7 The Riemann Integral on Rectangles

Definition

(1) Let $J_1, \ldots, J_n \neq \emptyset$ be compact intervals of $\mathbb{R}$. Then $R = J_1 \times \ldots \times J_n \subset \mathbb{R}^n$ is called a *(closed) n-rectangle* or simply *rectangle* of $\mathbb{R}^n$.

(2) Let P_i be a partition of $J_i = [a_i, b_i]$, to be more precise, let

$$P_i : a_i = c_{i0} < c_{i1} < \ldots < c_{ik_i} = b_i \,,$$

$k_i \in \mathbb{N}$, $i = 1, \ldots, n$. Then the set of all rectangles

$$[c_{1j_1}, c_{1j_1+1}] \times \cdots \times [c_{nj_n}, c_{nj_n+1}]$$

where $0 \le j_i \le k_i - 1$, $i = 1, \ldots, n$, defines the *partition* $P = P_1 \times \ldots \times P_n$ of the rectangle R.

(3) The *n–dimensional volume* of the rectangle R is defined by

$$\mathrm{vol}_n(R) := |R| := \prod_{i=1}^{n} (b_i - a_i).$$

Lemma 7.1 *Let $R \subset \mathbb{R}^n$ be a rectangle and let $P = \{S\}$ be a partition of R. Then*

$$|R| = \sum_{S \in P} |S| \,.$$

Proof Using the notation of the above definition

$$\sum_{S \in P} |S| = \sum_{j_1=0}^{k_1-1} \ldots \sum_{j_n=0}^{k_n-1} (c_{1j_1+1} - c_{1j_1}) \cdot \ldots \cdot (c_{nj_n+1} - c_{nj_n}) \,.$$

Writing the inner sum w.r.t. j_n in front of the term $(c_{nj_n+1} - c_{nj_n})$, we get

$$\sum_{j_1=0}^{k_1-1} \ldots \sum_{j_{n-1}=0}^{k_{n-1}-1} (\ldots) \cdot \ldots \cdot (\ldots)(b_n - a_n) \,.$$

Then by mathematical induction $\prod_{i=1}^{n} (b_i - a_i) = |R|$. ∎

For later use we recall some notions from topology of $\mathbb{R}^n$. Given $M \subset \mathbb{R}^n$, the *interior* $\overset{\circ}{M} = M^o$ is defined as the set of all interior points of M. It is easily seen that

$$M^o = \bigcup \{U \subset \mathbb{R}^n \,|\, U \subset M,\, U \text{ is open}\}$$

and that M^o is open. Furthermore, the *closure* $\overline{M}$ is defined as the union of M and of all its accumulation points. Obviously,

$$\overline{M} = \bigcup \{A \subset \mathbb{R}^n \mid M \subset A, A \text{ is closed}\}$$

is a closed set. Finally, the *boundary* ∂M is defined as the set of all points $x \in \mathbb{R}^n$ such that every neighborhood of x contains a point of M and of the complement of M. Thus for every $M \subset \mathbb{R}^n$

$$M^o = M \setminus \partial M \ \subset \ M \ \subset \ \overline{M} = M \cup \partial M \,.$$

Furthermore, $\partial M = \overline{M} \setminus M^o$ is closed; when M is bounded, ∂M is even compact.

Definition

(1) Let $f : R \to \mathbb{R}$ be a bounded function and let $P = \{S\}$ be a partition of the rectangle R. Then

$$L_R(P, f) = L(P, f) := \sum_{S \in P} \inf_S \, f \cdot |S|$$

and

$$U_R(P, f) = U(P, f) := \sum_{S \in P} \sup_S f \cdot |S|$$

are the *lower* and *upper Riemann sum* of f on R, resp.

(2) A partition $P' = \{S'\}$ of R is called a *refinement* of P, if for every $S' \in P'$ there exists an $S \in P$ such that $S' \subset S$.

