $x_m \in U$ for $m \geq n_U$.

For the case of metric spaces, this is easily seen to be equivalent with the usual definition of convergence of sequences in metric spaces, when it is applied to the corresponding topology τ_d.

Proposition 3.2.3. *A map $f\colon X \to Y$ between topological spaces is continuous in $p \in X$ if and only if for each net $(x_i)_{i \in I}$ in X with $x_i \to p$ we have $f(x_i) \to f(p)$.*

Proof. (a) Suppose first that f is continuous in p and that $x_i \to p$ in X. Let V be a neighborhood of $f(p)$ in Y. Then there exists a neighborhood U of p in X with $f(U) \subseteq V$. Pick $i_U \in I$ with $x_i \in U$ for $i \geq i_U$. Then $f(x_i) \in V$ for $i \geq i_U$ implies that $f(x_i) \to f(p)$.

(b) Now assume that for each net $x_i \to p$ we have $f(x_i) \to f(p)$. To show that f is continuous in p, we argue by contradiction. If f is not continuous in p, there exists a neighborhood V of $f(p)$ for which $f^{-1}(V)$ is not a neighborhood of p. We therefore find for each $U \in \mathfrak{U}(p)$ an element $x_U \in U$ with $f(x_U) \notin V$. For the directed set $(I, \leq) := (\mathfrak{U}(p), \supseteq)$ we then obtain a net $(x_U)_{U \in \mathfrak{U}(p)}$ converging to p (by construction), for which $f(x_U)$ does not converge to $f(p)$. This contradicts our assumption, hence proves that f is continuous in p. □

Proposition 3.2.4. *A topological space (X, τ) is hausdorff if and only if any two limit points of a convergent net are equal.*

Proof. Suppose first that X is hausdorff, $p, q \in X$, and that $(x_i)_{i \in I}$ is a net in X with $x_i \to p, q$. Let O_p and O_q be disjoint open subsets containing p, resp., q. If $x_i \in O_p$ for $i > i_p$ and $x_i \in Q_q$ for $i > i_q$, we arrive at a contradiction for any i with $i > i_p, i_q$.

Next we assume that X is not hausdorff and that $p, q \in X$ are points with the property that if (A, B) is a pair of open sets of X with $p \in A$ and $q \in B$, then $A \cap B \neq \emptyset$. Let

$$I := \{(A, B) \in \tau \times \tau \colon p \in A, q \in B\},$$

ordered by

$$(A, B) \leq (C, D) \quad \Longleftrightarrow \quad A \supseteq C \quad \text{and} \quad B \supseteq D.$$

Then $(I, \leq)$ is a directed set. For each $(A, B) \in I$ we now pick $x_{(A,B)} \in A \cap B$ and obtain a net in X. We claim that this net converges to p and q.

In fact, let U be an open neighborhood of p and pick any open neighborhood V of q. Then $(U, V) \in I$ and $(A, B) \geq (U, V)$ implies $x_{(A,B)} \in A \subseteq U$. Therefore $x_{(A,B)} \to p$. By symmetry, we also have $x_{(A,B)} \to q$. □

Exercises for Section 3.2

Exercise 3.2.1. Let $(x_i)_{i \in I}$ be a net in the topological space X.

For each $i \in I$, let $F_i := \{x_j \colon j \geq i\}$. Show that the F_i form a filter basis $\mathcal{F}$ on X for which $x_i \to p$ is equivalent to $\mathcal{F} \to p$.

Exercise 3.2.2. Let $(d_i)_{i \in I}$ be a family of semimetrics on the set X and $\tau := \bigcap_{i \in I} \tau_{d_i}$ be the topology defined by this family. Show that:

(a) The diagonal mapping $\eta \colon X \to \prod_{i \in I}(X, \tau_{d_i}), x \mapsto (x)_{i \in I}$ is a homeomorphism onto its image.

(b) A net $(x_j)_{j \in J}$ converges in (X, τ) to some $p \in X$ if and only if $d_i(x_j, p) \to 0$ holds for each $i \in I$.

(c) (X, τ) is Hausdorff if and only if for $x \neq y$ there exists an i with $d_i(x, y) \neq 0$.

Chapter 4

Compactness

As we know from the basic Analysis course, compactness is a key property for existence theorems to hold. A typical example is the Maximal Value Theorem, asserting that a real-valued continuous function on a compact metric space has a maximal value. In this chapter we shall see that the metric structure is irrelevant for these conclusions. This provides in particular the freedom to form arbitrary products of compact spaces. The central result is Tychonov's Theorem, that a product space is compact if and only if all factors are.

4.1 Compact Spaces

Definition 4.1.1. A topological space X is said to be *quasicompact* if each open covering of X has a finite subcovering, i.e., if $(U_i)_{i\in I}$ is a family of open sets of X with $\bigcup_{i\in I} U_i = X$, then there exists a finite subset $F \subseteq I$ with $\bigcup_{i\in F} U_i = X$.

A topological space X is said to be *compact* if it is quasicompact and separated.

Lemma 4.1.2. *A subset C of a topological space X is quasicompact with respect to the subspace topology if and only if every covering of C by open subsets of X has a finite subcovering, i.e., if the family $(U_i)_{i\in I}$ of open subsets of X satisfies $\bigcup_{i\in I} U_i \supseteq K$, then there exists a finite subset $F \subseteq I$ with $K \subseteq \bigcup_{i\in F} U_i$.*

Lemma 4.1.3. (a) *If X is separated and $C \subseteq X$ (quasi)compact, then C is closed.*

(b) *If X is compact and $C \subseteq X$ is closed, then C is compact.*

Proof. (a) Let $x \in C^c$. For each $c \in C$ we then have $c \neq x$, and since X is separated, there exists an open subset U_c of X and an open subset V_c of X with $c \in U_c$, $x \in V_c$ and $U_c \cap V_c = \emptyset$. Then we obtain an open covering $(U_c \cap C)_{c\in C}$ of C. Let $U_{c_1} \cap C, \ldots, U_{c_n} \cap C$ be a finite subcovering and $V := \bigcap_{i=1}^n V_{c_i}$. Then V intersects $\bigcup_{i=1}^n U_{c_i} \supseteq C$ trivially, and therefore $x \notin \overline{C}$. This proves that C is closed.

(b) Let $(U_i)_{i\in I}$ be an open covering of C and pick open subsets $O_i \subseteq X$ with $O_i \cap C = U_i$. Then the open subset C^c, together with the O_i, $i \in I$, form an open covering in X. Hence there exists a finite subcovering, and this implies the existence of a finite subset $F \subseteq I$ with $C \subseteq \bigcup_{i\in F} U_i$. □

Proposition 4.1.4. *If X is quasicompact and $f\colon X \to Y$ is continuous, then $f(X)$ is quasicompact.*

Proof. Let $(U_i)_{i\in I}$ be a covering of $f(X)$ by open subsets of Y. Then $(f^{-1}(U_i))_{i\in I}$ is an open covering of X, so that there exists a finite subset $F \subseteq I$ with $X \subseteq \bigcup_{i\in F} f^{-1}(U_i)$. Then

$$f(X) \subseteq \bigcup_{i\in F} f(f^{-1}(U_i)) \subseteq \bigcup_{i\in F} U_i.$$

In view of Lemma 4.1.2, this implies that $f(X)$ is quasicompact. □

Corollary 4.1.5. (Maximal Value Theorem) *If X is compact and $f\colon X \to \mathbb{R}$ a continuous function, then f is bounded and has a maximal value.*

Proof. Since $\mathbb{R}$ is separated, Proposition 4.1.4, combined with Lemma 4.1.3(a) implies that $f(X) \subseteq \mathbb{R}$ is a compact subset, hence bounded and closed. In particular, it has a maximal element. □

Lemma 4.1.6. *If $f : X \to Y$ is injective and continuous and Y is separated, then X is separated.*

Proof. Let $x \neq y$ be two points in X. Then $f(x) \neq f(y)$ implies the existence of two disjoint open subsets $U_x \subseteq Y$ and $U_y \subseteq Y$ with $f(x) \in U_x$ and $f(y) \in U_y$. Then the two sets $f^{-1}(U_x)$ and $f^{-1}(U_y)$ are open and disjoint with $x \in f^{-1}(U_x)$ and $y \in f^{-1}(U_y)$. Therefore X is separated. □

Proposition 4.1.7. *If $f : X \to Y$ is a bijective continuous map and X is quasicompact and Y is separated, then f is a homeomorphism.*

Proof. From Lemma 4.1.6 we derive that X is separated, hence compact. Let $A \subseteq X$ be a closed subset. Then A is compact by Lemma 4.1.3. Therefore $f(A) \subseteq Y$ is quasicompact, and since Y is separated, it is compact, hence closed by Lemma 4.1.3. Since f is continuous, $A \subseteq X$ is closed if and only if $f(A) \subseteq Y$ is closed, so that Proposition 1.2.8 implies that f is a homeomorphism. □

Example 4.1.8. Let A be a compact space and $\sim$ be an equivalence relation on A. Then the quotient space $X := A/\sim$ is quasicompact by Proposition 4.1.4. If $f\colon A/\sim \to Y$ is a continuous bijective map and Y is separated, then Proposition 4.1.7 applies and shows that $f\colon A/\sim \to Y$ is a homeomorphism.

Typical examples, where these arguments apply are:

(a) $A = [0,1]$, $Y = \mathbb{S}^1 \subseteq \mathbb{C}$ and $f([x]) = e^{2\pi i x}$. This means that we obtain the circle by identifying the two endpoints 0 and 1 of the unit interval.

(b) $A = [0,1]^2$, $Y = \mathbb{T}^2 \subseteq \mathbb{C}^2$ and $f([x,y]) = (e^{2\pi i x}, e^{2\pi i y})$. This means that we obtain the 2-torus $\mathbb{T}^2$ by identifying certain boundary points in the unit square A. We thus obtain $\mathbb{T}^2$ by a glueing construction from A. It is instructive to visualize this construction with paper.

Exercises for Section 4.1

Exercise 4.1.1. If (X, d) is a metric space and $C \subseteq X$ is a compact subset, then C is bounded and closed. Here boundedness of a subset $S \subseteq X$ means that

$$\operatorname{diam}(S) := \sup\{d(x, y) \colon x, y \in S\} < \infty.$$

Exercise 4.1.2. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the topological space X. Show that, if $\lim_{n\to\infty} x_n = x$, then the set $\{x_n \colon n \in \mathbb{N}\} \cup \{x\}$ is compact.

Exercise 4.1.3. (The cofinite topology) Let X be a set and

$$\tau := \{\emptyset\} \cup \{A \subseteq X \colon |A^c| < \infty\}$$

be the cofinite topology introduced in Exercise 1.1.12. Show that (X, τ) is quasicompact.

Exercise 4.1.4. Let X and Y be Hausdorff spaces. For a compact subset $K \subseteq X$ and an open subset $O \subseteq Y$, we write

$$W(K, O) := \{f \in C(X, Y) \colon f(K) \subseteq O\}.$$

The topology on $C(X, Y)$ generated by these sets is called the *compact open topology*. Show that:

(a) If $f \colon Z \to X$ is a continuous map, then

$$f^* \colon C(X, Y) \to C(Z, X), \quad g \mapsto g \circ f$$

is continuous with respect to the compact open topology on $C(X, Y)$, resp., $C(Z, X)$.

(b) If $f \colon Y \to Z$ is a continuous map, then

$$f_* \colon C(X, Y) \to C(X, Z), \quad g \mapsto f \circ g$$

is continuous with respect to the compact open topology on $C(X, Y)$, resp., $C(X, Z)$.

Exercise 4.1.5. On the compact space $X := [a, b]$, $-\infty < a < b < \infty$, we consider the equivalence relation defined by $x \sim y$ if either $x = y$ or $x = a$ and $y = b$. Show that

(a) The quotient space $X/\sim$ is Hausdorff and compact. Hint: Proposition 4.1.7.

(b) $X/\sim$ is homeomorphic to the circle $\mathbb{S}^1$. Hint: Consider the function $f \colon X/\sim \to \mathbb{S}^1$, $f([t]) := e^{2\pi i(t-a)/(b-a)}$.

