10 Integral Theorems in $\mathbb{R}^2$

The aim of Sections 10 and 11 is the generalization of the fundamental theorem of calculus

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

and of integration by parts

$$\int_a^b u'v\,dx = -\int_a^b uv'\,dx + uv\big|_a^b$$

to multidimensional integrals. Here the question arises which (partial) differential operator will replace $\frac{d}{dx}$ and which boundary terms will replace $uv|_a^b$.

Definition

(1) A mapping $F : B \subset \mathbb{R}^n \to \mathbb{R}^n$ is called a *vector field.*

(2) Let $B \subset \mathbb{R}^3$ be open and let $F : B \to \mathbb{R}^3$ be a C^1–vector field. Then the vector field

$$\operatorname{rot} F : B \to \mathbb{R}^3\,, \quad \operatorname{rot} F(x) = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}(x)\,,$$

is called the *rotation* or *curl* of F.

(3) Let $B \subset \mathbb{R}^2$ be open and let $F : B \to \mathbb{R}^2$ be a C^1–vector field. Then the (scalar–valued) *rotation* or curl is defined by

$$\operatorname{rot} F(x) = \partial_1 F_2(x) - \partial_2 F_1(x)\,.$$

Note that in this case $\operatorname{rot} F$ equals the third component of $\operatorname{rot}\widetilde{F}$, the rotation of the $3D$–vector field

$$\widetilde{F}(x_1, x_2, x_3) = (F_1(x_1, x_2)\,, F_2(x_1, x_2)\,, 0)^T\,.$$

Examples

(1) Vector fields occur e.g. as velocity fields in hydrodynamics, as displacement vectors of elastic bodies, as force fields in the theory of electromagnetism and of gravitation.

(2) For

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \quad \text{and} \quad F(x) = \frac{1}{2}\begin{pmatrix} \omega_2 x_3 - \omega_3 x_2 \\ \omega_3 x_1 - \omega_1 x_3 \\ \omega_1 x_2 - \omega_2 x_1 \end{pmatrix}$$

we get $\operatorname{rot} F \equiv \omega$. The velocity field F decribes a *rigid body rotation* around the axis $\omega \in \mathbb{R}^3$ with angular velocity $\frac{1}{2}|\omega|$, $|\cdot| = \|\cdot\|_2$.

(3) Define the vector field F by the scalar C^2–potential function $\varphi : B \to \mathbb{R}$, i.e., $F(x) = \nabla\varphi(x)$. Then $\operatorname{rot} F(x) \equiv 0$; in other words,

$$\text{gradient fields are irrotational, in short: } \operatorname{rot}\operatorname{grad} = 0.$$

(4) The vector field $F(x,y) = \frac{\omega}{2}(-y,x)^T$, $\omega \in \mathbb{R}$ describes a two-dimensional vortex around the origin with angular velocity $\frac{\omega}{2}$. Here $\operatorname{rot} F = \omega$. For every disc $B_r(0)$, $r > 0$, where $\partial B_r(0)$ will be considered in the mathematically positive sense,

$$\int_{B_r(0)} \operatorname{rot} F\, dx = \omega\pi r^2 = \int_{\partial B_r(0)} F \cdot dx\,.$$

It holds even for every compact rectangle $R \subset \mathbb{R}^2$

$$\int_R \operatorname{rot} F\, dx = \int_{\partial R} F \cdot dx\,.$$

Definition A function $\varphi : [a,b] \to \mathbb{R}$ is called a *function of bounded variation* (in short: *BV–function*, $\varphi \in BV[a,b]$,) provided there exists a constant $M \geq 0$ such that

$$\sum_{k=1}^{n} |\varphi(t_k) - \varphi(t_{k-1})| \leq M$$

for every partition $P : a = t_0 < t_1 < \ldots < t_n = b$ of $[a,b]$. In this case

$$V(\varphi) = \sup_P \sum_{k=1}^{n} |\varphi(t_k) - \varphi(t_{k-1})|$$

is called the *total variation* of φ on $[a,b]$.

Remark Obviously the Mean Value Theorem shows that $C^1[a,b] \subset BV[a,b]$, but note that

$$C[a,b] \not\subset BV[a,b] \not\subset C[a,b]\,.$$

The function $\varphi : [a,b] \to \mathbb{R}$ is of bounded variation iff the curve $\gamma(t) = \begin{pmatrix} t \\ \varphi(t) \end{pmatrix}$ is rectifiable; here, in contrast to the definiton of paths, we do not require that a curve is continuous.

Definition A set $B \subset \mathbb{R}^2$ is called a *BV–projected domain*, provided there exist continuous functions $\varphi_1 \leq \varphi_2 \in BV[a,b]$ and $\psi_1 \leq \psi_2 \in BV[c,d]$ such that

$$\begin{aligned} B &= \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b\,, \varphi_1(x) \leq y \leq \varphi_2(x)\} \\ &= \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d\,, \psi_1(y) \leq x \leq \psi_2(y)\}\,. \end{aligned}$$