Lemma 7.2 *Given a refinement P' of P*

$$L(P, f) \le L(P', f) \le U(P', f) \le U(P, f) \,.$$

Proof We only prove the first inequality, since the second one is trivial and the third one may be proved analogously. By the definition of a refinement, for every rectangle $S \in P$ there exist finitely many rectangles $S'_1, \ldots, S'_k \in P'$ such that

$$S = \bigcup_{j=1}^{k} S'_j \,, \quad |S| = \sum_{j=1}^{k} |S'_j|$$

(cf. Lemma 7.1). Then

$$\inf_S \, f \cdot |S| = \sum_{j=1}^{k} |S'_j| \cdot \inf_S f \le \sum_{j=1}^{k} |S'_j| \, \inf_{S'_j} f \,.$$

Summing over $S \in P$ we get $L(P, f) \le L(P', f)$. ∎

Definition

(1) Let $R \subset \mathbb{R}^n$ be a rectangle and let $f : R \to \mathbb{R}$ be bounded. Then

$$\int_{*R} f(x)\,dx := \sup_P L_R(P, f)$$

and

$$\int^*_R f(x)\,dx := \inf_P U_R(P, f)$$

are called *(Riemann's) lower–* and *upper integral* of f on R, resp. Here, for $\sup_P$ and $\inf_P$, all partitions P of R are considered.

(2) If $\int_{*R} f(x)\,dx = \int^*_R f(x)\,dx$, then f is called *Riemann integrable* on R, and its *Riemann integral* on R is defined by

$$\int_R f(x)\,dx := \int^*_R f(x)\,dx\,.$$

Example The characteristic function $f = \chi_Q$ of a rectangle $Q \subset \mathbb{R}^n$ is Riemann integrable on every rectangle $R \supset Q$, and

$$\int_R \chi_Q(x)\,dx = |Q|\,.$$

By Lemma 7.2 we get the following integrability criterion:

Lemma 7.3 (Riemann's integrability criterion) *A bounded function $f : R \to \mathbb{R}$ is integrable iff for every $\varepsilon > 0$ there exists a partition P of R such that*

$$U(P, f) - L(P, f) < \varepsilon\,.$$

Theorem 7.4 (1) *The Riemann integrable functions on a rectangle $R \subset \mathbb{R}^n$ define a vector space.*

(2) *The integral has the following properties:*

(i) *The map $f \mapsto \int_R f(x)\,dx$ is linear.*

(ii) *If $f \geq 0$, then $\int_R f(x)\,dx \geq 0$.*

(iii) *If $f \leq g$, then $\int_R f(x)\,dx \leq \int_R g(x)\,dx$.*

Proof We only prove the additivity of the integral; the assertion (ii) is trivial, (iii) is a consequence of (ii). Let f, g be integrable, let $\varepsilon > 0$, and let P, P' be partitions of R such that

$$U(P,f) - L(P,f) < \varepsilon, \quad U(P',g) - L(P',g) < \varepsilon\,, \tag{$*$}$$

cf. Lemma 7.3. Then there exists a further partition P'' of R such that $S \cap S' \in P''$ for all $S \in P$, $S' \in P'$ (if $(S \cap S')^o \neq \emptyset$), i.e. a refinement of P and of P'. Then, for $S \in P''$,

$$\inf_S f + \inf_S g \leq \inf_S (f+g) \leq \sup_S (f+g) \leq \sup_S f + \sup_S g$$

and consequently

$$L(P'',f) + L(P'',g) \leq L(P'',f+g) \leq U(P'',f+g) \leq U(P'',f) + U(P'',g)\,.$$

Since $(*)$ also holds for the finer partition P'',

$$U(P'',f+g) - L(P'',f+g) < 2\varepsilon\,.$$

Hence $f+g$ is integrable and $\int_R (f+g)(x)\,dx = \int_R f(x)\,dx + \int_R g(x)\,dx\,.$ ■

The computation of multi-dimensional integrals on rectangles may be reduced to the computation of one-dimensional integrals. Let $R \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ be compact rectangles, and let $f : R \times Q \to \mathbb{R}$ be integrable. Then, for every $x \in R$, consider the function

$$f_x : Q \to \mathbb{R}, \quad f_x(y) := f(x,y)\,.$$

If f_x is integrable for every $x \in R$ on Q, we may define the function $I : R \to \mathbb{R}$ by

$$I(x) := \int_Q f_x(y)\,dy\,.$$

Main Theorem 7.5 (Theorem of Fubini) *(Guido Fubini 1879 – 1943)*
Let $R \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ be compact rectangles, let $f : R \times Q \to \mathbb{R}$ be Riemann integrable and let $f_x = f(x,\cdot)$ be Riemann integrable on Q for every $x \in R$. Then $I(x) = \int_Q f_x(y)\,dy$ is Riemann integrable on R , and

$$\boxed{\int_{R\times Q} f(x,y)\,d(x,y) = \int_R \Big(\int_Q f(x,y)\,dy\Big)\,dx\,.}$$