Exercise 4.1.6. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{K}$-valued continuous functions on the topological space X ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and $f: X \to \mathbb{K}$ a function with

$$\|f_n - f\|_\infty \to 0,$$

i.e., the sequence f_n converges uniformly to f. Show that f is continuous. Conclude in particular that for a compact space X, the space $(C(X, \mathbb{K}), \|\cdot\|)$ is a Banach space.

4.2 Tychonov's Theorem

Before we turn to Tychonov's Theorem, we need the following characterization of compactness in terms of convergence of ultrafilters.

Proposition 4.2.1. *For a topological space X, the following are equivalent:*

(i) *X is quasicompact.*

(ii) *For each family $(A_i)_{i\in I}$ of closed subsets of X with $\bigcap_{i\in I} A_i = \emptyset$, there exists a finite subset $F \subseteq I$ with $\bigcap_{i\in F} A_i = \emptyset$.*

(iii) *Every ultrafilter on X converges.*

Proof. (i) $\Leftrightarrow$ (ii) follows by taking complements: The condition $\bigcap_{i\in I} A_i = \emptyset$ means that the family $(A_i^c)_{i\in I}$ of complements is an open covering of X because $X = \emptyset^c = \bigcup_{i\in I} A_i^c$. Similarly, $\bigcap_{i\in F} A_i = \emptyset$ means that $(A_i^c)_{i\in F}$ is a finite subcovering.

(ii) $\Rightarrow$ (iii): Let $\mathcal{F}$ be an ultrafilter on X. If $A_1, \ldots, A_n \in \mathcal{F}$ are closed subsets, then $A := \bigcap_{i=1}^n A_i \in \mathcal{F}$ (Remark 3.1.3), hence in particular $A \neq \emptyset$. Therefore (ii) implies that the intersection of all closed subsets in $\mathcal{F}$ is not empty. Let x be an element in this intersection and U be an open neighborhood of x. Then $X\backslash U$ is a closed subset of X, hence not contained in $\mathcal{F}$ because it does not contain x. Since $\mathcal{F}$ is an ultrafilter, we obtain $U \in \mathcal{F}$ (Proposition 3.1.14), and since U was arbitrary, this shows that $\mathcal{F} \to x$.

(iii) $\Rightarrow$ (ii): Let $(A_i)_{i\in I}$ be a family of closed subsets of X and assume that all finite intersections of the sets A_i are non-empty. Let

$$\mathcal{B} := \Big\{ \bigcap_{i\in F} A_i : F \subseteq I, |F| < \infty \Big\}$$

be the filter basis of finite intersections of the A_i. Clearly, $\mathcal{B}$ is intersection stable and satisfies (FB1) and (FB2), hence is a filter basis. Let $\mathcal{F} := \widehat{\mathcal{B}}$ be the filter generated by $\mathcal{B}$ and $\mathcal{U}$ be an ultrafilter containing $\mathcal{F}$. Then there exists an element $x \in X$ with $\mathcal{U} \to x$. Let $i \in I$ and U be a neighborhood of x. Then $A_i, U \in \mathcal{U}$ implies that $A_i \cap U \in \mathcal{U}$, so that $A_i \cap U \neq \emptyset$ by (F2). Therefore A_i intersects each neighborhood of x, which leads to $x \in \overline{A_i} = A_i$, so that $x \in \bigcap_{i\in I} A_i$. □

Lemma 4.2.2. *A topological product space $X = \prod_{i\in I} X_i$ of non-empty spaces is separated if and only if each X_i is separated.*

Proof. Exercise 4.2.1. □

Theorem 4.2.3. (Tychonov's Theorem) *If $(X_i)_{i\in I}$ is a family of non-empty topological spaces, then the topological product space $X = \prod_{i\in I} X_i$ is (quasi)compact if and only if each factor X_i is (quasi)compact.*

Proof. Let $p_i \colon X \to X_i$ denote the projection maps. If X is quasicompact, then the continuity of the projection maps p_i implies that $X_i = p_i(X)$ is quasicompact (Proposition 4.1.4). If, in addition, X is compact, then X is separated, so that Lemma 4.2.2 implies that each X_i is separated, and we thus obtain that each X_i is compact.

Suppose, conversely, that each space X_i is quasicompact and let $\mathcal{F}$ be an ultrafilter in X. Then each $p_i(\mathcal{F})$ is an ultrafilter in X_i (Proposition 3.1.14), hence convergent to some element x_i (Proposition 4.2.1). We claim that $\mathcal{F}$ converges to $x := (x_i)_{i\in I}$. In fact, if U is an open neighborhood of x, then there exists a finite subset $F \subseteq I$ and open neighborhoods U_i of x_i in X_i with

$$\prod_{i\in F} U_i \times \prod_{i\in F^c} X_i \subseteq U$$

(Remark 2.2.8). Pick $A_i \in \mathcal{F}$ with $p_i(A_i) \subseteq U_i$. Then $A := \bigcap_{i\in F} A_i \in \mathcal{F}$ satisfies $p_i(A) \subseteq U_i$ for each $i \in F$, and this implies that $A \subseteq U$. Now $U \in \mathcal{F}$ is a consequence of (F4), and we conclude that $\mathcal{F} \to x$. We have thus shown that every ultrafilter on X converges, and this implies that X is quasicompact (Proposition 4.2.1).

If, in addition, each X_i is compact, then Lemma 4.2.2 implies that the quasicompact space X is separated, hence compact. □

Exercises for Section 4.2

Exercise 4.2.1. Show that a topological product space $X = \prod_{i\in I} X_i$ of non-empty spaces X_i is separated if and only if each X_i is separated.

Exercise 4.2.2. (The Cantor Set as a product space) We consider the compact product space $\{0,1\}^{\mathbb{N}}$, where $\{0,1\}$ carries the discrete topology. The image C of the function

$$f \colon \{0,1\}^{\mathbb{N}} \to \mathbb{R}, \quad f(x) := 2\sum_{n=1}^{\infty} \frac{x_n}{3^n}$$

is called the *Cantor set*. Show that:

(a) f is continuous and injective.

(b) $f \colon \{0,1\}^{\mathbb{N}} \to C$ is a homeomorphism and C is compact.

(c) $C = \bigcap_{n\in\mathbb{N}} C_n$, where

$$C_1 = [0,1] \setminus \left]\frac{1}{3}, \frac{2}{3}\right[= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right],$$

each C_n is a union of 2^n closed intervals of length $\frac{1}{3^n}$, and C_{n+1} arises from C_n by deleting in each interval of C_n the open middle third.

Exercise 4.2.3. Let X and Y be topological spaces, and $A \subseteq X$ and $B \subseteq Y$ be compact subsets. Show that for each open subset $O \subseteq X \times Y$ containing $A \times B$ there exist open subsets $U \subseteq X$ and $V \subseteq Y$ with

$$A \times B \subseteq U \times V \subseteq O.$$

4.3 Compact Metric Spaces

In this section we discuss compactness for metric spaces. In particular, we shall see various characterizations that shed some new light on the concept.

Definition 4.3.1. Let (X, d) be a metric space and $S \subseteq X$ be a subset. For $r \geq 0$ we put

$$B_r(S) := \bigcup_{s\in S} B_r(s) = \{x \in X\colon (\exists s \in S)\ d(x,s) < r\}.$$

The metric space (X, d) is said to be *precompact* if for each $\varepsilon > 0$ there exists a finite subset $F \subseteq X$ with $X = B_\varepsilon(F)$. Then F is called an *ε-net* in X.

Lemma 4.3.2. *For a metric space (X, d), the following are equivalent:*

(1) *Every sequence $(x_n)_{n\in\mathbb{N}}$ in X possesses a Cauchy subsequence.*

(2) *X is precompact.*

Proof. (1) $\Rightarrow$ (2): We argue indirectly. Assume that there exists an $\varepsilon > 0$ such that for no finite subsets $F \subseteq X$, the relation $X = B_\varepsilon(F)$ holds. We now construct inductively a sequence having no Cauchy subsequence. Pick an arbitrary element $x_1 \in X$. Since $X \neq B_\varepsilon(x_1)$, there exists an element $x_2 \in B_\varepsilon(x_1)^c$. Now $X \neq B_\varepsilon(\{x_1, x_2\})$, so that we find

$$x_3 \in X \setminus B_\varepsilon(\{x_1, x_2\}).$$

Proceeding inductively, we thus obtain a sequence (x_n) with

$$x_{n+1} \in X \setminus B_\varepsilon(\{x_1, \ldots, x_n\})$$

for each $n \in \mathbb{N}$. Then $d(x_n, x_m) \geq \varepsilon$ holds for $n \neq m$, so that the sequence (x_n) contains no Cauchy subsequence.

(2) ⇒ (1): Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. For each $n \in \mathbb{N}$ the precompactness implies the existence of a finite subset $E_n \subseteq X$ with $X = B_{2^{-n}}(E_n)$. We choose inductively $y_k \in E_k$, such that
$\mathbb{N}_1 := \{m \in \mathbb{N} \colon x_m \in B_{2^{-1}}(y_1)\}$ is infinite,
$\mathbb{N}_2 := \{m \in \mathbb{N}_1 \colon x_m \in B_{2^{-2}}(y_2)\}$ is infinite, etc., so that for each $k \in \mathbb{N}$, the set $\mathbb{N}_{k+1} := \{m \in \mathbb{N}_k \colon x_m \in B_{2^{-k-1}}(y_{k+1})\}$ is infinite.

Now we select a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $n_1 \in \mathbb{N}_1$, $n_2 \in \mathbb{N}_2$ with $n_2 > n_1$, etc., such that $n_{k+1} > n_k$ and $n_{k+1} \in \mathbb{N}_{k+1}$. Then

$$d(x_{n_k}, x_{n_{k+1}}) \leq d(x_{n_k}, y_k) + d(y_k, x_{n_{k+1}}) < 2^{-k} + 2^{-k} = 2^{-k+1}$$

and thus

$$d(x_{n_k}, x_{n_{k+m}}) \leq 2^{-k+1} + 2^{-k} + 2^{-k-1} + \ldots \leq 2^{-k+2} \quad \text{for} \quad m \geq 1$$

because of the triangle inequality. Therefore $(x_{n_k})_{k\in\mathbb{N}}$ is a Cauchy subsequence. □

Proposition 4.3.3. *For a metric space (X, d), the following are equivalent:*

(1) *X is compact.*

(2) *X is sequentially compact, i.e., each sequence in X possesses a convergent subsequence.*

(3) *X is complete and precompact.*

Proof. (1) ⇒ (2): Let (x_n) be a sequence in X and $A_n := \overline{\{x_m \colon m \geq n\}}$. Then each finite intersection of elements of the sequence $(A_n)_{n\in\mathbb{N}}$ is a non-empty because $A_n \supseteq A_{n+1}$, and all sets are non-empty. Therefore the characterization of compactness in Proposition 4.2.1 implies the existence of an $x \in \bigcap_{n\in\mathbb{N}} A_n$. Hence we find inductively for each $k \in \mathbb{N}$ a natural number $n(k) > n(k-1)$ with $d(x_{n(k)}, x) < \frac{1}{k}$: In view of $x \in A_1$, there exists $n(1) \in \mathbb{N}$ with $d(x_{n(1)}, x) < 1$; then $x \in A_2$ implies the existence of $n(2) > n(1)$ with $d(x_{n(2)}, x) < \frac{1}{2}$ etc. This leads to $\lim_{k\to\infty} x_{n(k)} = x$, so that the sequence (x_n) possesses a convergent subsequence.

(2) ⇒ (3): (2) implies that every Cauchy sequence in X has a convergent subsequence, hence is convergent (Exercise). Therefore X is complete and the precompactness of X follows from Lemma 4.3.2.

(3) ⇒ (1): This implication is also proved indirectly. Assume that there exists an open covering $(U_i)_{i\in I}$ of X without a finite subcovering. We put $\varepsilon_n := 2^{-n}$. Since X is precompact, there exists a finite subset $E_n \subseteq X$ with $X = B_{\varepsilon_n}(E_n)$.

