Fachbereich Mathematik
Prof. Dr. W. Trebels
Dr. V. Gregoriades
Dr. A. Linshaw

TECHNISCHE UNIVERSITÄT DARMSTADT

29-04-2010

3rd Homework Sheet Analysis II (engl.)
Summer Semester 2010

(H3.1)

Let X denote the space $C^2([0,1],\mathbb{R})$ of continuously twice-differentiable, real-valued functions on the interval $[0,1]$. Given $f \in X$, recall the notation $\|f\|_\infty = sup_{x\in[0,1]}|f(x)|$.

1. Determine whether the following are norms on X:

a. $\|f\|_a = max\{\|f\|_\infty, \|f'\|_\infty, \|f''\|_\infty\}$,

b. $\|f\|_b = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty$,

c. $\|f\|_c = \|f''\|_\infty$,

d. $\|f\|_d = \int_0^1 |f(x)|dx$.

2. Which of these norms are equivalent?

(H3.2)

Given $x, y \in \mathbb{R}^2$, define $\tilde{d}(x,y) = \frac{\|x-y\|_2}{1+\|x-y\|_2}$.

1. Show that $\tilde{d}$ is a metric on $\mathbb{R}^n$, and describe the unit ball centered at the origin.
2. For $0 < \epsilon < \frac{1}{2}$, let $\tilde{U}_\epsilon(x)$ denote the ϵ-ball centered at x with respect to the metric $\tilde{d}$, and let $U_\epsilon(x)$ denote the ϵ-ball centered at x with respect to the Euclidean metric $d(x,y) = \|x-y\|_2$. Determine whether $\tilde{U}_\epsilon(x) \subset U_\epsilon(x)$, or $U_\epsilon(x) \subset \tilde{U}_\epsilon(x)$, or neither.

(H3.3)

Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic function defined by $f(x) = -\cos x$ for $x \in [-\pi, 0)$, $f(0) = 0$, and $f(x) = \cos x$ for $x \in (0, \pi)$.

1. Find all the Fourier coefficients $\hat{f}_k$ of f.

   Hint: Use the formula $\sin x \cos y = \frac{1}{2}(\sin(x-y) + \sin(x+y))$.
2. Determine for which $x \in \mathbb{R}$ the Fourier series converges pointwise to $f(x)$.