Proof For given $\varepsilon > 0$ Lemma 7.3 yields a partition P' of R and P'' of Q such that the partition

$$P := P' \times P'' = \{S' \times S'' : S' \in P',\ S'' \in P''\}$$

of $R \times Q$ satisfies the estimate

$$U_{R\times Q}(P,f) - L_{R\times Q}(P,f) < \varepsilon\,.$$

Then

$$\begin{aligned} L_{R\times Q}(P,f) &= \sum_{S\in P} \inf_S f\cdot|S| = \sum_{S'}\sum_{S''} \inf_{S'\times S''} f\cdot|S'|\,|S''| \\ &= \sum_{S'}\sum_{S''} \inf_{x\in S'}\big(\inf_{S''} f_x\big)\cdot|S'|\,|S''|\,. \end{aligned}$$

Since $\sum_{S''}\inf_{S'}(\ldots)\le \inf_{S'}\sum_{S''}(\ldots)$, we may continue as follows:

$$\begin{aligned} L_{R\times Q}(P,f) &\le \sum_{S'} \inf_{x\in S'} \underbrace{\Big(\sum_{S''}\inf_{S''} f_x\cdot|S''|\Big)}_{\le I(x)}|S'| \\ &\le L_R\big(P', I(\cdot)\big) \\ &\le U_R\big(P', I(\cdot)\big) \\ &\le \cdots \le U_{R\times Q}(P,f)\,. \end{aligned}$$

By the choice of the partition P and due to Lemma 7.3 we get that $I(x)$ is integrable on R. Moreover,

$$\int_R I(x)\,dx = \int_{R\times Q} f(x,y)\,d(x,y)\,.$$

■

Corollary 7.6 *Let $f: R\times Q\to\mathbb{R}$ be Riemann integrable. Assume that for every $x\in R$ the Riemann integral $\int_Q f(x,y)\,dy$ and that for every $y\in Q$ the Riemann integral $\int_R f(x,y)\,dx$ exist. Then*

$$\boxed{\int_R\Big(\int_Q f(x,y)\,dy\Big)\,dx = \int_{R\times Q} f(x,y)\,d(x,y) = \int_Q\Big(\int_R f(x,y)\,dx\Big)\,dy\,.}$$

Remark

(1) Under the assumptions of Corollary 7.6 the value of the iterated integrals $\int_R\int_Q$ and $\int_Q\int_R$ are independent of the order.

(2) Under suitable assumptions (e.g. for continuous functions, see below) Corollary 7.6 admits the calculation of the integral of f over the rectangle $R=[a_1,b_1]\times\cdots\times[a_n,b_n]$ by the calculation of n one-dimensional integrals:

$$\int_R f(x)\,dx = \int_{a_n}^{b_n}\Big(\int_{a_{n-1}}^{b_{n-1}}\cdots\big(\int_{a_1}^{b_1} f(x_1,\ldots,x_{n-1},x_n)\,dx_1\big)\cdots dx_{n-1}\Big)\,dx_n\,.$$

(3) The Riemann integrability of f over $R\times Q$ does not imply that f_x is Riemann integrable for every $x\in R$, see Theorem 7.8 below.

(4) Fubini's Theorem holds under much weaker assumptions when using another notion of integrability (Lebesgue's integral theory).

Definition

(1) A set $M \subset \mathbb{R}^n$ is called a *set of (Lebesgue) measure zero* iff for every $\varepsilon > 0$ there exist *countably many* (!) closed rectangles $(R_i)_{i\in\mathbb{N}}$ such that

$$M \subset \bigcup_{i=1}^{\infty} R_i \quad \text{and} \quad \sum_{i=1}^{\infty} |R_i| < \varepsilon\,.$$

[Obviously the rectangles may be chosen to be open since for every closed rectangle R and for every $\delta > 0$ there exists an open rectangle $\mathring{R}_\delta$ such that $\mathring{R} \subset R \subset \mathring{R}_\delta$ and $|\mathring{R}_\delta| := |R_\delta| = (1+\delta)|R|]$.