Since $(U_i)_{i\in I}$ has no finite subcover, there exists an element $x_1 \in E_1$ for which the ball $B_{\varepsilon_1}(x_1)$ is not covered by finitely many U_i. Otherwise each ball $B_{\varepsilon_1}(x)$, $x \in E_1$, is covered by finitely many U_i, and then the collection of these U_i form a finite subcover of X. Since

$$B_{\varepsilon_1}(x_1) = \bigcup_{x\in E_2} B_{\varepsilon_1}(x_1) \cap B_{\varepsilon_2}(x)$$

also is a finite union, the same argument as above implies the existence of an $x_2 \in E_2$, so that the set $B_{\varepsilon_1}(x_1) \cap B_{\varepsilon_2}(x_2)$ is not covered by finitely many U_i. Inductively, we thus obtain a sequence (x_n) with $x_n \in E_n$, for which no set of the form $\bigcap_{j=1}^n B_{\varepsilon_j}(x_j)$ can be covered by finitely many U_i. In particular, these sets are non-empty, and the triangle equality leads to

$$d(x_n, x_{n+1}) < \varepsilon_n + \varepsilon_{n+1} \leq 2\varepsilon_n = 2^{-n+1}.$$

The triangle inequality further yields

$$d(x_n, x_{n+k}) \leq 2^{-n+1}(1 + 2^{-1} + \cdots) \leq 2^{-n+2}.$$

Therefore (x_n) is a Cauchy sequence, hence, by assumption, convergent to some $x \in X$.

Since $(U_i)_{i\in I}$ is an open covering of X, there exists some $i_0 \in I$ with $x \in U_{i_0}$. As U_{i_0} is open, we have $B_\varepsilon(x) \subseteq U_{i_0}$ for some $\varepsilon > 0$. For $2^{-n+3} < \varepsilon$ we obtain with

$$d(x_n, x) = \lim_{k\to\infty} d(x_n, x_{n+k}) \leq 2^{-n+2} < \frac{\varepsilon}{2} \quad \text{and} \quad \varepsilon_n + \frac{\varepsilon}{2} < 2\frac{\varepsilon}{2} = \varepsilon$$

the relation

$$B_{\varepsilon_n}(x_n) \subseteq B_\varepsilon(x) \subseteq U_{i_0}.$$

This contradicts the fact that $\bigcap_{j=1}^n B_{\varepsilon_j}(x_j) \subseteq B_{\varepsilon_n}(x_n)$ cannot be covered by finitely many U_i. □

Corollary 4.3.4. *For a subset X of a complete metric space Y, the following are equivalent:*

(1) *X is relatively compact, i.e., $\overline{X}$ is compact.*

(2) *X is precompact.*

Proof. (1) ⇒ (2): Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Since $\overline{X}$ is compact, Proposition 4.3.3 implies the existence of a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, converging in $\overline{X}$. This subsequence is in particular a Cauchy sequence. Therefore the precompactness of X follows by Lemma 4.3.2.

(2) ⇒ (1): According to Proposition 4.3.3, it suffices to verify the sequential compactness of $\overline{X}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\overline{X}$. Then there exists for each $n \in \mathbb{N}$ a $y_n \in X$ with $d(x_n, y_n) < \frac{1}{n}$. Since X is precompact, there exists a Cauchy subsequence $(y_{n_k})_{k\in\mathbb{N}}$, and this sequence converges in the complete space Y to some element y. Now $y \in \overline{X}$, and $d(x_{n_k}, y_{n_k}) \leq n_k^{-1}$ implies that $\lim_{k\to\infty} x_{n_k} = y$. We have thus found a convergent subsequence of $(x_n)_{n\in\mathbb{N}}$. □

Exercises for Section 4.3

Exercise 4.3.1. Let (X, d) be a compact metric space. Show that:

(1) X is separable, i.e., X contains a countable dense subset.

(2) If Y is a metric space and $f\colon X \to Y$ is continuous, then f is *uniformly continuous*, i.e., for each $\varepsilon > 0$ there exists a $\delta > 0$ with $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ for each $x \in X$.

4.4 Locally Compact Spaces

Definition 4.4.1. A separated topological space X is called *locally compact* if each point $x \in X$ has a compact neighborhood.

Lemma 4.4.2. *If X is locally compact and $x \in X$, then each neighborhood U of x contains a compact neighborhood of x.*

Proof. Let K be a compact neighborhood of $x \in X$. Since it suffices to show that $U \cap K$ contains a compact neighborhood of x, we may w.l.o.g. assume that X is compact. Replacing U by its interior, we may further assume that U is open, so that its complement U^c is compact.

We argue by contradiction and assume that U does not contain any compact neighborhood of x. Then the family $\mathcal{F}$ of all intersections $C \cap U^c$, where C is a compact neighborhood of x, contains only non-empty sets and is stable under finite intersections. We thus obtain a family of closed subsets of the compact space U^c for which all finite intersections are non-empty, and therefore Proposition 4.2.1 implies that its intersection $\bigcap_C (C \cap U^c)$ contains a point y. Then $y \in U^c$ implies $x \neq y$, and since X is separated, there exist open neighborhoods U_x of x and U_y of y with $U_x \cap U_y = \emptyset$. Then U_y^c is a compact neighborhood of x, which leads to the contradiction $y \in U_y^c \cap U^c$ to $y \in U_y$. □

Definition 4.4.3. A subset A of a topological space X is said to be *relatively compact* if $\overline{A}$ is compact.

Lemma 4.4.4. *Let X be locally compact, $K \subseteq X$ compact and $U \supseteq K$ open. Then there exists a compact subset $V \subseteq X$ with*

$$K \subseteq V^0 \subseteq V \subseteq U.$$

Proof. For each $x \in K$ we choose a compact neighborhood $V_x \subseteq U$ (Lemma 4.4.2). Then there exist finitely many $x_1, \ldots, x_n$ with $K \subseteq \bigcup_{i=1}^n V_{x_i}^0$ and we put $V := \bigcup_{i=1}^n V_{x_i} \subseteq U$. □

Proposition 4.4.5. (Urysohn's Theorem) *Let X be locally compact, $K \subseteq X$ compact and $U \supseteq K$ be an open subset. Then there exists a continuous function $h \colon X \to \mathbb{R}$ with*

$$h|_K = 1 \quad \textit{and} \quad h|_{X \setminus U} = 0.$$

Proof. We put $U(1) := U$. With Lemma 4.4.4, we find an open, relatively compact subset $U(0)$ with $K \subseteq U(0) \subseteq \overline{U(0)} \subseteq U(1)$. Iterating this procedure leads to a subset $U(\frac{1}{2})$ with

$$\overline{U(0)} \subseteq U\Big(\frac{1}{2}\Big) \subseteq \overline{U\Big(\frac{1}{2}\Big)} \subseteq U(1).$$

Continuing like this, we find for each dyadic number $\frac{k}{2^n} \in [0,1]$ an open, relatively compact subset $U(\frac{k}{2^n})$ with

$$\overline{U\Big(\frac{k}{2^n}\Big)} \subseteq U\Big(\frac{k+1}{2^n}\Big) \quad \text{for} \quad k = 0, \ldots, 2^n - 1.$$

Let $\mathbb{D} := \{\frac{k}{2^n} : k = 0, \ldots, 2^n, n \in \mathbb{N}\}$ for the set of dyadic numbers in $[0,1]$. For $r \in [0,1]$, we put

$$U(r) := \bigcup_{s \leq r, s \in \mathbb{D}} U(s).$$

For $r = \frac{k}{2^n}$ this is consistent with the previous definition. For $t < t'$ we now find $r = \frac{k}{2^n} < r' = \frac{k+1}{2^n}$ in $\mathbb{D}$ with $t < r < r' < t'$, so that we obtain

$$\overline{U(t)} \subseteq \overline{U(r)} \subseteq U(r') \subseteq U(t').$$

We also put $U(t) = \emptyset$ for $t < 0$ and $U(t) = X$ for $t > 1$. Finally, we define

$$f(x) := \inf\{t \in \mathbb{R} : x \in U(t)\}.$$

Then $f(K) \subseteq \{0\}$ and $f(X \setminus U) \subseteq \{1\}$.

We claim that f is continuous. So let $x_0 \in X$, $f(x_0) = t_0$ and $\varepsilon > 0$. We put $V := U(t_0 + \varepsilon) \setminus \overline{U(t_0 - \varepsilon)}$ and note that this is a neighborhood of x_0. From $x \in V \subseteq \underline{U(t_0 + \varepsilon)}$ we derive $f(x) \leq t_0 + \varepsilon$. If $f(x) < t_0 - \varepsilon$, then also $x \in U(t_0 - \varepsilon) \subseteq \overline{U(t_0 - \varepsilon)}$, which is a contradiction. Therefore $|f(x) - f(x_0)| \leq \varepsilon$ holds on V, and this implies that f is continuous. Finally, we put $h := 1 - f$. □

Exercises for Section 4.4

Exercise 4.4.1. Let X, Y and Z be locally compact spaces and endow $C(X,Y)$ and $C(Y,Z)$ with the compact open topology (Exercise 4.4). Show that the composition map

$$C(Y,Z) \times C(X,Y) \to C(X,Z), \quad (f,g) \mapsto f \circ g$$

is continuous. Hint: If $f \circ g \in W(K,O)$, $K \subseteq X$ compact and $O \subseteq Z$ open, then there exists a compact neighborhood C of K in Y with $f(C) \subseteq O$ (Lemma 4.4.4).

Exercise 4.4.2. (One point compactification) Let X be a locally compact space. Show that:

(i) There exists a compact topology on the set $X_\omega := X \cup \{\omega\}$, where ω is a symbol of a point not contained in X. Hint: A subset $O \subseteq X_\omega$ is open if it either is an open subset of X or $\omega \in O$ and $X \setminus O$ is compact.

(ii) The inclusion map $\eta_X \colon X \to X_\omega$ is a homeomorphism onto an open subset of X_ω.

(iii) If Y is a compact space and $f \colon X \to Y$ a continuous map which is a homeomorphism onto the complement of a point in Y, then there exists a homeomorphism $F \colon X_\omega \to Y$ with $F \circ \eta_X = f$.

The space X_ω is called the *Alexandroff compactification* or the *one point compactification* of X.[1]

[1] Alexandroff, Pavel (1896–1982)

Exercise 4.4.3. Show that the one-point compactification of $\mathbb{R}^n$ is diffeomorphic to the n-dimensional sphere $\mathbb{S}^n$. Hint: Exercise 1.2.5.

Exercise 4.4.4. Show that the one-point compactification of an open interval $]a, b[\subseteq \mathbb{R}$ is homeomorphic to $\mathbb{S}^1$.

Exercise 4.4.5. Let X be a locally compact space and $Y \subseteq X$ be a subset. Show that Y is locally compact with respect to the subspace topology if and only if there exists an open subset $O \subseteq X$ and a closed subset A with $Y = O \cap A$. Hint: If Y is locally compact, write it as a union of compact subsets of the form $O_i \cap Y$, O_i open in X, where $\overline{O_i \cap Y}$ has compact closure, contained in Y. Then put $O := \bigcup_{i\in I} O_i$ and $A := \overline{Y \cap O}$.

Exercise 4.4.6. Show that a locally compact space is regular, i.e., a T_3-space. Hint:Urysohn's Theorem.

Exercise 4.4.7. Let X be a compact space and $A \subseteq X$ be a compact subset. The space X/A is defined as the topological quotient space $X/\sim$, defined by the equivalence relation $x \sim y$ if either $x = y$ or $x, y \in A$. This means that we are collapsing A to a point. Show that:

(i) X/A is compact. Hint: The main point is to see that X/A is hausdorff (Lemma 4.4.4, Proposition 4.1.7).

(ii) X/A is homeomorphic to the one-point compactification of the locally compact space $X \setminus A$.

Exercise 4.4.8. Let $(V, \|\cdot\|)$ be a normed space and $B := \{v \in \mathbb{R}^n : \|v\| \leq 1\}$ be the closed unit ball. Show that:

(i) The map

$$f : V \to B^0, \quad v \mapsto \frac{v}{1 + \|v\|}$$

is a homeomorphism whose inverse is given by $g(w) := \frac{w}{1-\|w\|}$.

(ii) If $\dim V = n < \infty$, then the quotient space $B/\partial B$ is homeomorphic to the n-dimensional sphere $\mathbb{S}^n$. Hint: Exercise 4.4.7 and 4.4.3.

Chapter 5

Applications to Function Spaces

In this chapter we prove two important theorems on spaces of continuous functions on compact spaces: the abstract version of the Weierstraß Approximation Theorem about dense subspaces in $C(X, \mathbb{R})$ and Ascoli's Theorem which provides a characterization of (relative) compactness of subsets of the Banach space $C(X, \mathbb{R})$.