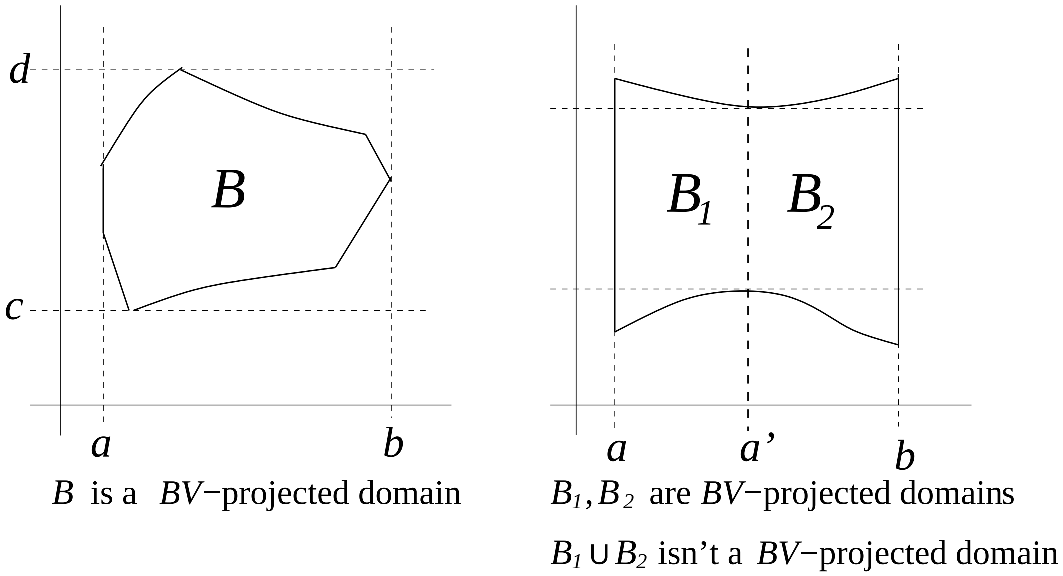

Fig. 10.1 BV–projected domains

Main Theorem 10.1 (Green's Theorem) *Let $B \subset \mathbb{R}^2$ be a BV–projected domain and let $\gamma = \partial B$ be the positively oriented boundary of B, i.e., the closed curve γ will be considered in the mathematically positive sense. Then for every C^1–vector field F defined on an open set $G \supset B$*

$$\boxed{\int_B \operatorname{rot} F(x)\, dx = \int_\gamma F(x) \cdot dx\,.}$$

Proof In a first step consider $F(x) = \begin{pmatrix} P(x) \\ 0 \end{pmatrix}$ yielding $\operatorname{rot} F(x) = -\partial_2 P(x)$. Since B is a BV–projected domain (with respect to the x–axis), we write $\gamma = \partial B$ in the form $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ using the rectifiable curves

$$\gamma_1(t) = \begin{pmatrix} t \\ \varphi_1(t) \end{pmatrix},\ t \in [a,b]; \qquad \gamma_2(t) = \begin{pmatrix} b \\ \varphi_1(b) + t(\varphi_2(b) - \varphi_1(b)) \end{pmatrix},\ t \in [0,1];$$

$$\gamma_3^-(t) = \begin{pmatrix} t \\ \varphi_2(t) \end{pmatrix},\ t \in [a,b]; \qquad \gamma_4^-(t) = \begin{pmatrix} a \\ \varphi_1(a) + t(\varphi_2(a) - \varphi_1(a)) \end{pmatrix},\ t \in [0,1]\,.$$

Here the notation γ_3^-, γ_4^- means that the curves γ_3, γ_4 originate from γ_3^-, γ_4^- by reversing the orientation, s. Fig. 10.2.

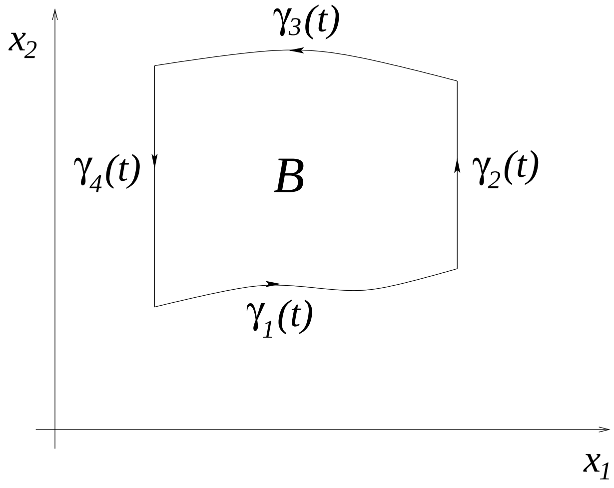

Fig. 10.2 The projected domain B

Now Theorem 8.14 and the Fundamental Theorem of Calculus yield

$$\begin{aligned}\int_B -\partial_2 P(x)\,dx &= \int_a^b \left(\int_{\varphi_1(x_1)}^{\varphi_2(x_1)} -\partial_2 P(x_1,x_2)\,dx_2\right) dx_1 \\ &= \int_a^b \big(P(x_1,\varphi_1(x_1)) - P(x_1,\varphi_2(x_1))\big)\,dx_1\,.\end{aligned}$$

For the first part of the right hand side we get that

$$\int_a^b P(t,\varphi_1(t))\,dt = \int_{\gamma_1} \begin{pmatrix} P \\ 0 \end{pmatrix} \cdot dx\,,$$

since for every partition $P: a = t_0 < t_1 < \ldots < t_n = b$ the Riemann sum

$$\sum_k \begin{pmatrix} P(t_k,\varphi_1(t_k)) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} t_k - t_{k-1} \\ \varphi_1(t_k) - \varphi_1(t_{k-1}) \end{pmatrix}$$

approximating $\int_{\gamma_1} (P,0)^T \cdot dx$ is a Riemann sum of $\int_a^b P(t,\varphi_1(t))\,dt$ as well. Analogously

$$-\int_a^b P(t,\varphi_2(t))\,dt = \int_{\gamma_3} (P,0)^T \cdot dx\,.$$

Since $\int_{\gamma_2} (P,0)^T \cdot dx$ and $\int_{\gamma_4} (P,0)^T \cdot dx$ vanish, we get for $F = (P,0)^T$ that

$$\int_B \mathrm{rot}\, F\,dx = \int_\gamma F \cdot dx\,.$$

Analogously we prove the assertion for the vector field $F(x) = (0, Q(x))^T$ using that B is a BV–projected domain w.r.t. the y–axis. Now the Theorem is completely proved. ∎