(2) A set $M \subset \mathbb{R}^n$ is called a *set of Jordan measure zero* iff for every $\varepsilon > 0$ there exist *finitely many* (!) closed rectangles $R_1, \ldots, R_N \subset \mathbb{R}^n$ ($N = N(\varepsilon, M) \in \mathbb{N}$) such that

$$M \subset \bigcup_{i=1}^{N} R_i \quad \text{and} \quad \sum_{i=1}^{N} |R_i| < \varepsilon\,.$$

Lemma 7.7 (1) *M set of Jordan measure zero $\Rightarrow$ M set of Lebesgue measure zero.*

(2) *M_j set of Lebesgue measure zero for every $j \in \mathbb{N} \Rightarrow \bigcup_{j=1}^{\infty} M_j$ set of Lebesgue measure zero.*

(3) *M_j set of Jordan measure zero for $j = 1, \ldots, N \Rightarrow \bigcup_{j=1}^{N} M_j$ set of Jordan measure 0.*

(4) *$M \subset \mathbb{R}^n$ compact (!) set of Lebesgue measure zero $\Rightarrow$ M set of Jordan measure zero.*

Proof The assertion (1) is trivial. To prove (2) consider a set of Lebesgue measure zero M_j, $j \in \mathbb{N}$ and let $\varepsilon > 0$. Then for every M_j there exist rectangles R_j^i, $i \in \mathbb{N}$, such that

$$M_j \subset \bigcup_i R_j^i\,, \quad \sum_{i=1}^{\infty} |R_j^i| < \frac{\varepsilon}{2^j}\,, \quad j \in \mathbb{N}\,.$$

Consequently, $M = \bigcup_j M_j$ is covered by the union of the countably many rectangles R_j^i, $j \in \mathbb{N}$, $i \in \mathbb{N}$. Moreover, for every (finite) family of R_j^i's

$$\sum_{i,j} |R_j^i| \le \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |R_j^i| \le \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon\,.$$

Thus $\sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} |R^i_j| \le \varepsilon$ independently of the enumeration of (R^i_j).

(3) is proved analogously to (2). In (4) consider for every $\varepsilon > 0$ open (!) rectangles $(R_i)_{i\in\mathbb{N}}$ such that $M \subset \bigcup_i R_i$ and $\sum_i |R_i| < \varepsilon$. Since M is compact, there exist finitely many rectangles $R_{i_1}, \ldots, R_{i_N}$ such that $M \subset \bigcup_{j=1}^N R_{i_j}$. For these R_{i_j} we get that $\sum_{j=1}^N |R_{i_j}| < \varepsilon$. ■

Consequence *Finite* subsets of $\mathbb{R}^n$ are sets of Jordan measure zero, *countable* subsets such as $\mathbb{Q}^n \subset \mathbb{R}^n$ are sets of Lebesgue measure zero. The boundary of a rectangle is a set of Jordan measure zero: For a lateral side of R we get that

$$\{a_1\} \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset R' := [a_1 - \varepsilon, a_1 + \varepsilon] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

and $|R'| \le 2\varepsilon \cdot \prod_{j=2}^n (b_j - a_j)$ for every $\varepsilon > 0$.

Main Theorem 7.8 (**Lebesgue's Integrability Criterion for Riemann Integrals**). *Let $R \subset \mathbb{R}^n$ be a rectangle. A bounded function $f : R \to \mathbb{R}$ is Riemann integrable on R iff f is* **almost everywhere continuous**, *i.e., there exists a set of Lebesgue measure zero $M \subset R$ such that f is continuous at all points of $R \backslash M$:*

$$\forall x \in R\backslash M \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \; : \; |f(x) - f(y)| < \varepsilon \quad \forall y \in B_\delta(x) \cap R\,.$$