5.1 The Stone–Weierstraß Theorem

Definition 5.1.1. (a) Let M be a set and $\mathcal{A} \subseteq \mathbb{K}^M$ be a set of functions $M \to \mathbb{K}$. We say that $\mathcal{A}$ *separates the points of* M if for two points $x \neq y$ in X there exists some $f \in \mathcal{A}$ with $f(x) \neq f(y)$.

(b) A linear subspace $\mathcal{A} \subseteq \mathbb{K}^M$ is called an *algebra* if it is closed under pointwise multiplication.

Theorem 5.1.2. (Dini's Theorem) [1] *Let X be a compact space and $(f_n)_{n\in\mathbb{N}}$ be a monotone sequence of functions in $C(X, \mathbb{R})$. If $(f_n)_{n\in\mathbb{N}}$ converges pointwise to some $f \in C(X, \mathbb{R})$, then the convergence is uniform, i.e., $\|f_n - f\|_\infty \to 0$.*

Proof. Idea: First we find for each $x \in X$ and each $\varepsilon > 0$ a neighborhood U_x and an $n_x \in \mathbb{N}$ with $|f(x) - f_n(y)| < \varepsilon$ for $y \in U_x$ and $n \geq n_x$. Then X is covered by finitely many such U_x and the monotony is used.

Here are the details: Replacing f_n by $f - f_n$ or $f_n - f$, we may w.l.o.g. assume that $f = 0$ and $f_n \geq f_{n+1} \geq 0$ for $n \in \mathbb{N}$. For $\varepsilon > 0$ and $x \in X$ we now find an $n_x \in \mathbb{N}$ with

$$(\forall n \geq n_x) \quad 0 \leq f_m(x) \leq \frac{\varepsilon}{3}.$$

[1] Dini, Ulisse (1845–1918)

The continuity of f and f_{n_x} yields a neighborhood U_x of x with

$$(\forall y \in U_x) \quad |f_{n_x}(x) - f_{n_x}(y)| \leq \frac{\varepsilon}{3}.$$

We thus obtain

$$(\forall y \in U_x) \quad 0 \leq f_{n_x}(y) \leq \varepsilon.$$

Now we choose $x_1, \ldots, x_k \in X$ such that the U_{x_j} cover X and put $n_0 := \max\{n_{x_1}, \ldots, n_{x_k}\}$. Then, by monotony of the sequence,

$$0 \leq f_{n_0}(x) \leq f_{n_{x_j}}(x) \leq \varepsilon \quad \text{for} \quad x \in U_{x_j}$$

and thus

$$(\forall n \geq n_0)(\forall x \in X) \quad 0 \leq f_n(x) \leq f_{n_0}(x) \leq \varepsilon.$$

This completes the proof. □

Lemma 5.1.3. *There exists an increasing sequence of real polynomials p_n which converges in $[0,1]$ uniformly to the square root function $x \mapsto \sqrt{x}$.*

Proof. Idea: We start with $p_1 := 0$ and construct p_n inductively by the rule

$$p_{n+1}(x) := p_n(x) + \frac{1}{2}(x - p_n(x)^2). \tag{5.1}$$

Then we show that this sequence is monotone and bounded. The iteration procedure produces an equation for the limit which turns out to be $\sqrt{x}$. Then we apply Dini's Theorem.

Details: We prove by induction that that

$$(\forall n \in \mathbb{N})(\forall x \in [0,1]) \quad 0 \leq p_n(x) \leq \sqrt{x} \leq 1.$$

In fact,

$$\begin{aligned} \sqrt{x} - p_{n+1}(x) &= \sqrt{x} - p_n(x) - \frac{1}{2}\big(x - p_n(x)^2\big) \\ &= (\sqrt{x} - p_n(x))\Big(1 - \frac{1}{2}\big(\sqrt{x} + p_n(x)\big)\Big) \end{aligned}$$

and $p_n(x) \leq \sqrt{x}$ yields

$$(\forall x \in [0,1]) \qquad 0 \leq \frac{1}{2}(\sqrt{x} + p_n(x)) \leq \sqrt{x} \leq 1.$$

Therefore the definition of p_{n+1} yields $p_n \leq p_{n+1}$ on $[0,1]$, so that our claim follows by induction. Therefore the sequence $(p_n)_{n\in\mathbb{N}}$ is increasing on $[0,1]$ and bounded, hence converges pointwise to some function $f\colon [0,1] \to [0,1]$. Passing in (5.1) to the limit on both sides, we obtain the relation $f(x)^2 = x$, which proves that $f(x) = \sqrt{x}$. Now Dini's Theorem 5.1.2 implies that the convergence $p_n \to f$ is uniform. □

Theorem 5.1.4. (Stone-Weierstraß) [2] [3] *Let X be a compact space and $\mathcal{A} \subseteq C(X, \mathbb{R})$ be a point separating subalgebra containing the constant functions. Then $\mathcal{A}$ is dense in $C(X, \mathbb{R})$ w.r.t. $\|\cdot\|_\infty$.*

Proof. Let $\mathcal{B} := \overline{\mathcal{A}}$ denote the closure of $\mathcal{A}$ in the Banach space

$$(C(X, \mathbb{R}), \|\cdot\|_\infty).$$

Then $\mathcal{B}$ also contains the constant functions, separates the points and is a subalgebra (Exercise 5.1.1). We have to show that $\mathcal{B} = C(X, \mathbb{R})$.

Here is the idea of the proof. First we use Lemma 5.1.3 to see that for $f, g \in \mathcal{B}$, also $|f|$, $\min(f, g)$ and $\max(f, g)$ are contained in $\mathcal{B}$. Then we use the point separation property to approximate general continuous functions locally by elements of $\mathcal{B}$. Now the compactness of X permits to complete the proof.

Here are the details: Let $(p_n)_{n\in\mathbb{N}}$ be the sequence of polynomials from Lemma 5.1.3. For $f \in \mathcal{B}$, we consider the functions $p_n\big(\frac{f^2}{\|f\|_\infty^2}\big)$, which also belong to $\mathcal{B}$. In view of Lemma 5.1.3, they converge uniformly to $\sqrt{\frac{f^2}{\|f\|_\infty^2}} = \frac{|f|}{\|f\|_\infty}$, so that $|f| \in \mathcal{B}$.

Now let $f, g \in \mathcal{B}$. Then

$$\min(f, g) = \frac{1}{2}(f + g - |f - g|) \quad \text{and} \quad \max(f, g) = \frac{1}{2}(f + g + |f - g|),$$

so that the preceding argument implies that $\min(f, g), \max(f, g) \in \mathcal{B}$.

Next let $x \neq y$ in X and $r, s \in \mathbb{R}$. According to our assumption, there exists a function $g \in \mathcal{B}$ with $g(x) \neq g(y)$. For

$$h := r + (s - r)\frac{g - g(x)}{g(y) - g(x)} \in \mathcal{B}$$

we then have $h(x) = r$ and $h(y) = s$.

Claim: For $f \in C(X, \mathbb{R})$, $x \in X$ and $\varepsilon > 0$, there exists a function $g_x \in \mathcal{B}$ with

$$f(x) = g_x(x) \quad \text{and} \quad (\forall y \in X) \quad g_x(y) \leq f(y) + \varepsilon.$$

To verify this claim, pick for each $z \in X$ a function $h_z \in \mathcal{B}$ with $h_z(x) = f(x)$ and $h_z(z) \leq f(z) + \frac{\varepsilon}{2}$. Then there exists a neighborhood U_z of z with

$$(\forall y \in U_z) \quad h_z(y) \leq f(y) + \varepsilon.$$

Since X is compact, it is covered by finitely many $U_{z_1}, \ldots, U_{z_k}$ of these neighborhoods. Then $g_x := \min\{h_{z_1}, \ldots, h_{z_k}\}$ is the desired function.

Now we complete the proof by showing that $\mathcal{B} = C(X, \mathbb{R})$. So let $f \in C(X, \mathbb{R})$ and $\varepsilon > 0$. For each $x \in X$, pick $g_x \in \mathcal{B}$ with

$$(\forall y \in X) \quad f(x) = g_x(x) \quad \text{and} \quad g_x(y) \leq f(y) + \varepsilon.$$

[2] Stone, Marshall (1903–1989)
[3] Weierstraß, Karl (1815–1897)

Then the continuity of f and g_x yield neighborhoods U_x of x with

$$\forall y \in U_x \quad : \quad g_x(y) \geq f(y) - \varepsilon.$$

Now the compactness of X implies the existence of finitely many points $x_1, \ldots, x_k$ such that $X \subseteq U_{x_1} \cup \cdots \cup U_{x_k}$. We now put $\varphi_\varepsilon := \max\{g_{x_1}, \ldots, g_{x_k}\} \in \mathcal{B}$. Then

$$\forall y \in X \quad : \quad f(y) - \varepsilon \leq \varphi_\varepsilon(y) \leq f(y) + \varepsilon.$$

This implies that $\|f - \varphi_\varepsilon\|_\infty \leq \varepsilon$ and since ε was arbitrary, $f \in \mathcal{B}$. □

Corollary 5.1.5. *Let X be a compact space and $\mathcal{A} \subseteq C(X, \mathbb{C})$ be a point separating subalgebra containing the constant functions which is invariant under complex conjugation, i.e., $f \in \mathcal{A}$ implies $\overline{f} \in \mathcal{A}$. Then $\mathcal{A}$ is dense in $C(X, \mathbb{C})$ w.r.t. $\| \cdot \|_\infty$.*

Proof. Let $\mathcal{A}_\mathbb{R} := \mathcal{A} \cap C(X, \mathbb{R})$. Since $\mathcal{A}$ is conjugation invariant, we have $\mathcal{A} = \mathcal{A}_\mathbb{R} \oplus i\mathcal{A}_\mathbb{R}$. This implies that $\mathcal{A}_\mathbb{R}$ contains the real constants and separates the points of X. Now Theorem 5.1.4 implies that $\mathcal{A}_\mathbb{R}$ is dense in $C(X, \mathbb{R})$, and therefore $\mathcal{A}$ is dense in $C(X, \mathbb{C}) = C(X, \mathbb{R}) + iC(X, \mathbb{R})$. □

Exercises for Section 5.1

Exercise 5.1.1. If X is a compact topological space and $\mathcal{A} \subseteq C(X, \mathbb{R})$ is a subalgebra, then its closure also is a subalgebra. Hint: If $f_n \to f$ and $g_n \to g$ uniformly, then also $f_n + g_n \to f + g, \lambda f_n \to \lambda f$ and $f_n g_n \to fg$ uniformly.

Exercise 5.1.2. Let $[a, b] \subseteq \mathbb{R}$ be a compact interval. Show that the space

$$\mathcal{A} := \Big\{ f|_{[a,b]} \colon (\exists a_0, \ldots, a_n \in \mathbb{R}, n \in \mathbb{N})\ f(x) = \sum_{i=0}^n a_i x^i \Big\}$$

of polynomial functions on $[a, b]$ is dense in $C([a, b], \mathbb{R})$ with respect to $\| \cdot \|_\infty$.

Exercise 5.1.3. Let $K \subseteq \mathbb{R}^n$ be a compact subset. Show that the space $\mathcal{A}$ consisting of all restrictions of polynomial functions

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{R}, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

to K is dense in $C(X, \mathbb{R})$ with respect to $\| \cdot \|_\infty$.

Exercise 5.1.4. Let $\mathbb{S}^1 = \{z \in \mathbb{C} \colon |z| = 1\}$ and

$$\mathcal{A} := \Big\{ f|_{\mathbb{S}^1} \colon (\exists a_0, \ldots, a_n \in \mathbb{C}, n \in \mathbb{N})\ f(z) = \sum_{j=0}^n a_n z^n \Big\}.$$

Show that $\mathcal{A}$ is not dense in $C(\mathbb{S}^1, \mathbb{C})$. Hint: Consider the function $f(z) := z^{-1}$ on $\mathbb{S}^1$ and try to approximate it by elements f_n of $\mathcal{A}$; then consider the complex path integrals $\int_{|z|=1} f_n(z)\, dz$. Why does the Stone–Weierstraß Theorem not apply?