Remarks

(1) The meaning of Green's Theorem becomes evident when considering the projected domain B as a union $B = \bigcup_i Q_i$ of many small non-overlapping squares Q_i. Replacing $F(x)$ on Q_i by its linear approximation $F(x) = F_i + a_i x + b_i x^\perp$ with constant $a_i, b_i \in \mathbb{R}$ and $x^\perp = (-x_2, x_1)^T$, we get that

$$\int_{Q_i} \operatorname{rot} F\, dx = \int_{Q_i} 2b_i\, dx = \int_{\partial Q_i} F \cdot dx\,,$$

since $F_i + a_i x = \nabla\big(F_i \cdot x + \frac{a_i}{2}|x|^2\big)$ has a potential function; consequently $\operatorname{rot}(F_i + a_i x) = 0$ and $\int_{\partial Q_i}(F_i + a_i x)\cdot dx = 0$. Passing from Q_i to a neighboring square Q_j the integrals of the tangential parts of F along $\partial Q_i \cap \partial Q_j$ vanish due to the opposite orientations of the boundaries and the continuity of F:

$$\int_{\partial Q_i \cap \partial Q_j} F|_{Q_i} \cdot dx + \int_{\partial Q_i \cap \partial Q_j} F|_{Q_j} \cdot dx = 0\,.$$

Hence

$$\int_{Q_i \cup Q_j} \operatorname{rot} F\, dx \doteq \int_{\partial(Q_i \cup Q_j)} F \cdot dx$$

and, summing up, even $\int_B \operatorname{rot} F\, dx = \int_{\partial B} F \cdot dx$. Here the first integral yields the overall vorticity integrated on B and the second integral yields the tangential flux of F along ∂B.

(2) For $F = \begin{pmatrix} P \\ Q \end{pmatrix}$ Green's Theorem may also be written in the form

$$\int_B (\partial_x Q - \partial_y P)\, d(x,y) = \int_{\partial B} (P\, dx + Q dy)\,.$$

(3) Let $P(x,y) = -y$, $Q(x,y) = x$ such that $\operatorname{rot}(P,Q)^T = 2$. Then Green's Theorem yields for a BV–projected domain B the formula

$$\boxed{|B| = \frac{1}{2}\int_{\partial B} (x\, dy - y\, dx)}$$

computing the area $|B|$ of B by a line integral on ∂B.

Corollary 10.2 *Let $G \subset \mathbb{R}^2$ be open and let $F : G \to \mathbb{R}^2$ be a C^1–vector field. Suppose that the set $B \subset G$ can be written as the union of two non-overlapping BV–projected domains with just one joint boundary component. Then*

$$\int_B \operatorname{rot} F(x)\, dx = \int_\gamma F(x) \cdot dx ,$$

where $\gamma = \partial B$ denotes the positively oriented boundary of B.

Supplement: *The assertion also holds when B is the union of finitely many non-overlapping sets $B_1, \ldots, B_N$ such that every B_j is a BV–projected domain w.r.t. some Euclidean coordinate system obtained by a suitable rotation if necesary and such that B_j and B_{j+1} have just one joint boundary component for every $1 \le j \le N-1$; furthermore B_N and B_1 are supposed to be disjoint.*

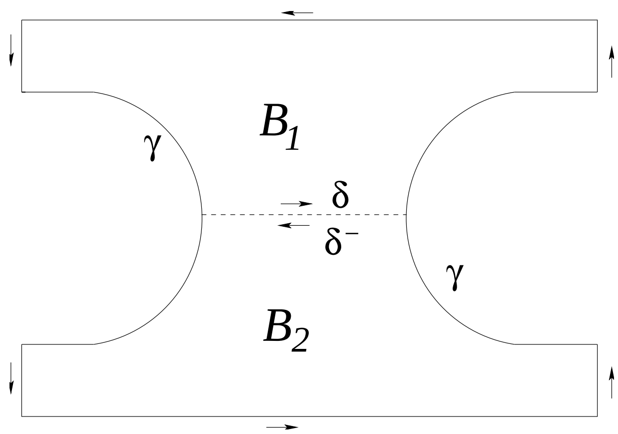

Fig. 10.3 Non-overlapping BV–projected domains

Proof Let $\delta = \partial B_1 \cap \partial B_2$ be the joint, rectifiable boundary component of the decomposition of $B = B_1 \cup B_2$ into BV–projected domains B_1 and B_2. By Theorem 10.1 and Corollary 8.7

$$\begin{aligned} \int_B \operatorname{rot} F\, dx &= \int_{B_1} \operatorname{rot} F\, dx + \int_{B_2} \operatorname{rot} F\, dx \\ &= \int_{\partial B_1} F \cdot dx + \int_{\partial B_2} F \cdot dx . \end{aligned}$$

Since the sum of the line integrals $\int_\delta F \cdot dx$ and $\int_{\delta^-} F \cdot dx$ vanishes, we are left with the line integral $\int_{\partial B} F \cdot dx$. ■

Definition

(1) Let $B \subset \mathbb{R}^n$ be open and let $F : B \to \mathbb{R}^n$ be a C^1–vector field. Then

$$\operatorname{div} F : B \to \mathbb{R} , \;\; \operatorname{div} F(x) = \sum_{i=1}^n \partial_i F_i(x)$$

(in short: $\operatorname{div} F = \nabla \cdot F$) is called the *divergence* of F.

(2) Let $B \subset \mathbb{R}^2$ be a BV–projected domain, the positively oriented boundary $\gamma = \partial B$ of which may be written as a piecewise C^1–path. If γ is continuously differentiable at t, we define the *exterior normal vector* $N(t) = (\gamma_2'(t), -\gamma_1'(t))$ pointing outward, and, if $\gamma'(t) \neq 0$, the *exterior normal unit vector*

$$n(t) = \frac{N(t)}{|N(t)|}\,;$$

as usual $|\cdot| = \|\cdot\|_2$.