Proof: „$\Leftarrow$" Let $M \subset R$ be the set (of Lebesgue measure zero) of all points of discontinuity of f and let $\varepsilon > 0$. Then there exist countably many open (!) rectangles $(R_j)_{j\in\mathbb{N}}$ such that

$$M \subset \bigcup_{j=1}^\infty R_j \quad \text{and} \quad \sum_{j=1}^\infty |R_j| < \varepsilon\,.$$

Furthermore, at every point of continuity $x \in R\backslash M$ of f there exists an open (!) rectangle U_x such that $x \in U_x$ and

$$|f(x') - f(x'')| < \varepsilon \quad \forall x', x'' \in \overline{U}_x \cap R\,.$$

Consequently $R \subset \bigcup_j R_j \cup \bigcup_{x\in R\backslash M} U_x$. Since R is compact, we find finitely many R_j, $1 \le j \le N$, and $U_i = U_{x_i}$, $1 \le i \le N$, which cover R:

$$R \subset \bigcup_{j=1}^N R_j \cup \bigcup_{i=1}^N U_i\,.$$

To satisfy Riemann's integrability criterion in Lemma 7.3 choose a partition P of R such that every $S \in P$ is completely contained in at least a set $\overline{R}_j$ or $\overline{U}_i$. Hence

$$U(P,f) - L(P,f) = \sum_{S\in P} (\sup_S f - \inf_S f) \cdot |S| =: \textstyle\sum_1 + \sum_2\,,$$

where

$$\begin{aligned}\textstyle\sum_1 &= \text{Sum of all terms such that } S \text{ is contained in } \overline{R}_j \\ \textstyle\sum_2 &= \text{Sum of all other terms}.\end{aligned}$$

Now we estimate $\sum_1$ and $\sum_2$ as follows:

$$\begin{aligned}\textstyle\sum_1 &\le 2\|f\|_\infty \sum_j \sum_{S\subset\overline{R}_j} |S| \le 2\|f\|_\infty \sum_j |R_j| < 2\varepsilon\|f\|_\infty\,, \\ \textstyle\sum_2 &\le \varepsilon \sum_{S\in P} |S| = \varepsilon|R|\,;\end{aligned}$$

here, for $\sum_2$, we use the fact that every set S is completely contained in one of the sets $\overline{U}_i$ implying $\sup_S f - \inf_S f \le \varepsilon$. Summarizing both estimates we conclude that $U(P,f) - L(P,f) < \varepsilon(2\|f\| + |R|)$ and that f is Riemann integrable. (■)

„⇒" For the Riemann integrable function $f : R \to \mathbb{R}$ consider the set M of all points of discontinuity. Obviously, f is discontinuous at x iff

$$w(x) := \lim_{\delta\to 0+} \sup\{|f(x') - f(x'')| : x', x'' \in B_\delta(x) \cap R\} > 0\,;$$

here $B_\delta(x)$ denotes the ball with center x and radius δ (note that the expression $\sup\{\ldots\}$ is increasing in $\delta > 0$; thus the limit for $\delta \to 0+$ exists). Hence

$$M = \{x \in R : w(x) > 0\} = \bigcup_{k=1}^{\infty} \{x \in R : w(x) \ge \frac{1}{k}\} =: \bigcup_{k=1}^{\infty} M_k\,.$$

By Lemma 7.7 (2) it suffices to prove the following claim:

Claim: $M_k = \{x \in R : w(x) \ge \frac{1}{k}\}$ is a set of Lebesgue measure zero (and even of Jordan measure zero).

Proof: For given $\varepsilon > 0$ Lemma 1.3 yields a partition P of R such that

$$U(P,f) - L(P,f) < \frac{\varepsilon}{2k}\,.$$

First consider $x \in M_k$ lying in the interior $\overset{\circ}{S}$ of a rectangle $S \in P$. By the definition of $w(x)$ we get the estimate

$$(\sup_S f - \inf_S f) \ge w(x) \ge \frac{1}{k}\,.$$

Then for $P' = \{S \in P : \exists x \in M_k \text{ such that } x \in \overset{\circ}{S}\}$ the following chain of inequalities holds:

$$\begin{aligned}\frac{1}{k}\sum_{S\in P'} |S| &\le \sum_{S\in P'} (\sup_S f - \inf_S f)\,|S| \\ &\le U(P,f) - L(P,f) < \frac{\varepsilon}{2k}\end{aligned}$$

and consequently

$$\sum_{S\in P'} |S| < \frac{\varepsilon}{2}\,.$$

Furthermore there exist points $x \in M_k$ lying on the boundary ∂S of a set $S \in P$. Summarizing we get that

$$M_k \subset \bigcup_{S\in P'} S \cup \bigcup_{S\in P} \partial S\,,$$

where $\bigcup_S \partial S$ is a set of Jordan measure zero. Hence M_k may be covered by finitely many rectangles of volume $< \varepsilon$. (■) ■

