Fachbereich Mathematik
Prof. Dr. W. Trebels
Dr. V. Gregoriades
Dr. A. Linshaw

TECHNISCHE UNIVERSITÄT DARMSTADT

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6th Homework Sheet Analysis II (engl.)

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(H6.1)

Which of the following statements are correct? (You have to give a complete justification. If a statement is correct you can refer to a theorem given in the lecture and if a statement is false you can refer to a counterexample given in the lecture/tutorials/exercises/homework.)

Let U be an open subset of $\mathbb{R}^2$ and let $f : U \to \mathbb{R}$ be a function.

1. If f is differentiable, then f is continuous.
2. If all partial derivatives of f exist, then f is differentiable.
3. If f is differentiable, then all partial derivatives of f exist.
4. If the partial derivatives of f exist and are continuous in a neighborhood of a point x, then f is differentiable at x.
5. If f is differentiable, then all partial derivatives of f are continuous.
6. If f is not differentiable, then f is not continuous.

(H6.2)

1. We are given the functions $f : \mathbb{R}^3 \to \mathbb{R}^2 : f(x, y, z) = (\cos(x^3), yz)$ and $g : \mathbb{R}^2 \to \mathbb{R}^2 : g(a, b) = (e^{a^2+b^2}, ab)$. Verify the Chain Rule (Theorem 2.1 Chap. VII) for the composition $h : \mathbb{R}^3 \to \mathbb{R}^2 : h = g \circ f$ of the functions f and g. [This means that you have to compute the derivatives of f, g, h and check that for all $\overrightarrow{x} \in \mathbb{R}^3$ we have that $Dh(\overrightarrow{x}) = Dg(f(\overrightarrow{x})) \cdot Df(\overrightarrow{x})$]
2. Find all natural numbers $n \geq 2$ such that the function $N(n) \equiv N : \mathbb{R}^n \setminus \{(0, 0)\} \to \mathbb{R} : N(\overrightarrow{x}) = \dfrac{1}{\|\overrightarrow{x}\|^{n-2}}$ satisfies the equation $\Delta N = 0$.

   (Recall that Δ is the Laplace operator, see Example 3.4 (a) Chap. VII).

(H6.3)

1. Prove Corollary 3.3 Chap. VII:

   Let $U \subseteq \mathbb{R}^n$ be open, $i_1, \ldots, i_k \in \{1, \ldots, n\}$ and let $f \in C^k(U, \mathbb{R})$ for some $k \in \mathbb{N}$. Then

   $$\frac{\partial}{\partial x_{i_k}} \cdots \frac{\partial f}{\partial x_{i_1}} = \frac{\partial}{\partial x_{i_{\pi(k)}}} \cdots \frac{\partial f}{\partial x_{i_{\pi(1)}}}$$

   for every permutation $\pi : \{1, \ldots, k\} \to \{1, \ldots, k\}$.

   Hint. Use induction on k.

   (Recall that a function $\pi : \{1, \ldots, k\} \to \{1, \ldots, k\}$ is called a permutation if it is injective and surjective).

2. Prove that the function $D : (C^1([0,1]), \|\cdot\|_\infty) \to (C([0,1]), \|\cdot\|_\infty)$ such that $D(f) = f'$ is not continuous. (Recall that $\|f\|_\infty = \{|f(x)| : x \in [0,1]\}$).

   Hint. For all $n \in \mathbb{N}$ take the function $p_n(x) = \frac{1}{n} \cdot x^n$ for $0 \leq x \leq 1$.