Remark

(1) The divergence denotes the magnitude of a source or sink of a velocity field or of a force field. The velocity field $F(x) = \frac{1}{2\pi}\frac{x}{|x|^2}$, $x \in \mathbb{R}^2 \setminus \{0\}$, describes a solenoidal, i.e. divergence–free flow in $\mathbb{R}^2 \setminus \{0\}$ with a source of magnitude 1 at the origin. The same assertion holds for the vector field $F(x) = \frac{1}{4\pi}\frac{x}{|x|^3}$, $x \in \mathbb{R}^3 \setminus \{0\}$.

(2) A vector field of type $F(x) = \text{rot}\, A(x)$ in $\mathbb{R}^3$ is solenoidal, in short:

$$\text{div rot} = 0\,;$$

here the vector field $A(x)$ is called a *vector potential* of F.

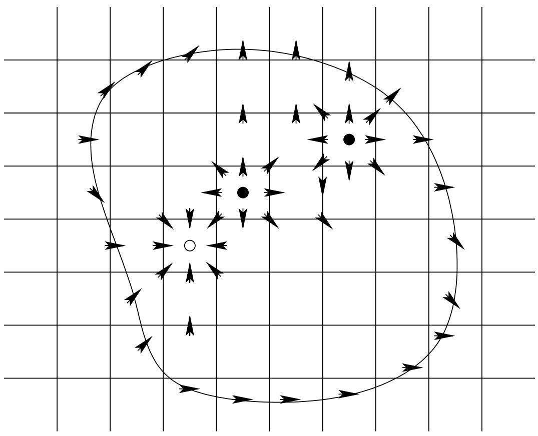

Fig. 10.4 Vector field with source, sink and flux on the boundary

Main Theorem 10.3 (Gauss' Theorem in $\mathbb{R}^2$) *Let $B \subset \mathbb{R}^2$ be a BV–projected domain the positively oriented boundary $\gamma = \partial B$ of which is a piecewise regular C^1–path. Then for a C^1–vector field $F : G \to \mathbb{R}^2$ defined on an open set $G \supset B$*

$$\boxed{\int_B \text{div } F(x)\, dx = \int_{\partial B} F(x) \cdot n(x)\, d\sigma(x)\,,}$$

where, using a parameterization of γ on $[a, b]$,

$$\int_{\partial B} F \cdot n\, d\sigma := \int_a^b F(\gamma(t)) \cdot n(\gamma(t))|\gamma'(t)|\, dt\,.$$

Remark

(1) The proof will show that Theorem 10.3 holds under similar assumptions as 10.1 or 10.2.

(2) The new "line integral" $\int_\gamma F \cdot n\, d\sigma$ is also written in the form $\int_\gamma F \cdot n\, do$ or $\int_\gamma F \cdot n\, dS$ using the area- or curve length element $do = d\sigma = dS = |\gamma'(t)|\, dt$.

(3) In Gauß' Theorem $\int_B \text{div } F\, dx$ yields the overall magnitude of the source (or sink) $\text{div}\, F$ of F integrated on B while $\int_{\partial B} F \cdot n\, d\sigma$ denotes the outward net flux of F through ∂B.

Proof For $\widetilde{F}(x) = (-F_2(x),\ F_1(x))^T$

$$\text{rot } \widetilde{F} = \partial_1 \widetilde{F}_2 - \partial_2 \widetilde{F}_1 = \partial_1 F_1 + \partial_2 F_2 = \text{div } F.$$

Then 10.1 or 10.2 yield

$$\begin{aligned}
\int_B \text{div } F\, dx &= \int_B \text{rot } \widetilde{F}\, dx = \int_\gamma \widetilde{F} \cdot dx = \int_a^b \widetilde{F}(\gamma(t)) \cdot \gamma'(t)\, dt \\
&= \int_a^b (-F_2 \cdot \gamma_1' + F_1 \cdot \gamma_2')\, dt = \int_a^b F \cdot N\, dt \\
&= \int_a^b F \cdot n\, |\gamma'|\, dt = \int_{\partial B} F \cdot n\, d\sigma\,.
\end{aligned}$$

□

11 Integral Theorems in $\mathbb{R}^3$

Before generalizing Green's and Gauß' Theorem to three dimensions we define area and surface integrals in $\mathbb{R}^3$.

Definition Let $K \neq \emptyset$ be a compact Jordan–measurable subset of $\mathbb{R}^2$, let $G \supset K$ be open and let $\phi : G \to \mathbb{R}^3$ be a C^1–map. Then

$$S = \{\phi(u) : u \in K\}$$

is called a *parametric surface in* $\mathbb{R}^3$ *with parametrization* (ϕ, K) *and domain of parameters* K. If for every parameter $u \in G$ the functional matrix

$$D\phi(u) = \begin{pmatrix} \partial_1\phi_1 & \partial_2\phi_1 \\ \partial_1\phi_2 & \partial_2\phi_2 \\ \partial_1\phi_3 & \partial_2\phi_3 \end{pmatrix}(u)$$

has rank 2, i.e., the column vectors of $D\phi(u)$ are linearly independent, the parametrization is called *regular* or *non-degenerate* at $u \in G$.

