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Repetition Sheet Analysis II (engl.)
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NORMED AND METRIC SPACES

(R.1)
Let (X, d) be a metric space, and let $V, W \subseteq X$ be disjoint (i.e. $V \cap W = \emptyset$), nonempty and closed. Prove that there exist disjoint open $V' \subseteq X$ and $W' \subseteq X$ such that $V \subseteq V'$ and $W \subseteq W'$.

(R.2)

1. Prove that the closed unit ball in $(C([0, 1], \mathbb{R}), \|\cdot\|_\infty)$ is not compact.
2. Let $T : \ell^2 \to \ell^2$ be defined by
$$T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$
Prove that T is continuous.

FOURIER SERIES

(R.3) Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function defined by $f(x) = e^{|x|}$ for $x \in [-\pi, \pi)$. Find all Fourier coefficients $\tilde{f}_n$ of f.

DIFFERENTIABILITY

(R.4)
We define $M \subset \mathbb{R}^2$ by
$$M = \{(x, y) \in \mathbb{R}^2 : y \leq 0 \text{ or } y \geq x^2\}.$$
Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote the characteristic function of M, i.e. $f(x) = 1$ if $x \in M$ and $f(x) = 0$ if $x \notin M$. Prove that all directional derivatives in $(0, 0)$ of f exist. Prove that f is discontinuous in $(0, 0)$.

(R.5) (Chain Rule)

Let $f(u,v) = \log(u^2+v^2)$ for $u^2+v^2>0$, $g_1(x,y) = xy$ and $g_2(x,y) = \frac{\sqrt{x}}{y}$ for $x,y>0$. Define for all $x,y>0$

$$\Phi(x,y) = f(g_1(x,y), g_2(x,y)).$$

Compute $\Phi'(x,y)$ in two different ways:

(i) compute Φ and then differentiate;

(ii) use the Chain Rule.

(R.6)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Prove the following assertions.

1. The partial derivatives $\partial_j f(0,0)$ exist for $j = 1,2$.
2. The directional derivative $D_v f(0,0)$ does not exist if v is not a multiple of the standard unit vectors e_1, e_2.
3. The function f is not differentiable.

(R.7) (Taylor series)

Find the third-order Taylor polynomial for $f(x,y) = e^{2x}\cos(x+y)$ at the point $(0,0)$.

(R.8) (Polar coordinates)

The polar coordinates are given by

$$P : U = (0,\infty) \times (0, 2\pi) \to \mathbb{R}^2, \quad P(r,\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

(i) Prove that P is injective and find the range $P(U)$ of P.

(ii) Calculate the Jacobi matrix $P'(r,\varphi)$ of P. What is the rank of $P'(r,\varphi)$?

(iii) Compute the inverse function $Q : P(U) \to U$ and its Jacobi matrix $Q'(x,y)$.

(iv) Calculate the Jacobi matrix of Q once again (without computing Q, but assuming that Q is differentiable) by applying the chain rule to $P \circ Q = id_{P(U)}$.

EXTREMUM PROBLEMS

(R.9)
Let

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = 3x^4 - 4x^2y + y^2.$$

1. Does f attain a local minimum at $(0,0)$?
2. Prove that the restriction of f to a line through the origin $(0,0)$ has a local minimum.

(R.10) Let $f(x,y) = 3x - x^3 - 2y^2 + y^4$. Find all critical points of f, and classify each critical point as a local minimum, local maximum, or neither

(R.11) Using Lagrange multipliers, find the maximum and minimum values of the function $f(x,y) = 4x + 6y$ on the circle $x^2 + y^2 = 13$.

DIFFERENTIATION OF INTEGRALS W/PARAMETERS

(R.12)
Let $h : (0,\infty) \to \mathbb{R}$ be defined by

$$h(x) = \int_1^{x^2+1} \frac{1}{t} e^{-(xt)^2}\, dt.$$

Calculate the derivative h'.

INVERSE/IMPLICIT FUNCTION THEOREM

(R.13)
Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, let $U \subseteq \mathbb{R}^n$ be open and bounded, and let $f : \overline{U} \to \mathbb{R}^n$ be continuous. Assume further that f is continuously differentiable on U and that $Df(x)$ is invertible for each $x \in U$.

Prove that each $y \in f(U) \setminus f(\partial U)$ has finitely many inverse images under f.