Exercise 5.1.5. For a locally compact space X, we consider the Banach space $C_0(X)$ of all continuous functions $f\colon X \to \mathbb{C}$ vanishing at infinity, i.e., with the property that for each $\varepsilon > 0$ there exists a compact subset $C_\varepsilon \subseteq X$ with $|f(x)| \leq \varepsilon$ for $x \notin C_\varepsilon$. Suppose that $\mathcal{A} \subseteq C_0(X)$ is a complex subalgebra satisfying

(a) $\mathcal{A}$ is invariant under conjugation.

(b) $\mathcal{A}$ has no zeros, i.e., for each $x \in X$ there exists an $f \in \mathcal{A}$ with $f(x) \neq 0$.

(c) $\mathcal{A}$ separates the points of X.

Show that $\mathcal{A}$ is dense in $C_0(X)$ with respect to $\|\cdot\|_\infty$. Hint: Let X_ω be the one-point compactification of X (Exercise 4.4.3). Then each function $f \in C_0(X)$ extends to a continuous function $\widetilde{f}$ on X_ω by $\widetilde{f}(\omega) := 0$, and this leads to bijection

$$C_*(X_\omega) := \{f \in C(X_\omega)\colon f(\omega) = 0\} \to C_0(X), \quad f \mapsto f|_X.$$

Use the Stone-Weierstraß Theorem to show that the algebra

$$\widetilde{\mathcal{A}} := \mathbb{C}\mathbf{1} + \{\widetilde{a}\colon a \in \mathcal{A}\}$$

is dense in $C(X_\omega)$ and show that if $\widetilde{f}_n + \lambda\mathbf{1} \to \widetilde{f}$ for $\lambda_n \in \mathbb{C}, f \in C_0(X), f_n \in \mathcal{A}$, then $\lambda_n \to 0$ and $f_n \to f$.

5.2 Ascoli's Theorem

Throughout this section $\mathbb{K}$ may be either $\mathbb{R}$ or $\mathbb{C}$.

Theorem 5.2.1. (Ascoli's Theorem) *Let X be a compact space, $C(X, \mathbb{K})$ be the Banach space of all continuous functions $f\colon X \to \mathbb{K}$, endowed with the sup-norm*

$$\|f\| := \sup\{|f(x)|\colon x \in X\}$$

and $M \subseteq C(X, \mathbb{K})$ a subset. Then M is relatively compact if and only if

(a) *M is pointwise bounded, i.e., $\sup\{|f(x)|\colon f \in M\} < \infty$ for each $x \in X$.*

(b) *M is equicontinuous, i.e., for each $\varepsilon > 0$ and each $x \in X$ there exists a neighborhood U_x with*

$$|f(x) - f(y)| \leq \varepsilon \quad \text{for} \quad f \in M, y \in U_x.$$

Proof. First we observe that, in view of the completeness of the Banach space $C(X, \mathbb{K})$ and Corollary 4.3.4, M is precompact if and only if it is relatively compact.

Step 1: First we assume that M is relatively compact. For each $x \in X$ the evaluation map

$$\mathrm{ev}_x\colon C(X, \mathbb{K}) \to \mathbb{K}, \quad f \mapsto f(x)$$

is continuous, hence maps the relatively compact subset M into a relatively compact subset of $\mathbb{K}$, and this implies that $\mathrm{ev}_x(M)$ is bounded.

Since M is precompact, there exists, for any $\varepsilon > 0$, finitely many elements $f_1, \ldots, f_n \in M$ with

$$M \subseteq \bigcup_{j=1}^{n} B_{\varepsilon/3}(f_j).$$

For $x \in X$ we now find a neighborhood U_x with

$$|f_j(x) - f_j(y)| \leq \frac{\varepsilon}{3} \quad \text{for} \quad y \in U_x, j = 1, \ldots, n.$$

For $f \in M$ we now pick j with $\|f - f_j\|_\infty < \varepsilon/3$ and obtain for $y \in U_x$:

$$|f(x)-f(y)| \leq |f(x)-f_j(x)|+|f_j(x)-f_j(y)|+|f_j(y)-f(y)| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.$$

Therefore M is equicontinuous.

Step 2: Now we assume that (a) and (b) are satisfied. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in M. To see that M is relatively compact, we have to prove the existence of a subsequence converging uniformly to some element of $C(X, \mathbb{K})$ (Lemma 4.3.2).

Fix $k \in \mathbb{N}$. Then there exists for each $x \in X$ an open neighborhood V_x^k with

$$|f(x) - f(y)| < \frac{1}{k} \quad \text{for} \quad f \in M, y \in V_x^k.$$

Then $(V_x^k)_{x\in X}$ is an open covering of X, and the compactness of X implies the existence of a finite subcover. This leads to points $x_1^k, \ldots, x_{m_k}^k$ in X and neighborhoods $V_j^k := V_{x_j}^k$, $j = 1, \ldots, m_k$, of these points, such that $X \subseteq \bigcup_{i=1}^{m_k} V_i^k$ and

$$|f(x) - f(x_i^k)| < \frac{1}{k} \qquad \text{for} \quad f \in M, x \in V_i^k, i = 1, \ldots, m_k.$$

We order the countable set $\{x_i^k : k \in \mathbb{N}, i = 1, \ldots, m_k\}$ as follows to a sequence $(y_m)_{m\in\mathbb{N}}$:

$$x_1^1, \ldots, x_{m_1}^1, x_1^2, \ldots, x_{m_2}^2, \ldots$$

For each y_m, the set $\{f_n(y_m) : n \in \mathbb{N}\} \subseteq \mathbb{K}$ is bounded, hence contains a subsequence (f_n^1), converging in y_1. This sequence has a subsequence (f_n^2), converging in y_2, etc. The sequence $(f_n^n)_{n\in\mathbb{N}}$ is a subsequence of the original sequence, converging on the set $\{y_m : m \in \mathbb{N}\} = \{x_j^k : k \in \mathbb{N}, j = 1, \ldots, m_k\}$. To simplify notation, we may now assume that the sequence f_n converges pointwise on this set.

Next we show that the sequence (f_n) converges pointwise. Pick $x \in X$. In view of the completeness of $\mathbb{K}$, it suffices to show that the sequence $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy. So let $\varepsilon > 0$. Then there exists a $k \in \mathbb{N}$ with $\frac{3}{k} < \varepsilon$ and some x_j^k with $x \in V_j^k$, so that

$$|f_n(x) - f_n(x_j^k)| < \frac{1}{k} \quad \text{for} \quad n \in \mathbb{N}.$$

Since the sequence $(f_n(x_j^k))_{n\in\mathbb{N}}$ converges, there exists an $n_0 \in \mathbb{N}$, such that

$$|f_n(x_j^k) - f_{n'}(x_j^k)| < \frac{1}{k} \quad \text{for} \quad n, n' > n_0.$$

Then

$$\begin{aligned} |f_n(x) - f_{n'}(x)| &\le |f_n(x) - f_n(x_j^k)| + |f_n(x_j^k) - f_{n'}(x_j^k)| + |f_{n'}(x_j^k) - f_{n'}(x)| \\ &\le \frac{3}{k} \le \varepsilon. \end{aligned}$$

Let $F(x) := \lim_{n\to\infty} f_n(x)$. It remains to show that f_n converges uniformly to F. Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ with $\frac{3}{k} < \varepsilon$. We pick $n_0 \in \mathbb{N}$ so large that

$$|f_n(x_i^k) - F(x_i^k)| \le \frac{1}{k} \qquad \text{for} \quad n \ge n_0, i = 1, \ldots, m_k.$$

Since each element $x \in X$ is contained in one of the sets V_i^k,

$$|f_n(x) - F(x)| \le |f_n(x) - f_n(x_i^k)| + |f_n(x_i^k) - F(x_i^k)| + |F(x_i^k) - F(x)| \le \frac{3}{k} \le \varepsilon,$$

because $|F(x_i^k) - F(x)| = \lim_{n\to\infty} |f_n(x_i^k) - f_n(x)| \le \frac{1}{k}$. This proves that f_n converges uniformly to F, and the proof is complete. □

Chapter 6

Covering Theory

In this appendix we provide the main results on coverings of topological spaces needed in particular to calculate fundamental groups and to prove the existence of simply connected covering spaces.

6.1 The Fundamental Group

To define the notion of a simply connected space, we first have to define its fundamental group. The elements of this group are homotopy classes of loops. The present section develops this concept and provides some of its basic properties.

Definition 6.1.1. Let X be a topological space, $I := [0,1]$, and $x_0, x_1 \in X$. We write

$$P(X, x_0) := \{\gamma \in C(I, X) \colon \gamma(0) = x_0\}$$

and

$$P(X, x_0, x_1) := \{\gamma \in P(X, x_0) \colon \gamma(1) = x_1\}.$$

We call two paths $\alpha_0, \alpha_1 \in P(X, x_0, x_1)$ *homotopic*, written $\alpha_0 \sim \alpha_1$, if there exists a continuous map

$$H \colon I \times I \to X \quad \text{with} \quad H_0 = \alpha_0, \quad H_1 = \alpha_1$$

(for $H_t(s) := H(t, s)$) and

$$(\forall t \in I) \quad H(t, 0) = x_0, \ H(t, 1) = x_1.$$

It is easy to show that $\sim$ is an equivalence relation (Exercise 6.1.2), called *homotopy*. The homotopy class of α is denoted by $[\alpha]$.

We write $\Omega(X, x_0) := P(X, x_0, x_0)$, for the set of loops based at x_0. For $\alpha \in P(X, x_0, x_1)$ and $\beta \in P(X, x_1, x_2)$ we define a product $\alpha * \beta$ in $P(X, x_0, x_2)$ by

$$(\alpha * \beta)(t) := \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Lemma 6.1.2. *If* $\varphi\colon [0,1] \to [0,1]$ *is a continuous map with* $\varphi(0) = 0$ *and* $\varphi(1) = 1$*, then for each* $\alpha \in P(X, x_0, x_1)$ *we have* $\alpha \sim \alpha \circ \varphi$*.*

Proof. Use $H(t,s) := \alpha(ts + (1-t)\varphi(s))$. □

Proposition 6.1.3. *The following assertions hold:*

(1) $\alpha_1 \sim \alpha_2$ *and* $\beta_1 \sim \beta_2$ *implies* $\alpha_1 * \beta_1 \sim \alpha_2 * \beta_2$*, so that we obtain a well-defined product*

$$[\alpha] * [\beta] := [\alpha * \beta]$$

of homotopy classes.

(2) *If* x *also denotes the constant map* $I \to \{x\} \subseteq X$*, then*

$$[x_0] * [\alpha] = [\alpha] = [\alpha] * [x_1] \quad \textit{for} \quad \alpha \in P(X, x_0, x_1).$$

(3) *(Associativity)* $[\alpha * \beta] * [\gamma] = [\alpha] * [\beta * \gamma]$ *for* $\alpha \in P(X, x_0, x_1)$, $\beta \in P(X, x_1, x_2)$ *and* $\gamma \in P(X, x_2, x_3)$*.*

(4) *(Inverse) For* $\alpha \in P(X, x_0, x_1)$ *and* $\overline{\alpha}(t) := \alpha(1-t)$ *we have*

$$[\alpha] * [\overline{\alpha}] = [x_0].$$

(5) *(Functoriality) For any continuous map* $\varphi\colon X \to Y$ *and* $\alpha \in P(X, x_0, x_1), \beta \in P(X, x_1, x_2)$*, we have*

$$(\varphi \circ \alpha) * (\varphi \circ \beta) = \varphi \circ (\alpha * \beta),$$

and $\alpha \sim \beta$ *implies* $\varphi \circ \alpha \sim \varphi \circ \beta$*.*

Proof. (1) If H^α is a homotopy from α_1 to α_2 and H^β a homotopy from β_1 to β_2, then we put

$$H(t,s) := \begin{cases} H^\alpha(t, 2s) & \text{for } 0 \le s \le \frac{1}{2} \\ H^\beta(t, 2s-1) & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

(cf. Exercise 6.1.1).