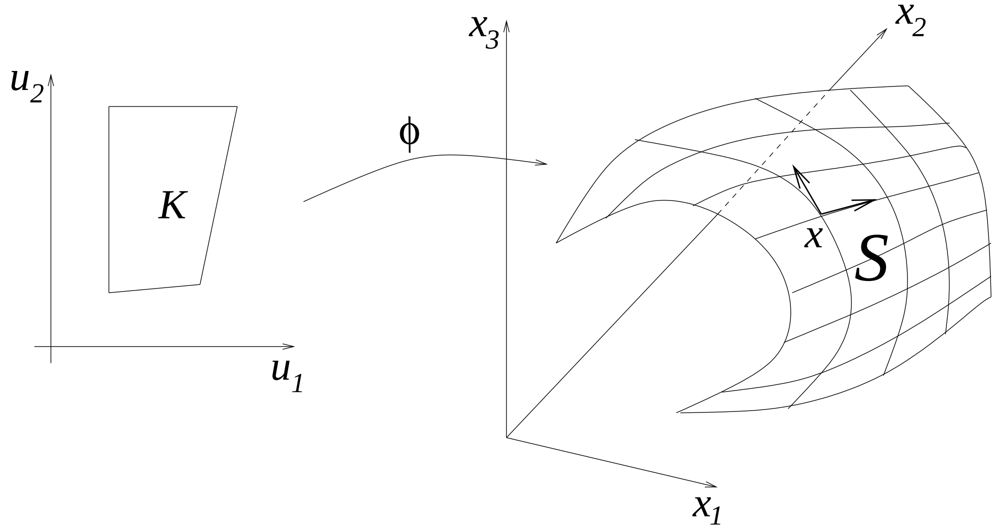

Fig. 11.1 A domain of parameters in $\mathbb{R}^3$

Remark The condition on the rank of $D\phi$ guarantees that at every point $x = \phi(u)$ of the parametric surface S two linearly independent tangential vectors, namely $\partial_1\phi(u)$ and $\partial_2\phi(u)$, exist. Using the vector product of $\partial_1\phi$ and $\partial_2\phi$ we will define the normal vector at x.

The *vector product*, also called *wedge product or exterior product*,

$$\wedge : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 , \quad x, y \longmapsto x \wedge y = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} ,$$

has the following properties (for all $x, y, z \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$):

(i) $x \wedge y = -y \wedge x$, $x \wedge x = 0$

(ii) $(\lambda x) \wedge y = x \wedge (\lambda y) = \lambda(x \wedge y)$

(iii) $x \wedge (y+z) = x \wedge y + x \wedge z$, $\quad (x+y) \wedge z = x \wedge z + y \wedge z$

(iv) $|x \wedge y| = |x||y||\sin \measuredangle(x,y)|$

Since $|y||\sin \measuredangle(x,y)|$ is the height of the parallelogram spanned by the vectors x and y, the Euclidean length $|x \wedge y|$ of $x \wedge y$ is the area of this parallelogram.

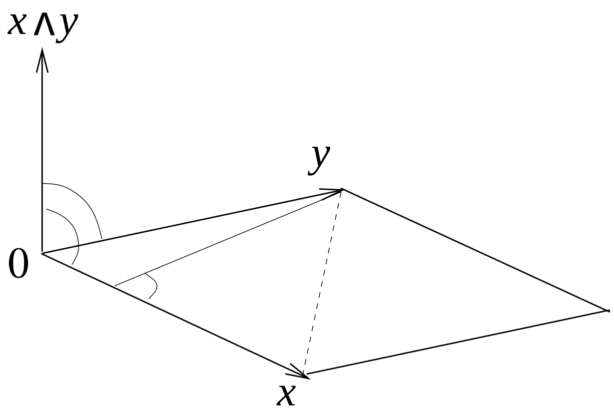

Fig. 11.2 The vector product

Hence $\frac{1}{2}|x \wedge y|$ is the area of the triangle with vertices $0, x$ and y.

(v) $x \wedge y = 0 \Leftrightarrow x$ and y are linearly dependent

(vi) $< x, x \wedge y > = < y, x \wedge y > = 0$

The vector $x \wedge y$ is orthogonal to x and y. Furthermore, every vector orthogonal to x and y is a scalar multiple of $x \wedge y$.

Definition Let $S \subset \mathbb{R}^3$ be a parametric surface in $\mathbb{R}^3$ with parametrization (ϕ, K), $\phi \in C^1(G)$, $G \supset K$ open. Then we define at $x = \phi(u) \in S$ the tangential vectors $\partial_1\phi(u)$ and $\partial_2\phi(u)$ as well as the *normal vector*

$$N(u) = \partial_1\phi(u) \wedge \partial_2\phi(u)$$

and the *normal unit vector*

$$n(u) = \begin{cases} \frac{N(u)}{|N(u)|} & \text{, falls } N(u) \neq 0 \\ 0 & \text{, falls } N(u) = 0\,. \end{cases}$$

Examples

(1) Let $f \in C^1(G)$ and let $S \subset \mathbb{R}^3$ denote the surface given as the graph of f, i.e., $S = \{\phi(u) : u \in K\}$ with $\phi(u) = (u_1, u_2, f(u))^T$. Then

$$D\phi(u) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_1 f & \partial_2 f \end{pmatrix}$$

has rang 2 at every $u \in K$ and

$$N(u) = \begin{pmatrix} 1 \\ 0 \\ \partial_1 f \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ \partial_2 f \end{pmatrix} = \begin{pmatrix} -\partial_1 f \\ -\partial_2 f \\ 1 \end{pmatrix} .$$

(2) The unit sphere $\partial U_1(0)$ in $\mathbb{R}^3$ may be parameterized over $K = [0, 2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \subset G = \mathbb{R}^2$ by

$$\phi(u) = \begin{pmatrix} \cos u_1 \cos u_2 \\ \sin u_1 \cos u_2 \\ \sin u_2 \end{pmatrix} .$$

Then

$$D\phi(u) = \begin{pmatrix} -\sin u_1 \cos u_2 & -\cos u_1 \sin u_2 \\ \cos u_1 \cos u_2 & -\sin u_1 \sin u_2 \\ 0 & \cos u_2 \end{pmatrix} , \quad N(u) = \begin{pmatrix} \cos u_1 \cos^2 u_2 \\ \sin u_1 \cos^2 u_2 \\ \sin u_2 \cos u_2 \end{pmatrix} ,$$

yielding $N = \cos u_2 \cdot \phi(u_1, u_2)$ and $|N| = \cos u_2$. Evidently N is parallel to the vector ϕ; however, the unit sphere is degenerate at the north and south pole ($u_2 = \frac{\pi}{2}$ and $u_2 = -\frac{\pi}{2}$, resp.) for this parametrization.