(R.14)

1. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ as follows

$$f(x,y) = \begin{cases} \dfrac{x^3y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   Which of the partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ is continuous? Compute the derivative of the function $F : [0,1] \to \mathbb{R} : F(x) = \int_0^1 f(x,y)dy$ at $x = 0$.

2. Prove that we can solve the following system

$$\begin{aligned} xy^2 + xzu + yv^2 &= 3 \\ u^3yz + 2xv - u^2v^2 &= 2 \end{aligned}$$

with u and v as differentiable functions of (x, y, v) close to the point $(x, y, z, u, v) = (1, 1, 1, 1, 1)$. Compute the partial derivative $\dfrac{\partial v}{\partial y}$ at $(1, 1, 1)$.

PATHS, LINE INTEGRALS

(R.15)

1. Let $U \subset \mathbb{R}^n$ be open, $\gamma : [a, b] \to U$ be a rectifiable path and $f, g : U \to \mathbb{R}^n$ be continuous vector fields. Prove the following:

   (a) $\displaystyle\int_\gamma (f + g)(x)\, dx = \int_\gamma f(x)\, dx + \int_\gamma g(x)\, dx.$

   (b) Let γ^- denote the path obtained from γ by reversing the orientation, that is

   $$\gamma^- : [a, b] \to U, \quad \gamma^-(t) = \gamma(a + b - t).$$

   Then

   $$\int_{\gamma^-} f(x)\, dx = -\int_\gamma f(x)\, dx.$$

2. Consider the vector field $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

   $$f(x_1, x_2, x_3) = (x_1, x_2, 0),$$

   and the path $\gamma : [0, 1] \to \mathbb{R}^3$, $\gamma(t) = \left(\frac{t^4}{4}, \sin^3\left(\frac{t\pi}{2}\right), 0\right)$. Evaluate the line integral $\int_\gamma f(x)\, dx$.

INTEGRATION IN HIGHER DIMENSIONS

(R.16) Use Fubini's theorem to evaluate the double integral

$$\int_0^2 \int_0^{2y} e^{x^2+1}\, dx\, dy.$$

(R.17)

Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ as follows

$$f(x,y) = \begin{cases} x^2, & \text{if } x \leq y \\ y^2 & \text{if } x > y. \end{cases}$$

Verify that $\int_0^1 \int_0^1 f(x,y)\ dx\ dy = \int_0^1 \int_0^1 f(x,y)\ dy\ dx$.

(R.18) (The Volume of the Unit Ball)
Calculate the volume c_n of the n-dimensional ball $B_n := U_1(0) \subseteq \mathbb{R}^n$ with radius 1. Conclude that $c_n \to 0$ as $n \to \infty$.

Hint: Consider the intersections with the $(n-1)$-dimensional hyperplane $\mathbb{R}^{n-1} \times \{s\}$ for $-1 < s < 1$, and use Fubini's Theorem to obtain a recursive formula for c_n. For the integrals which appear in this formula use the substitution $s(t) = \sin t$ and integration by parts to obtain again a recursive formula for these integrals. Use mathematical induction and combine all results.

(R.19) Consider the integral of the function $f(x,y,z) = xyz$ over the region $W \subset \mathbb{R}^3$ lying in the octant $\{(x,y,z) |\ x \geq 0,\ y \geq 0,\ z \geq 0\}$, outside the sphere of radius 2, and inside the sphere of radius 3.

1. Using cylindrical coordinates, describe W as the union of two regions W_1 and W_2 of the form
$$\{(r,\varphi,z) |\ a \leq r \leq b,\ c \leq \varphi \leq d,\ \gamma_1(r,\theta) \leq z \leq \gamma_2(r,\theta)\}.$$
2. Using Part (1), express $\int_W xyz\ d(x,y,z)$ as a sum of two integrals in cylindrical coordinates. (You don't need to evaluate either integral).

(R.20)

1. Compute the following integrals: $I_1 = \int_0^1 \frac{1}{x^2-9} dx$, $I_2 = \int_0^1 \frac{2x}{x^2+5} dx$, $I_3 = \int_1^\infty \frac{1}{x^2+4} dx$.
2. Define the functions F_1, F_2 and F_3 as follows: $F_1(y) = \int_0^1 \frac{1}{x^2-y^2} dx,\ y \in [2,3]$, $F_2(y) = \int_0^1 \frac{2x}{x^2+y} dx,\ y \in [2,3]$, $F_3(y) = \int_1^\infty \frac{1}{x^2+y^2} dx,\ y \geq 0$.
   Find a simpler form for the functions F_1, F_2 and F_3.