(2) For the first assertion we use Lemma 6.1.2 and

$$x_0 * \alpha = \alpha \circ \varphi \quad \text{for} \quad \varphi(t) := \begin{cases} 0 & \text{for } 0 \le t \le \frac{1}{2} \\ 2t-1 & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

For the second, we have

$$\alpha * x_1 = \alpha \circ \varphi \quad \text{for} \quad \varphi(t) := \begin{cases} 2t & \text{for } 0 \le t \le \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

(3) We have $(\alpha * \beta) * \gamma = (\alpha * (\beta * \gamma)) \circ \varphi$ for

$$\varphi(t) := \begin{cases} 2t & \text{for } 0 \le t \le \frac{1}{4} \\ \frac{1}{4} + t & \text{for } \frac{1}{4} \le t \le \frac{1}{2} \\ \frac{t+1}{2} & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

(4)

$$H(t,s) := \begin{cases} \alpha(2s) & \text{for } s \leq \frac{1-t}{2} \\ \alpha(1-t) & \text{for } \frac{1-t}{2} \leq s \leq \frac{1+t}{2} \\ \overline{\alpha}(2s-1) & \text{for } s \geq \frac{1+t}{2}. \end{cases}$$

(5) is trivial. □

Definition 6.1.4. From the preceding definition, we derive in particular that the set

$$\pi_1(X, x_0) := \Omega(X, x_0)/\sim$$

of homotopy classes of loops in x_0 carries a natural group structure, given by

$$[\alpha][\beta] := [\alpha * \beta]$$

(Exercise). This group is called the *fundamental group of X with respect to x_0*.

A pathwise connected space X is called *simply connected* if $\pi_1(X, x_0)$ vanishes for some $x_0 \in X$ (which implies that is trivial for each $x_0 \in X$; Exercise 6.1.4).

Lemma 6.1.5. (Functoriality of the fundamental group) *If $f\colon X \to Y$ is a continuous map with $f(x_0) = y_0$, then*

$$\pi_1(f, x_0)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [f \circ \gamma]$$

is a group homomorphism. Moreover, we have

$$\pi_1(\mathrm{id}_X, x_0) = \mathrm{id}_{\pi_1(X,x_0)} \quad \text{and} \quad \pi_1(f \circ g, x_0) = \pi_1(f, g(x_0)) \circ \pi_1(g, g(x_0)).$$

Proof. This follows directly from Proposition 6.1.3(5). □

Remark 6.1.6. The map

$$\sigma\colon \pi_1(X, x_0) \times \big(P(X, x_0)/\sim\big) \to P(X, x_0)/\sim, \quad ([\alpha], [\beta]) \mapsto [\alpha * \beta] = [\alpha] * [\beta]$$

defines an action of the group $\pi_1(X, x_0)$ on the set $P(X, x_0)/\sim$ of homotopy classes of paths starting in x_0 (Proposition 6.1.3).

Remark 6.1.7. (a) Suppose that the topological space X is contractible, i.e., there exists a continuous map $H\colon I \times X \to X$ and $x_0 \in X$ with $H(0, x) = x$ and $H(1, x) = x_0$ for $x \in X$. Then $\pi_1(X, x_0) = \{[x_0]\}$ is trivial (Exercise).
(b) $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ (Exercise).
(c) $\pi_1(\mathbb{R}^n, 0) = \{0\}$ because $\mathbb{R}^n$ is contractible.

More generally, if the open subset $\Omega \subseteq \mathbb{R}^n$ is starlike with respect to x_0, then $H(t, x) := x + t(x - x_0)$ yields a contraction to x_0, and we conclude that $\pi_1(\Omega, x_0) = \{\mathbf{1}\}$.

The following lemma implies in particular, that fundamental groups of topological groups are always abelian.

Lemma 6.1.8. *Let G be a topological group and consider the identity element $\mathbf{1}$ as a base point. Then the path space $P(G,\mathbf{1})$ also carries a natural group structure given by the pointwise product $(\alpha\cdot\beta)(t) := \alpha(t)\beta(t)$ and we have*

(1) *$\alpha \sim \alpha'$, $\beta \sim \beta'$ implies $\alpha\cdot\beta \sim \alpha'\cdot\beta'$, so that we obtain a well-defined product*

$$[\alpha]\cdot[\beta] := [\alpha\cdot\beta]$$

of homotopy classes, defining a group structure on $P(G,\mathbf{1})/\sim$.

(2) *$\alpha \sim \beta \Longleftrightarrow \alpha\cdot\beta^{-1} \sim \mathbf{1}$, the constant map.*

(3) *(Commutativity) $[\alpha]\cdot[\beta] = [\beta]\cdot[\alpha]$ for $\alpha,\beta \in \Omega(G,\mathbf{1})$.*

(4) *(Consistency) $[\alpha]\cdot[\beta] = [\alpha]*[\beta]$ for $\alpha \in \Omega(G,\mathbf{1})$, $\beta \in P(G,\mathbf{1})$.*

Proof. (1) follows by composing homotopies with the multiplication map m_G.
(2) follows from (1) by multiplication with β^{-1}.
(3)

$$[\alpha][\beta] = [\alpha*\mathbf{1}][\mathbf{1}*\beta] = [(\alpha*\mathbf{1})(\mathbf{1}*\beta)] = [(\mathbf{1}*\beta)(\alpha*\mathbf{1})] = [\mathbf{1}*\beta][\alpha*\mathbf{1}] = [\beta][\alpha].$$

(4) $[\alpha][\beta] = [(\alpha*\mathbf{1})(\mathbf{1}*\beta)] = [\alpha*\beta] = [\alpha]*[\beta]$. □

As a consequence of (4), we can calculate the product of homotopy classes as a pointwise product of representatives and obtain:

Proposition 6.1.9. (Hilton's Lemma) *For each topological group G, the fundamental group $\pi_1(G) := \pi_1(G,\mathbf{1})$ is abelian.*

Proof. We only have to combine (3) and (4) in Lemma 6.1.8 for loops $\alpha,\beta \in \Omega(G,\mathbf{1})$. □

Exercises for Section 6.1

Exercise 6.1.1. If $f\colon X \to Y$ is a map between topological spaces and $X = X_1 \cup \ldots \cup X_n$ holds with closed subsets $X_1,\ldots,X_n$, then f is continuous if and only if all restrictions $f|_{X_i}$ are continuous.

Exercise 6.1.2. Show that the homotopy relation on $P(X,x_0,x_1)$ is an equivalence relation. Hint: Exercise 6.1.1 helps to glue homotopies.

Exercise 6.1.3. Show that for $n > 1$ the sphere $\mathbb{S}^n$ is simply connected. For the proof, proceed along the following steps:

(a) Let $\gamma : [0,1] \to \mathbb{S}^n$ be continuous. Then there exists an $m \in \mathbb{N}$ such that $\|\gamma(t)-\gamma(t')\| < \frac{1}{2}$ for $|t-t'| < \frac{1}{m}$.

(b) Define $\widetilde{\alpha} : [0,1] \to \mathbb{R}^{n+1}$ as the piecewise affine curve with $\widetilde{\alpha}(\frac{k}{m}) = \gamma(\frac{k}{m})$ for $k = 0,\ldots,m$. Then $\alpha(t) := \frac{1}{\|\widetilde{\alpha}(t)\|}\widetilde{\alpha}(t)$ defines a continuous curve $\alpha : [0,1] \to \mathbb{S}^n$.

(c) $\alpha \sim \gamma$. Hint: Consider $H(t,s) := \frac{(1-s)\gamma(t)+s\alpha(t)}{\|(1-s)\gamma(t)+s\alpha(t)\|}$.

(d) α is not surjective. The image of α is the central projection of a polygonal arc on the sphere.

(e) If $\beta \in \Omega(\mathbb{S}^n, y_0)$ is not surjective, then $\beta \sim y_0$ (it is homotopic to a constant map). Hint: Let $p \in \mathbb{S}^n \setminus \operatorname{im}\beta$. Using stereographic projection, where p corresponds to the point at infinity, show that $\mathbb{S}^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$, hence contractible.

(f) $\pi_1(\mathbb{S}^n, y_0) = \{[y_0]\}$ for $n \geq 2$ and $y_0 \in \mathbb{S}^n$.

Exercise 6.1.4. Let X be a topological space, $x_0, x_1 \in X$ and $\alpha \in P(X, x_0, x_1)$ a path from x_0 to x_1. Show that the map

$$C \colon \pi_1(X, x_1) \to \pi_1(X, x_0), \quad [\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}]$$

is an isomorphism of groups. In this sense the fundamental group does not depend on the base point if X is arcwise connected.

Exercise 6.1.5. Let $\sigma \colon G \times X \to X$ be a continuous action of the topological group G on the topological space X and $x_0 \in X$. Then the orbit map $\sigma^{x_0} \colon G \to X, g \mapsto \sigma(g, x_0)$ defines a group homomorphism

$$\pi_1(\sigma^{x_0}) \colon \pi_1(G) \to \pi_1(X, x_0).$$

Show that the image of this homomorphism is central, i.e., lies in the center of $\pi_1(X, x_0)$. Hint: Mimic the argument in the proof of Lemma 6.1.8.

6.2 Coverings

In this section we discuss the concept of a covering map. One of its main applications is that it provides a means to calculate fundamental groups in terms of suitable coverings.

Definition 6.2.1. Let X and Y be topological spaces. A continuous map $q \colon X \to Y$ is called a *covering* if each $y \in Y$ has an open neighborhood U such that $q^{-1}(U)$ is a non-empty disjoint union of open subsets $(V_i)_{i \in I}$, such that for each $i \in I$ the restriction $q|_{V_i} \colon V_i \to U$ is a homeomorphism. We call any such U an *elementary* open subset of X.

Note that this condition implies in particular that q is surjective and that the fibers of q are discrete subsets of X.

Examples 6.2.2.

(a) The exponential function $\exp \colon \mathbb{C} \to \mathbb{C}^\times, z \mapsto e^z$ is a covering map.

(b) The map $q \colon \mathbb{R} \to \mathbb{T}, x \mapsto e^{ix}$ is a covering.

(c) The power maps $p_k \colon \mathbb{C}^\times \to \mathbb{C}^\times, z \mapsto z^k$ are coverings.

(d) If $q \colon G \to H$ is a surjective continuous open homomorphism of topological groups with discrete kernel, then q is a covering (Exercise 6.2.2). All the examples (a)-(c) are of this type.

Lemma 6.2.3. (Lebesgue number) [1] *Let (X, d) be a compact metric space and $(U_i)_{i\in I}$ an open cover. Then there exists a positive number $\lambda > 0$, called a Lebesgue number of the covering, such that any subset $S \subseteq X$ with diameter $\leq \lambda$ is contained in some U_i.*

Proof. Let us assume that such a number λ does not exist. Then there exists for each $n \in \mathbb{N}$ a subset S_n of diameter $\leq \frac{1}{n}$ which is not contained in some U_i. Pick a point $s_n \in S_n$. Then the sequence (s_n) has a subsequence converging to some $s \in X$ (Proposition 4.3.3). Then s is contained in some U_i, and since U_i is open, there exists an $\varepsilon > 0$ with $U_\varepsilon(s) \subseteq U_i$. If $n \in \mathbb{N}$ is such that $\frac{1}{n} < \frac{\varepsilon}{2}$ and $d(s_n, s) < \frac{\varepsilon}{2}$, we arrive at the contradiction $S_n \subseteq U_{\varepsilon/2}(s_n) \subseteq U_\varepsilon(s) \subseteq U_i$. ☐

Remark 6.2.4. (1) If $(U_i)_{i\in I}$ is an open cover of the unit interval $[0, 1]$, then there exists an $n > 0$ such that all subsets of the form $\left[\frac{k}{n}, \frac{k+1}{n}\right]$, $k = 0, \ldots, n-1$, are contained in some U_i.

(2) If $(U_i)_{i\in I}$ is an open cover of the unit square $[0, 1]^2$, then there exists an $n > 0$ such that all subsets of the form

$$\left[\frac{k}{n}, \frac{k+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right], \quad k, j = 0, \ldots, n-1,$$

are contained in some U_i.