Remark Given a parametric surface S with parametrization (ϕ, K) and a point $x = \phi(u)$ on S the parallelogram

$$P = \{\phi(u) + s_1 \partial_1 \phi(u) + s_2 \partial_2 \phi(u) : 0 \le s_1 \le \varepsilon_1, 0 \le s_2 \le \varepsilon_2\}$$

linearly approximating S at x has the area

$$|\varepsilon_1 \partial_1 \phi(u) \wedge \varepsilon_2 \partial_2 \phi(u)| = \varepsilon_1 \varepsilon_2 |N(u)| .$$

Therefore, the term

$$do(u) = |N(u)|\, du$$

is called *surface element* in the definition of surface integrals below.

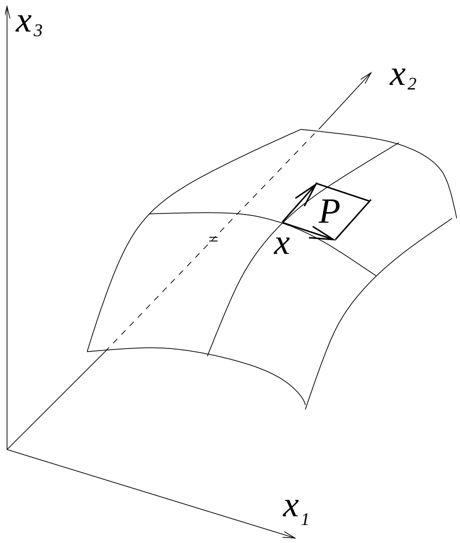

Fig. 11.3 Approximating parallelogram

Definition Let $S \subset \mathbb{R}^3$ be a parametric surface with parametrization (ϕ, K) and let $f : \phi(K) \to \mathbb{R}$ continuous. Then

$$\boxed{\int_S f\, do := \int_K f(\phi(u))|N(u)|\, du}$$

is called the *surface integral of f on the surface S*. For $f \equiv 1$

$$|S| = \int_S 1\, do = \int_K |N(u)|\, du$$

is called the *area* of S.

Since a surface S has different parametrizations, we have to show that the previous definition of $\int_S f\, do$ is independent of the parametrization (cf. the analogous result for the arc length and for line integrals).

Lemma 11.1 *Let $S \subset \mathbb{R}^3$ be a surface with parametrizations (ϕ, K), $G \supset K$ open, and (ϕ', K'), $G' \supset K'$ open, and let $g : G' \to G$ be an injective C^1–map with $g(K') = K$ and $\phi'(s) = \phi(g(s))$ on G'. The normal vectors are denoted by $N(u)$ at $u \in K$ and by $N'(s)$ at $s \in K'$. Moreover, let* $\det Dg$ *be either strictly positive or strictly negative on G'. Then*

$$\int_K f(\phi(u))|N(u)|\, du = \int_{K'} f(\phi'(s))|N'(s)|\, ds\,.$$

Proof Since

$$\begin{aligned}\partial_1\phi' &= \partial_1\phi\cdot\partial_1 g_1 + \partial_2\phi\cdot\partial_1 g_2\,,\\ \partial_2\phi' &= \partial_1\phi\cdot\partial_2 g_1 + \partial_2\phi\cdot\partial_2 g_2\,,\end{aligned}$$

the normal vector $N'(s)$ at $x=\phi'(s)=\phi(u)$, $u=g(s)$, equals

$$\begin{aligned}N'(s) &= \partial_1\phi'\wedge\partial_2\phi'\\ &= (\partial_1 g_1)(\partial_2 g_1)\underbrace{\partial_1\phi\wedge\partial_1\phi}_{=0}+(\partial_1 g_1)(\partial_2 g_2)\underbrace{\partial_1\phi\wedge\partial_2\phi}_{=N(u)}\\ &\quad+(\partial_1 g_2)(\partial_2 g_1)\underbrace{\partial_2\phi\wedge\partial_1\phi}_{=-N(u)}+(\partial_1 g_2)(\partial_2 g_2)\underbrace{\partial_2\phi\wedge\partial_2\phi}_{=0}\\ &= (\partial_1 g_1\cdot\partial_2 g_2-\partial_1 g_2\cdot\partial_2 g_1)\,N(u)\,,\end{aligned}$$

hence $|N'(s)| = |\det Dg(s)|\,|N(u)|$. Then by the Change of Variable Formula

$$\begin{aligned}\int_K f(\phi(u))\,|N(u)|\,du &= \int_{K'} f(\phi(g(s)))|\det Dg(s)|\,|N(g(s))|\,ds\\ &= \int_{K'} f(\phi'(s))\,|N'(s)|\,ds\,.\end{aligned}$$

□

Example Consider the parametrization of the unit sphere $S=\partial U_1(0)$ in $\mathbb{R}^3$ as above. Since $K=[0,2\pi]\times[-\frac{\pi}{2}\,,\,\frac{\pi}{2}]$ and $|N(u)|=\cos u_2$,

$$|\partial U_1(0)| = \int_K \cos u_2\,du = \int_0^{2\pi} du_1\cdot\int_{-\pi/2}^{\pi/2}\cos u_2\,du_2 = 4\pi\,.$$

Analogously we get that $|\partial U_R(0)| = 4\pi R^3$.