Consequence Every continuous function $f : R \to \mathbb{R}^n$ is Riemann integrable.

Theorem 7.9 *Let f and g be Riemann integrable functions on the rectangle $R \subset \mathbb{R}^n$.*

(1) *The functions $|f|$, f^+, f^-, $\max(f,g)$, $\min(f,g)$, $f\cdot g$ are Riemann integrable.*

(2) $\displaystyle\Big|\int_R f(x)\,dx\Big| \le \int_R |f(x)|\,dx\,.$

(3) *If $f = g$* **almost everywhere** *in R, i.e., there exists a set of Lebesgue measure zero $M \subset R$ such that $f(x) = g(x)$ for all $x \in R\backslash M$, then*

$$\int_R f(x)\,dx = \int_R g(x)\,dx\,.$$

Warning: Assertion (3) does not claim that the integrability of f and the property $f = g$ almost everywhere yield the integrability of g.

Proof (1) is proved by Theorem 7.8. (2) is a consequence of Theorem 7.4.

(3) Apply the ideas of the part „$\Leftarrow$“ of the proof of Theorem 7.8 to $h = f - g$ and show that $\int_R h\,dx = 0$. To be more precise, let M_f and M_g be the set of all points of discontinuity of f and g, resp., and let

$$M = M_f \cup M_g \cup \{x \in R : f(x) \ne g(x)\}\,.$$

Since M is a set of Lebesgue measure zero by Theorem 7.8, there exist open rectangles R_j such that

$$M \subset \bigcup_j R_j\,, \quad \sum_j |R_j| < \varepsilon\,.$$

Furthermore, at every $x \in R\backslash M$ – a point at which h is continuous and $h(x) = 0$ – we find an open rectangle U_x such that $x \in U_x$ and

$$|h(y)| < \varepsilon \quad \text{for all } y \in \overline{U}_x\,.$$

Since R is compact, there exists a finite subcovering, say

$$R \subset \bigcup_{j=1}^{N_1} R_j \cup \bigcup_{i=1}^{N_2} U_i \,.$$

Then we choose a suitable partition P of R such that every $S \in P$ is completely contained in some $\overline{R}_j$ or in some $\overline{U}_i$. Now we obtain the following estimate:

$$\begin{array}{rcl} |U(P,h)| & \leq & \big|\sum_{j=1}^{N_1} \sum_{S\in P, S\subset \overline{R}_j} \sup_S h \cdot |S|\big| \quad + \quad \big|\sum_{i=1}^{N_2} \sum_{S\in P, S\subset \overline{U}_i} \sup_S \, h \cdot |S|\big| \\ & \leq & \|h\|_\infty \cdot \varepsilon \qquad\qquad\qquad\qquad\quad + \quad \varepsilon |R| \,. \end{array}$$

Analogously, we prove that $|L(P,h)| \leq \varepsilon(\|h\|_\infty + |R|)$. Consequently, the integral of h over R vanishes. ∎

8 The Riemann Integral on Jordan Measurable Sets

Definition

(1) Let $A \subset \mathbb{R}^n$ be an arbitrary set and let $f : A \to \mathbb{R}$. Then the *extension* of f by 0 onto $\mathbb{R}^n$ is denoted by $f_A : \mathbb{R}^n \to \mathbb{R}$. i.e.,

$$f_A(x) = \begin{cases} f(x)\,, & x \in A \\ 0\,, & x \notin A \end{cases}.$$

(2) Let $\emptyset \neq A \subset \mathbb{R}^n$ be bounded and let $R \supset A$ be a closed rectangle. A bounded function $f : A \to \mathbb{R}$ is called *(Riemann) integrable* iff the extension f_A onto R is Riemann integrable. In this case the *(Riemann) integral* of f on A is defined by

$$\int_A f(x)dx := \int_R f_A(x)\,dx.$$

[It is easily seen that this definition does not depend on the choice of the rectangle $R \supset A$.]