Theorem 6.2.5. (The Path Lifting Property) *Let $q\colon X \to Y$ be a covering map and $\gamma\colon [0, 1] \to Y$ a path. Let $x_0 \in X$ be such that $q(x_0) = \gamma(0)$. Then there exists a unique path $\widetilde{\gamma}\colon [0, 1] \to X$ such that*

$$q \circ \widetilde{\gamma} = \gamma \quad \text{and} \quad \widetilde{\gamma}(0) = x_0.$$

Proof. Cover Y by elementary open set $U_i, i \in I$. By Remark 6.2.4, applied to the open covering of I by the sets $\gamma^{-1}(U_i)$, there exists an $n \in \mathbb{N}$ such that all sets $\gamma(\left[\frac{k}{n}, \frac{k+1}{n}\right])$, $k = 0, \ldots, n-1$, are contained in some U_i. We now use induction to construct $\widetilde{\gamma}$. Let $V_0 \subseteq q^{-1}(U_0)$ be an open subset containing x_0 for which $q|_{V_0}$ is a homeomorphism onto U_0 and define $\widetilde{\gamma}$ on $\left[0, \frac{1}{n}\right]$ by

$$\widetilde{\gamma}(t) := (q|_{V_0})^{-1} \circ \gamma(t).$$

Assume that we have already constructed a continuous lift $\widetilde{\gamma}$ of γ on the interval $\left[0, \frac{k}{n}\right]$ and that $k < n$. Then we pick an elementary open subset U_i containing $\gamma([\frac{k}{n}, \frac{k+1}{n}])$ and an open subset $V_k \subseteq X$ containing $\widetilde{\gamma}(\frac{k}{n})$ for which $q|_{V_k}$ is a homeomorphism onto U_i. We then define $\widetilde{\gamma}$ for $t \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$ by

$$\widetilde{\gamma}(t) := (q|_{V_k})^{-1} \circ \gamma(t).$$

We thus obtain the required lift $\widetilde{\gamma}$ of γ on $[0, \frac{k}{n+1}]$.

If $\widehat{\gamma}\colon [0, 1] \to X$ is any continuous lift of γ with $\widehat{\gamma}(0) = x_0$, then $\widehat{\gamma}(\left[0, \frac{1}{n}\right])$ is a connected subset of $q^{-1}(U_0)$ containing x_0, hence contained in V_0, showing that $\widetilde{\gamma}$ coincides with $\widehat{\gamma}$ on $\left[0, \frac{1}{n}\right]$. Applying the same argument at each step of the induction, we obtain $\widehat{\gamma} = \widetilde{\gamma}$, so that the lift $\widetilde{\gamma}$ is unique. ☐

[1] Lebesgue, Henri (1875–1941)

Theorem 6.2.6. (The Covering Homotopy Theorem) *Let $I := [0,1]$ and $q\colon X \to Y$ be a covering map and $H\colon I^2 \to Y$ be a homotopy with fixed endpoints of the paths $\gamma := H_0$ and $\eta := H_1$. For any lift $\widetilde{\gamma}$ of γ there exists a unique lift $G\colon I^2 \to X$ of H with $G_0 = \widetilde{\gamma}$. Then $\widetilde{\eta} := G_1$ is the unique lift of η starting in the same point as $\widetilde{\gamma}$ and G is a homotopy from $\widetilde{\gamma}$ to $\widetilde{\eta}$. In particular, lifts of homotopic curves in Y starting in the same point are homotopic in X.*

Proof. Using the Path Lifting Property (Theorem 6.2.5), we find for each $t \in I$ a unique continuous lift $I \to X, s \mapsto G(s,t)$, starting in $\widetilde{\gamma}(t)$ with $q(G(s,t)) = H(s,t)$. It remains to show that the map $G\colon I^2 \to X$ obtained in this way is continuous.

So let $s \in I$. Using Remark 6.2.4, we find a natural number n such that for each connected neighborhood W_s of s of diameter $\leq \frac{1}{n}$ and each $i = 0,\ldots,n$, the set $H\big(W_s \times \big[\frac{k}{n}, \frac{k+1}{n}\big]\big)$ is contained in some elementary subset U_k of Y. Assuming that G is continuous in $W_s \times \{\frac{k}{n}\}$, G maps this set into a connected subset of $q^{-1}(U_k)$, hence into some open subset V_k for which $q|_{V_k}$ is a homeomorphism onto U_k. But then the lift G on $W_s \times \big[\frac{k}{n}, \frac{k+1}{n}\big]$ must be contained in V_k, so that it is of the form $(q|_{V_k})^{-1} \circ H$, hence continuous. This means that G is continuous on $W_s \times \big[\frac{k}{n}, \frac{k+1}{n}\big]$. Now an inductive argument shows that G is continuous on $W_s \times I$ and hence on the whole square I^2.

Since the fibers of q are discrete and the curves $s \mapsto H(s,0)$ and $s \mapsto H(s,1)$ are constant, the curves $G(s,0)$ and $G(s,1)$ are also constant. Therefore $\widetilde{\eta}$ is the unique lift of η starting in $\widetilde{\gamma}(0) = G(0,0) = G(1,0)$ and G is a homotopy with fixed endpoints from $\widetilde{\gamma}$ to $\widetilde{\eta}$. □

Corollary 6.2.7. *If $q\colon X \to Y$ is a covering with $q(x_0) = y_0$, then the corresponding group homomorphism*

$$\pi_1(q, x_0)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [q \circ \gamma]$$

is injective.

Proof. If γ, η are loops in x_0 with $[q \circ \gamma] = [q \circ \eta]$, then the Covering Homotopy Theorem 6.2.6 implies that γ and η are homotopic. Therefore $[\gamma] = [\eta]$ shows that $\pi_1(q, x_0)$ is injective. □

Corollary 6.2.8. *If Y is simply connected and X is arcwise connected, then each covering map $q\colon X \to Y$ is a homeomorphism.*

Proof. Since q is an open continuous map, it remains to show that q is injective. So pick $x_0 \in X$ and $y_0 \in Y$ with $q(x_0) = y_0$. If $x \in X$ also satisfies $q(x) = y_0$, then there exists a path $\alpha \in P(X, x_0, x)$ from x_0 to x. Now $q \circ \alpha$ is a loop in Y, hence contractible because Y is simply connected. Now the Covering Homotopy Theorem implies that the unique lift α of $q \circ \alpha$ starting in x_0 is a loop, and therefore that $x_0 = x$. This proves that q is injective. □

The following theorem provides a more powerful tool, from which the preceding corollary easily follows. We recall that a topological space X is called *locally arcwise connected* if each neighborhood U of a point $x \in X$ contains some arcwise connected neighborhood V of x (cf. Exercise 1.3.6).

Theorem 6.2.9. (The Lifting Theorem) *Assume that* $q\colon X \to Y$ *is a covering map with* $q(x_0) = y_0$*, that* W *is arcwise connected and locally arcwise connected, and that* $f\colon W \to Y$ *is a given map with* $f(w_0) = y_0$*. Then a continuous map* $g\colon W \to X$ *with*

$$g(w_0) = x_0 \quad \text{and} \quad q \circ g = f \tag{6.1}$$

exists if and only if

$$\pi_1(f, w_0)(\pi_1(W, w_0)) \subseteq \pi_1(q, x_0)(\pi_1(X, x_0)), \ \ i.e.\ \operatorname{im}(\pi_1(f, w_0)) \subseteq \operatorname{im}(\pi_1(q, x_0)). \tag{6.2}$$

If g *exists, then it is uniquely determined by* (6.1)*. Condition* (6.2) *is in particular satisfied if* W *is simply connected.*

Proof. If g exists, then $f = q \circ g$ implies that the image of the homomorphism $\pi_1(f, w_0) = \pi_1(q, x_0) \circ \pi_1(g, w_0)$ is contained in the image of $\pi_1(q, x_0)$.

Let us, conversely, assume that this condition is satisfied. To define g, let $w \in W$ and $\alpha_w\colon I \to W$ be a path from w_0 to w. Then $f \circ \alpha_w\colon I \to Y$ is a path which has a continuous lift $\beta_w\colon I \to X$ starting in x_0. We claim that $\beta_w(1)$ does not depend on the choice of the path α_w. Indeed, if α'_w is another path from w_0 to w, then $\alpha_w * \overline{\alpha'_w}$ is a loop in w_0, so that $(f \circ \alpha_w) * (f \circ \overline{\alpha'_w})$ is a loop in y_0. In view of (6.2), the homotopy class of this loop is contained in the image of $\pi_1(q, x_0)$, so that it has a lift $\eta\colon I \to X$ which is a loop in x_0. Since the reverse of the second half $\eta|_{[\frac{1}{2},1]}$ of η is a lift of $f \circ \alpha'_w$, starting in x_0, it is β'_w, or, more precisely

$$\beta'_w(t) = \eta\Big(1 - \frac{t}{2}\Big) \quad \text{for} \quad 0 \leq t \leq 1.$$

We thus obtain

$$\beta'_w(1) = \eta\Big(\frac{1}{2}\Big) = \beta_w(1).$$

We now put $g(w) := \beta_w(1)$, and it remains to see that g is continuous. This is where we shall use the assumption that W is locally arcwise connected. Let $w \in W$ and put $y := f(w)$. Further, let $U \subseteq Y$ be an elementary neighborhood of y and V be an arcwise connected neighborhood of w in W such that $f(V) \subseteq U$. Fix a path α_w from w_0 to w as before. For any point $w' \in W$ we choose a path $\gamma_{w'}$ from w to w' in V, so that $\alpha_w * \gamma_{w'}$ is a path from w_0 to w'. Let $\widetilde{U} \subseteq X$ be an open subset of X for which $q|_{\widetilde{U}}$ is a homeomorphism onto U and $g(w) \in \widetilde{U}$. Then the uniqueness of lifts implies that

$$\beta_{w'} = \beta_w * \big((q|_{\widetilde{U}})^{-1} \circ (f \circ \gamma_{w'})\big).$$

We conclude that

$$g(w') = (q|_{\widetilde{U}})^{-1}(f(w')) \in \widetilde{U},$$

hence that $g|_V$ is continuous.

We finally show that g is unique. In fact, if $h\colon W \to X$ is another lift of f satisfying $h(w_0) = x_0$, then the set $S := \{w \in W : g(w) = h(w)\}$ is non-empty and closed. We claim that it is also open. In fact, let $w_1 \in S$ and U be a connected open elementary neighborhood of $f(w_1)$ and V an arcwise connected neighborhood of w_1 with $f(V) \subseteq U$. If $\widetilde{U} \subseteq q^{-1}(U)$ is the open subset on which q is a homeomorphism containing $g(w_1) = h(w_1)$, then the arcwise connectedness of V implies that $g(V), h(V) \subseteq \widetilde{U}$, and hence that $V \subseteq S$. Therefore S is open, closed and non-empty, so that the connectedness of W yields $S = W$, i.e., $g = h$. □

Corollary 6.2.10. (Uniqueness of Simply Connected Coverings) *Suppose that Y is locally arcwise connected. If $q_1\colon X_1 \to Y$ and $q_2\colon X_2 \to Y$ are two simply connected arcwise connected coverings, then there exists a homeomorphism $\varphi\colon X_1 \to X_2$ with $q_2 \circ \varphi = q_1$.*

Proof. Since Y is locally arcwise connected, both covering spaces X_1 and X_2 also have this property. Pick points $x_1 \in X_1$, $x_2 \in X_2$ with $y := q_1(x_1) = q_2(x_2)$. According to the Lifting Theorem 6.2.9, there exists a unique lift $\varphi\colon X_1 \to X_2$ of q_1 with $\varphi(x_1) = x_2$. We likewise obtain a unique lift $\psi\colon X_2 \to X_1$ of q_2 with $\psi(x_2) = x_1$. Then $\varphi \circ \psi\colon X_1 \to X_1$ is a lift of id_Y fixing x_1, so that the uniqueness of lifts implies that $\varphi \circ \psi = \mathrm{id}_{X_1}$. The same argument yields $\psi \circ \varphi = \mathrm{id}_{X_2}$, so that φ is a homeomorphism with the required properties. □

Definition 6.2.11. A topological space X is called *semilocally simply connected* if each point $x_0 \in X$ has a neighborhood U such that each loop $\alpha \in \Omega(U, x_0)$ is homotopic to $[x_0]$ in X, i.e., the natural homomorphism

$$\pi(i_U)\colon \pi_1(U, x_0) \to \pi_1(X, x_0), \quad [\gamma] \mapsto [i_U \circ \gamma]$$

induced by the inclusion map $i_U\colon U \to X$ is trivial.

Theorem 6.2.12. *Let Y be arcwise connected and locally arcwise connected. Then Y has a simply connected covering space if and only if Y is semilocally simply connected.*

Proof. If $q\colon X \to Y$ is a simply connected covering space and $U \subseteq Y$ is a pathwise connected elementary open subset. Then each loop γ in U lifts to a loop $\widetilde{\gamma}$ in X, and since $\widetilde{\gamma}$ is homotopic to a constant map in X, the same holds for the loop $\gamma = q \circ \widetilde{\gamma}$ in Y.

