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7th Homework Sheet Analysis II (engl.)

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(H7.1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = x^3 + y^3 + \frac{3}{2}x^2 - 6x - 3y + 1$.

1. Compute *grad* $f(x, y)$, and find all critical points of f.
2. Compute the Hessian of f.
3. Determine if f has a local minimum, local maximum, or neither at each critical point.

(H7.2) Recall from Example 2.7, p. 167 of the script that a funtion $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is called positively homogeneous of degree $\alpha \in \mathbb{R}$ if $f(tx) = t^\alpha f(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $t \in (0, \infty)$.

1. Prove *Euler's relation*
$$\langle grad\ f(x), x\rangle = \alpha f(z).$$
2. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is positively homogeneous of degree 3, but is *not* a polynomial function.

(H7.3)

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a k times differentiable function. Let $T_k(f) : \mathbb{R}^n \to \mathbb{R}$ denote the kth order Taylor approximation to f at the origin in $\mathbb{R}^n$. Prove that $f(x) = T_k(f)(x)$ for all $x \in \mathbb{R}^n$ if and only if f is a polynomial function of degree $d \le k$.
2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable to all orders. Show that the second-order Taylor polynomial $T_2(g \circ f)$ for the composition $g \circ f : \mathbb{R}^2 \to \mathbb{R}$ at the point $a \in R^n$, can be obtained by substituting the Taylor polynomial for $T_2(f)$ into the (single-variable) Taylor polynomial $T_2(g)$, and then collecting the terms of degree at most 2.
3. Assuming that for all k we can compute the Taylor polynomial $T_k(g \circ f)$ of a composite function $g \circ f$ by using substitution as above, compute the fourth order Taylor polynomial for the function $f(x, y) = e^{x+xy+x^2y^2}$ at $(0, 0)$.