Main Theorem 11.2 (Stokes' Integral Theorem in $\mathbb{R}^3$) *Let $G\subset\mathbb{R}^2$ be open and let $K\subset G$ be a BV–projected domain the boundary ∂K of which has a piecewise C^1–parametrization γ. Furthermore, let $\phi: G\to\mathbb{R}^3$ be a C^2–map and let $S=\{\phi(u): u\in K\}$ be a parametric surface in $\mathbb{R}^3$, the 'boundary' of which is defined by $\partial S=\phi(\partial K)$.*

If $F: U\to\mathbb{R}^3$ is a C^1–vector field on an open set U containing S, then

$$\boxed{\int_S \operatorname{rot} F\cdot n\,do = \int_{\partial S} F\cdot dx\,.}$$

Using parametrizations Stokes' Integral Theorem reads as follows:

$$\int_K \operatorname{rot} F(\phi(u))\cdot N(u)\,du = \int_{\phi(\gamma)} F\cdot dx\,.$$

Proof Consider a vector field $F=(P,0,0)^T$; the proof for $F=(0,Q,0)^T$ and $F=(0,0,R)^T$ will follow the same line.

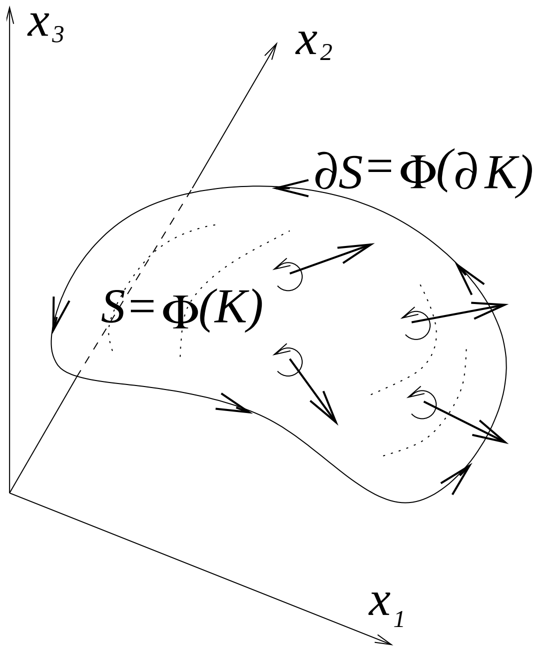

Fig. 11.4 Concerning Stokes' Integral Theorem

First the line integral $\int_{\partial S} F \cdot dx = \int_{\phi(\gamma)} F \cdot dx$ along the path $\phi(\gamma) \subset \mathbb{R}^3$ will be rewritten as a line integral along the path $\gamma \subset \mathbb{R}^2$. Given a piecewise C^1–parametrization $\gamma : [0,1] \to \partial K$ of ∂K we get that

$$\begin{aligned}\int_{\phi(\gamma)} \begin{pmatrix} P \\ 0 \\ 0 \end{pmatrix} \cdot dx &= \int_0^1 P(\phi(\gamma(t)))\,(\phi_1 \circ \gamma)'(t)\,dt \\ &= \int_0^1 (P \circ \phi)(\gamma(t))\,(\partial_1\phi_1 \cdot \gamma_1' + \partial_2\phi_1 \cdot \gamma_2')(t)\,dt \\ &= \int_\gamma P \circ \phi \begin{pmatrix} \partial_1\phi_1 \\ \partial_2\phi_1 \end{pmatrix} \cdot dx\end{aligned}$$

and, using Green's 10.1, that

$$\int_\gamma (P \circ \phi)\nabla\phi_1 \cdot dx = \int_K \operatorname{rot}\big((P \circ \phi)\nabla\phi_1\big)\,du\,.$$

By the definition of the 'scalar' rotation in $\mathbb{R}^2$

$$\begin{aligned}
\mathrm{rot}\,((P\circ\phi)\nabla\phi_1) &= \partial_1((P\circ\phi)\partial_2\phi_1) - \partial_2((P\circ\phi)\partial_1\phi_1) \\
&= (\partial_1(P\circ\phi))\partial_2\phi_1 - (\partial_2(P\circ\phi))\partial_1\phi_1 + (P\circ\phi)\underbrace{[\partial_1\partial_2\phi_1 - \partial_2\partial_1\phi_1]}_{=0} \\
&= (\partial_1 P\cdot\partial_1\phi_1 + \partial_2 P\cdot\partial_1\phi_2 + \partial_3 P\cdot\partial_1\phi_3)\partial_2\phi_1 \\
&\qquad - (\partial_1 P\cdot\partial_2\phi_1 + \partial_2 P\cdot\partial_2\phi_2 + \partial_3 P\cdot\partial_2\phi_3)\partial_1\phi_1 \\
&= \partial_2 P\underbrace{(\partial_1\phi_2\cdot\partial_2\phi_1 - \partial_1\phi_1\cdot\partial_2\phi_2)}_{=-N_3} + \partial_3 P\underbrace{(\partial_1\phi_3\cdot\partial_2\phi_1 - \partial_2\phi_3\cdot\partial_1\phi_1)}_{=N_2},
\end{aligned}$$

since $\phi_1 \in C^2$ and $N = \partial_1\phi\wedge\partial_2\phi$. Now the identity $\mathrm{rot}F = (0, \partial_3 P, -\partial_2 P)^T$ yields

$$\mathrm{rot}((P\circ\phi)\nabla\phi_1) = ((\mathrm{rot}F)\circ\phi)\cdot N\,.$$

Summarizing we get

$$\begin{aligned}
\int_{\partial S} F\cdot dx &= \int_\gamma (P\circ\phi)\nabla\phi_1\,dx &&= \int_K \mathrm{rot}((P\circ\phi)\nabla\phi_1)\,du \\
&= \int_K ((\mathrm{rot}F)\circ\phi)\cdot N\,du &&= \int_S \mathrm{rot}F\cdot n\,do\,.
\end{aligned}$$

□

Definition

(1) A compact set $V\subset\mathbb{R}^3$ is called *C^1–projected domain w.r.t. the x_1x_2–plane* iff there exist a compact set $K\subset\mathbb{R}^2$ and C^1–functions $\varphi_1\le\varphi_2$ on an open set containing K such that

$$V = \{(x_1,x_2,x_3) = (x',x_3)\in\mathbb{R}^3 : x'\in K\,,\ \varphi_1(x')\le x_3\le\varphi_2(x')\}$$

and ∂K is parametrized by a piecewise C^1–path. Analogously we define C^1–projected domains w.r.t. the x_2x_3– and the x_1x_3–plane.