Conversely, let us assume that Y is semilocally simply connected. We choose a base point $y_0 \in Y$ and let

$$\widetilde{Y} := P(Y, y_0)/\sim$$

be the set of homotopy classes of paths starting in y_0. We shall topologize $\widetilde{Y}$ in such a way that the map

$$q\colon \widetilde{Y} \to Y, \quad [\gamma] \mapsto \gamma(1)$$

defines a simply connected covering of Y.

Let $\mathcal{B}$ denote the set of all arcwise connected open subsets $U \subseteq Y$ for which each loop in U is contractible in Y and note that our assumptions on Y imply that $\mathcal{B}$ is a basis for the topology of Y, i.e., each open subset is a union of elements of $\mathcal{B}$. If $\gamma \in P(Y, y_0)$ satisfies $\gamma(1) \in U \in \mathcal{B}$, let

$$U_{[\gamma]} := \{[\eta] \in q^{-1}(U) \colon (\exists \beta \in C(I, U))\ \eta \sim \gamma * \beta\}.$$

We shall now verify several properties of these definitions, culminating in the proof of the theorem.

(1) $[\eta] \in U_{[\gamma]} \Rightarrow U_{[\eta]} = U_{[\gamma]}$.

To prove this, let $[\zeta] \in U_{[\eta]}$. Then $\zeta \sim \eta * \beta$ for some path β in U. Further $\eta \sim \gamma * \beta'$ for some path β' in U. Now $\zeta \sim \gamma * \beta' * \beta$, and $\beta' * \beta$ is a path in U, so that $[\zeta] \in U_{[\gamma]}$. This proves $U_{[\eta]} \subseteq U_{[\gamma]}$. We also have $\gamma \sim \eta * \overline{\beta'}$, so that $[\gamma] \in U_{[\eta]}$, and the first part implies that $U_{[\gamma]} \subseteq U_{[\eta]}$.

(2) q maps $U_{[\gamma]}$ injectively onto U.

That $q(U_{[\gamma]}) = U$ is clear since U and Y are arcwise connected. To show that that it is one-to-one, let $[\eta], [\eta'] \in U_{[\gamma]}$, which we know from (1) is the same as $U_{[\eta]}$. Suppose $\eta(1) = \eta'(1)$. Since $[\eta'] \in U_{[\eta]}$, we have $\eta' \sim \eta * \alpha$ for some loop α in U. But then α is contractible in Y, so that $\eta' \sim \eta$, i.e., $[\eta'] = [\eta]$.

(3) $U, V \in \mathcal{B}$, $\gamma(1) \in U \subseteq V$, implies $U_{[\gamma]} \subseteq V_{[\gamma]}$.

This is trivial.

(4) The sets $U_{[\gamma]}$ for $U \in \mathcal{B}$ and $[\gamma] \in \widetilde{Y}$ form a basis of a topology on $\widetilde{Y}$.

Suppose $[\gamma] \in U_{[\eta]} \cap V_{[\eta']}$. Let $W \subseteq U \cap V$ be in $\mathcal{B}$ with $\gamma(1) \in W$. Then $[\gamma] \in W_{[\gamma]} \subseteq U_{[\gamma]} \cap V_{[\gamma]} = U_{[\eta]} \cap V_{[\eta']}$.

(5) q is open and continuous.

We have already seen in (2) that $q(U_{[\gamma]}) = U$, and these sets form a basis of the topology on $\widetilde{Y}$, resp., Y. Therefore q is an open map. We also have for $U \in \mathcal{B}$ the relation

$$q^{-1}(U) = \bigcup_{\gamma(1) \in U} U_{[\gamma]},$$

which is open. Hence q is continuous.

(6) $q|_{U_{[\gamma]}}$ is a homeomorphism.

This is because it is bijective, continuous and open.

At this point we have shown that $q \colon \widetilde{Y} \to Y$ is a covering map. It remains to see that $\widetilde{Y}$ is arcwise connected and simply connected.

(7) Let $H \colon I \times I \to Y$ be a continuous map with $H(t, 0) = y_0$. Then $h_t(s) := H(t, s)$ defines a path in Y starting in y_0. Let $\widetilde{h}(t) := [h_t] \in \widetilde{Y}$. Then $\widetilde{h}$ is a path in $\widetilde{Y}$ covering the path $t \mapsto h_t(1) = H(t, 1)$ in Y. We claim that $\widetilde{h}$ is continuous. Let $t_0 \in I$. We shall prove continuity at t_0. Let $U \in \mathcal{B}$ be a neighborhood of $h_{t_0}(1)$. Then there exists an interval $I_0 \subseteq I$ which is a neighborhood of t_0 with $h_t(1) \in U$ for $t \in I_0$. Then $\alpha(s) := H(t_0 + s(t - t_0), 1)$ is a continuous curve in U with $\alpha(0) = h_{t_0}(1)$ and $\alpha(1) = h_t(1)$, so that $h_{t_0} * \alpha$

is curve with the same endpoint as h_t. Applying Exercise 6.2.1 to the restriction of H to the interval between t_0 and t, we see that $h_t \sim h_{t_0} * \alpha$, so that $\widetilde{h}(t) = [h_t] \in U_{[h_{t_0}]}$ for $t \in I_0$. Since $q|_{U_{[h_{t_0}]}}$ is a homeomorphism, $\widetilde{h}$ is continuous in t_0.

(8) $\widetilde{Y}$ is arcwise connected.

For $[\gamma] \in \widetilde{Y}$ put $h_t(s) := \gamma(st)$. By (7), this yields a path $\widetilde{\gamma}(t) = [h_t]$ in $\widetilde{Y}$ from $\widetilde{y}_0 := [y_0]$ (the class of the constant path) to the point $[\gamma]$.

(9) $\widetilde{Y}$ is simply connected.

Let $\widetilde{\alpha} \in \Omega(\widetilde{Y}, \widetilde{y}_0)$ be a loop in $\widetilde{Y}$ and $\alpha := q \circ \widetilde{\alpha}$ its image in Y. Let $h_t(s) := \alpha(st)$. Then we have the path $\widetilde{h}(t) = [h_t]$ in $\widetilde{Y}$ from (7). This path covers α since $h_t(1) = \alpha(t)$. Further, $\widetilde{h}(0) = \widetilde{y}_0$ is the constant path. Also, by definition, $\widetilde{h}(1) = [\alpha]$. From the uniqueness of lifts we derive that $\widetilde{h} = \widetilde{\alpha}$ is closed, so that $[\alpha] = [y_0]$. Therefore the homomorphism

$$\pi_1(q, y_0) \colon \pi_1(\widetilde{Y}, \widetilde{y}_0) \to \pi_1(Y, y_0)$$

vanishes. Since it is also injective (Corollary 6.2.7), $\pi_1(\widetilde{Y}, \widetilde{y}_0)$ is trivial, i.e., $\widetilde{Y}$ is simply connected. □

Definition 6.2.13. Let $q \colon X \to Y$ be a covering. A homeomorphism $\varphi \colon X \to X$ is called a *deck transformation* of the covering if $q \circ \varphi = q$. This means that φ permutes the elements in the fibers of q. We write $\mathrm{Deck}(X, q)$ for the group of deck transformations.

Example 6.2.14. For the covering map $\exp \colon \mathbb{C} \to \mathbb{C}^\times$, the deck transformations have the form

$$\varphi(z) = z + 2\pi i n, \quad n \in \mathbb{Z}.$$

Proposition 6.2.15. *Let $q \colon \widetilde{Y} \to Y$ be a simply connected covering of the locally arcwise connected space Y. Pick $\widetilde{y}_0 \in \widetilde{Y}$ and put $y_0 := q(\widetilde{y}_0)$. For each $[\gamma] \in \pi_1(Y, y_0)$ we write $\varphi_{[\gamma]} \in \mathrm{Deck}(\widetilde{Y}, q)$ for the unique lift of id_X mapping $\widetilde{y}_0$ to the endpoint $\widetilde{\gamma}(1)$ of the lift $\widetilde{\gamma}$ of γ starting in $\widetilde{y}_0$. Then the map*

$$\Phi \colon \pi_1(Y, y_0) \to \mathrm{Deck}(\widetilde{Y}, q), \quad \Phi([\gamma]) = \varphi_{[\gamma]}$$

is an isomorphism of groups.

Proof. For $\gamma, \eta \in \Omega(Y, y_0)$, the composition $\varphi_{[\gamma]} \circ \varphi_{[\eta]}$ is a deck transformation mapping $\widetilde{y}_0$ to the endpoint of $\varphi_{[\gamma]} \circ \widetilde{\eta}$ which coincides with the endpoint of the lift of η starting in $\widetilde{\gamma}(1)$. Hence it also is the endpoint of the lift of the loop $\gamma * \eta$. This leads to $\varphi_{[\gamma]} \circ \varphi_{[\eta]} = \varphi_{[\gamma * \eta]}$, so that Φ is a group homomorphism.

To see that Φ is injective, we note that $\varphi_{[\gamma]} = \mathrm{id}_{\widetilde{Y}}$ implies that $\widetilde{\gamma}(1) = \widetilde{y}_0$, so that $\widetilde{\gamma}$ is a loop, and hence that $[\gamma] = [y_0]$.

For the surjectivity, let φ be a deck transformation and $y := \varphi(\widetilde{y}_0)$. If α is a path from $\widetilde{y}_0$ to y, then $\gamma := q \circ \alpha$ is a loop in y_0 with $\alpha = \widetilde{\gamma}$, so that $\varphi_{[\gamma]}(\widetilde{y}_0) = y$, and the uniqueness of lifts (Theorem 6.2.9) implies that $\varphi = \varphi_{[\gamma]}$. □

Example 6.2.16. With Example 6.2.14 and the simple connectedness of $\mathbb{C}$ we derive that

$$\pi_1(\mathbb{C}^\times, 1) \cong \mathrm{Deck}(\mathbb{C}, \exp) \cong \mathbb{Z}.$$

Exercises for Section 6.2

Exercise 6.2.1. Let $F\colon I^2 \to X$ be a continuous map with $F(0,s) = x_0$ for $s \in I$ and define

$$\gamma(t) := F(t,0), \quad \eta(t) := F(t,1), \quad \alpha(t) := F(1,t), \quad t \in I.$$

Show that $\gamma * \alpha \sim \eta$. Hint: Consider the map

$$G\colon I^2 \to I^2, \quad G(t,s) := \begin{cases} (2t,s) & \text{for } 0 \le t \le \frac{1}{2}, s \le 1-2t, \\ (1, 2t-1) & \text{for } \frac{1}{2} \le t \le 1, s \le 2t-1, \\ (t + \frac{1-s}{2}, s) & \text{else} \end{cases}$$

and show that it is continuous. Take a look at the boundary values of $F \circ G$.

Exercise 6.2.2. Let $q\colon G \to H$ be an morphism of topological groups with discrete kernel Γ. Show that:

(1) If $V \subseteq G$ is an open $\mathbf{1}$-neighborhood with $(V^{-1}V) \cap \Gamma = \{\mathbf{1}\}$ and q is open, then $q|_V \colon V \to q(V)$ is a homeomorphism.

(2) If q is open and surjective, then q is a covering.

(3) If q is open and H is connected, then q is surjective, hence a covering.

Exercise 6.2.3. A map $f\colon X \to Y$ between topological spaces is called a *local homeomorphism* if each point $x \in X$ has an open neighborhood U such that $f|_U \colon U \to f(U)$ is a homeomorphism onto an open subset of Y.

(1) Show that each covering map is a local homeomorphism.

(2) Find a surjective local homeomorphism which is not a covering. Can you also find an example where X is connected?

Exercise 6.2.4. Let X be a topological space. The *cone over* X is the space

$$C(X) := (X \times [0,1])/(X \times \{1\}).$$

Show that $C(X)$ is always contractible.

Exercise 6.2.5. In the euclidean plane $\mathbb{R}^2$, we write

$$C_r(m) := \{x \in \mathbb{R}^2 \colon \|x - m\|_2 = r\}$$

for the circle of radius r and center m. Consider the union

$$X := \bigcup_{n \in \mathbb{N}} C_{1/n}(\frac{1}{n}, 0).$$

Show that X is arcwise connected but not semilocally simply connected. Hint: Consider the point $(0,0) \in X$.