(3) A non-void bounded set $A \subset \mathbb{R}^n$ is *Jordan measurable* iff its characteristic function $\chi_A : \mathbb{R}^n \to \mathbb{R}$, i.e., $\chi_A(x) = 1$ for $x \in A$, $\chi_A(x) = 0$ for $x \notin A$, is Riemann integrable. In this case

$$|A| := \int_A 1\,dx = \int_R \chi_A(x)\,dx$$

is called the n-dimensional *Jordan content* (area, volume) of A. Finally let $|\emptyset| := 0$.

Lemma 8.1 *A bounded set $A \subset \mathbb{R}^n$ is Jordan measurable iff its boundary ∂A is a set of measure zero in the sense of Lebesgue (or in the sense of Jordan).*

Proof Choose a closed rectangle $R \supset A$. Then the definition of Jordan measurability and Theorem 7.8 imply that A is measurable iff the set of all points of discontinuity of χ_A in R, i.e. the set ∂A, is a set of Lebesgue measure zero. Since $\partial A = \overline{A} \backslash \mathring{A}$ is closed and consequently even compact, by Lemma 7.7 A is Jordan measurable iff ∂A is a set of Jordan measure zero. ∎

Theorem 8.2 *Let $\emptyset \neq A \subset \mathbb{R}^n$ be Jordan measurable. A bounded function $f : A \to \mathbb{R}$ is Riemann integrable on A iff f is almost everywhere continuous on A. In particular, a continuous function on a compact Jordan measurable set is Riemann integrable.*

Proof „⇒“ If f is integrable on A, then by definition f_A is integrable on a rectangle $R \supset A$. Due to Theorem 7.8 the set of points of discontinuity M_{f_A} of f_A is a set of Lebesgue measure zero in R. Hence, also the set of points of discontinuity of f in A is a set of Lebesgue measure zero.
„⇐“ By assumption the set of points of discontinuity M_f of f in A and also ∂A are sets of Lebesgue measure zero. Hence $M_{f_A} \subset M_f \cup \partial A$ is set of Lebesgue measure zero in R implying that f is integrable. ∎

Theorem 8.3 *Let $\emptyset \neq A \subset \mathbb{R}^n$ be Jordan measurable, let $f, g : A \to \mathbb{R}$ be Riemann integrable on A and let $c \in \mathbb{R}$. Then*

$$f + g,\ cf,\ f^+,\ f^-,\ |f|,\ \max(f, g),\ \min(f, g)$$

are Riemann integrable on A as well. Furthermore,

$$\int_A (f + g) dx = \int_A f\, dx + \int_A g\, dx, \qquad \int_A cf\, dx = c \int_A f\, dx$$

and

$$\Big| \int_A f(x)\, dx \Big| \leq \int_A |f(x)|\, dx\,.$$

Proof All statements are consequences of Theorem 7.4. E.g., $(f + g)_A = f_A + g_A$, $(f^+)_A = (f_A)^+$. For a closed rectangle $R \supset A$ the estimate

$$\int_A f\, dx = \int_R f_A\, dx \leq \int_R |f_A|\, dx = \int_R |f|_A\, dx = \int_A |f|\, dx$$

holds. Analogously, $-\int_A f \leq \int_A |f|$. ∎

Lemma 8.4 *If $A, B \subset \mathbb{R}^n$ are Jordan measurable, then $A \cup B$, $A \cap B$ and $A \backslash B$ are Jordan measurable as well.*

Proof Since $\partial(A \cup B) \subset \partial A \cup \partial B$, $\partial(A \cap B) \subset \partial A \cup \partial B$ and $\partial(A \backslash B) \subset \partial A \cup \partial B$, the assertion follows from Lemma 8.1. ∎

Theorem 8.5 *Let $A, B \subset \mathbb{R}^n$ be Jordan measurable sets, and let $f : A \cup B \to \mathbb{R}$ be Riemann integrable on both A and B. Then f is Riemann integrable on both $A \cup B$ and $A \cap B$, and*

$$\boxed{\int_{A \cup B} f\, dx = \int_A f\, dx + \int_B f\, dx - \int_{A \cap B} f\, dx\,.}$$

In the case that $A \cap B = \emptyset$ the definition $\int_\emptyset f\, dx = 0$ has to be used.

