Fachbereich Mathematik
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4th Homework Sheet Analysis II (engl.)

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(H4.1)

1. Define the following functions on $\mathbb{R}\times\mathbb{R}$: $d_1(x,y)=|xy|$, $d_2(x,y)=|x+y|$, $d_3(x,y)=|x|+|y|$, $d_4(x,y)=x^2-y^2$. Which properties of a metric on $\mathbb{R}$ do these functions have?

2. Equipped with the metric $d(x,y)=\frac{|x-y|}{1+|x-y|}$, $\mathbb{R}$ is a complete metric space. Consider the function $\tilde{d}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, given by $\tilde{d}(x,y)=\left|\frac{x}{1+|x|}-\frac{y}{1+|y|}\right|$. Show that $\tilde{d}$ is a metric on $\mathbb{R}$. Is $(\mathbb{R},\tilde{d})$ a complete metric space?

(H4.2) Regard the set of rational numbers $\mathbb{Q}$ as a metric space with metric $d(x,y)=|y-x|$. Let E be the set of all $p\in\mathbb{Q}$ such that $2<p^2<3$. Show that E is closed and bounded in $\mathbb{Q}$, but E is not compact. Is E open in $\mathbb{Q}$?

(H4.3)

For $1<p<\infty$, define ℓ^p to be the set of sequences $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty}|a_n|^p$ converges.

1. Prove that for $p=2$ and $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in ℓ^2, the sequence $\sum_{n=1}^{\infty}a_nb_n$ converges absolutely, and
$$\sum_{n=1}^{\infty}|a_nb_n|\leq\left(\sum_{n=1}^{\infty}|a_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}|b_n|^2\right)^{1/2}.$$
(Hint: Use Holder's inequality).

2. For $(a_n)_{n\in\mathbb{N}}$ in ℓ^p, define
$$\|a\|_p:=\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{1/p}.$$
Prove that this defines a norm on ℓ^p.