(2) The compact set V is called *C^1–projected domain* iff V is a C^1–projected domain w.r.t. the x_1x_2–, the x_2x_3– and the x_1x_3–plane.

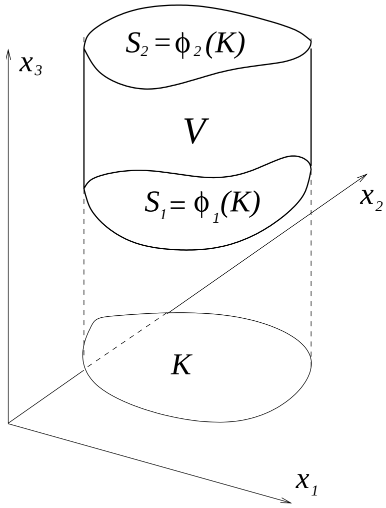

Fig. 11.5 A C^1–projected domain w.r.t. the x_1x_2–plane

Remark In (1) the 'upper lid' $S_2 = \phi_2(K)$ defines a regular C^1–surface in $\mathbb{R}^3$ with parametrization

$$\phi_2 : K \to \mathbb{R}^3 \, , \quad u = x' \mapsto \phi_2(x') = (x', \varphi_2(x')) \, .$$

The normal vector $N(x') = (-\partial_1\varphi_2, -\partial_2\varphi_2, 1)^T$ defines the exterior normal unit vector

$$n(x') = \frac{N(x')}{|N(x')|}$$

w.r.t. V. However, for the 'lower lid' $S_1 = \phi_1(K)$ the exterior normal unit vector (w.r.t. V) is $n(x') = -N(x')/|N(x')|$.

Main Theorem 11.3 (Gauss' Divergence Theorem in $\mathbb{R}^3$) *Let V be a C^1–projected domain in $\mathbb{R}^3$ and let F be a C^1–vector field on an open set containing V. Then*

$$\boxed{\int_V \operatorname{div} F \, dx = \int_{\partial V} F \cdot n \, do \, .}$$

Proof Consider $F = (0, 0, R)^T$. Since div $F = \partial_3 R$, by 8.14

$$\int_V \operatorname{div} F \, dx = \int_K \left(\int_{\varphi_1(x')}^{\varphi_2(x')} \partial_3 R(x', x_3) \, dx_3 \right) dx'$$

$$= \int_K \big(R(x', \varphi_2(x')) - R(x', \varphi_1(x'))\big) \, dx' \, .$$

On the upper lid $S_2 = \phi_2(K)$ we have $F \cdot N = R\, N_3 = R$. Thus

$$\int_K R(x', \varphi_2(x'))\, dx' = \int_K F \cdot N\, dx' = \int_{S_2} F \cdot n\, do\,;$$

analogously

$$-\int_K R(x', \varphi_1(x'))\, dx' = \int_{S_1} F \cdot n\, do\,.$$

On the remaining part of the boundary $\partial V \setminus S_1 \setminus S_2$, i.e., on

$$S_3 = \{(x', x_3) : x' \in \partial K,\, \varphi_1(x') \le x_3 \le \varphi_2(x')\}\,,$$

the exterior normal unit vector n exists (except for finitely many segments $\{x'\} \times [\varphi_1(x'),\, \varphi_2(x')]$ with $x' \in \partial K$, where the parametrization of ∂K is not differentiable). Since $n_3 = 0$ for $x \in S_3$, we see that $F \cdot n = Rn_3 = 0$ on S_3.

Summarizing we get that

$$\int_V \operatorname{div} F\, dx = \sum_{j=1}^{3} \int_{S_j} F \cdot n\, do = \int_{\partial V} F \cdot n\, do\,.$$

An analogous proof will be used when $F = (P, 0, 0)$ and when $F = (0, Q, 0)$. □

Corollary 11.4 *(Integration by parts in $\mathbb{R}^3$) Let $V \subset \mathbb{R}^3$ be a C^1–projected domain and let u, v be C^1–functions on an open set containing V. Then for $i = 1, 2, 3$*

$$\boxed{\int_V u\, \partial_i v\, dx = -\int_V (\partial_i u)\, v\, dx + \int_{\partial V} u\, v\, n_i\, do\,.}$$

Proof Applying Theorem 11.3 to the vector field $F(x) := u\, v\, e_i$ we get the assertion, since $\operatorname{div} F = u\, \partial_i v + (\partial_i u)\, v$ and $F \cdot n = u\, v\, n_i$. □

Remarks

(1) Theorem 11.3 and Corollary 11.4 hold under weaker assumptions on V: The compact set V can be written as the union of finitely many non-overlapping C^1–projected domains. The lids of these projected domains are assumed to be ‘piecewise C^1 ’ only.

(2) An essentially weaker assumptions on V has a *local* character: At every point $x \in \partial V$ there exists a neighborhood $U \subset \mathbb{R}^3$ such that – after a suitable rotation of the coordinate system – $U \cap \partial V$ may be written as graph of a Lipschitz continuous function. In this case the normal vector exists only ’almost everywhere’.

(3) Theorem 11.3 and Corollary 11.4 hold for every space dimension $n \ge 2$.