Proof By Theorem 8.2 and by Lemma 8.4 f is Riemann integrable on both $A \cup B$ and $A \cap B$.

Firstly let $A \cap B = \emptyset$ and let R be a closed rectangle such that $A \cup B \subset R$. Since $f_{A\cup B} = f_A + f_B$,

$$\int_{A\cup B} f = \int_R f_{A\cup B} = \int_R f_A + \int_R f_B = \int_A f + \int_B f\,.$$

If $A \cap B \neq \emptyset$, we apply the identities

$$\begin{aligned} &A = (A\backslash B)\,\dot{\cup}\,(A\cap B)\,, \quad B = (B\backslash A)\,\dot{\cup}\,(B\cap A)\,, \\ &A\cup B = (A\backslash B)\,\dot{\cup}\,(B\backslash A)\,\dot{\cup}\,(A\cap B)\,; \end{aligned}$$

here $\dot{\cup}$ indicates that the sets are pairwise disjoint. Now the part proved above implies that

$$\int_A f + \int_B f = \underbrace{\Big(\int_{A\backslash B} f + \int_{A\cap B} f\Big) + \Big(\int_{B\backslash A} f}_{=\int_{A\cup B} f} + \int_{A\cap B} f\Big) = \int_{A\cup B} f + \int_{A\cap B} f\,.$$

■

Corollary 8.6 *Given Jordan measurable sets $A, B \subset \mathbb{R}^n$*

$$\begin{aligned} &|A\cup B| = |A| + |B| - |A\cap B|\,, \\ &A\subset B \;\Rightarrow\; |A| \le |B|\,. \end{aligned}$$

Corollary 8.7 *Let $A, B \subset \mathbb{R}^n$ be Jordan measurable and* non-overlapping, *i.e., $A\cap B \subset \partial A \cup \partial B$ (or equivalently $\mathring{A} \cap \mathring{B} = \emptyset$). Then $|A\cup B| = |A| + |B|$. If $f : A\cup B \to \mathbb{R}$ is Riemann integrable on both A and B, then*

$$\int_{A\cup B} f\,dx = \int_A f\,dx + \int_B f\,dx\,.$$

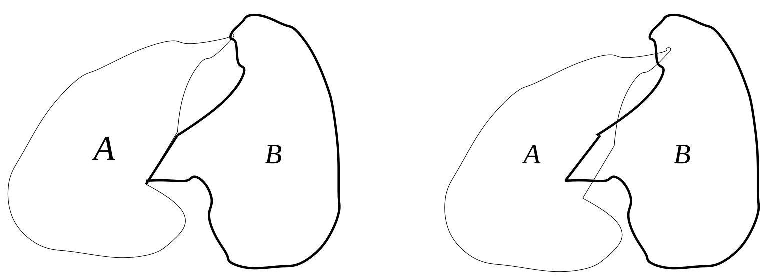

Fig. 8.1 Non-overlapping and overlapping sets

Proof If $\mathring{A} \cap \mathring{B} = \emptyset$, then

$$A \cap B \subset (\mathring{A} \cup \partial A) \cap (\mathring{B} \cup \partial B) \subset (\mathring{A} \cap \mathring{B}) \cup \partial A \cup \partial B = \partial A \cup \partial B\,.$$

Since $\mathring{A} \cap \mathring{B} \subset A \cap B$ and $\mathring{A} \cap \partial A = \emptyset$, $\mathring{B} \cap \partial B = \emptyset$, the assumption $A \cap B \subset \partial A \cup \partial B$ yields $\mathring{A} \cap \mathring{B} = \emptyset$. Hence both conditions for non-overlapping sets are equivalent.

To prove the integral identity it suffices, due to Theorem 8.5, to show that $\int_{A\cap B} f\,dx = 0$. By assumption $A \cap B (\subset \partial A \cup \partial B)$ is a set of Jordan measure zero. Hence, given $\varepsilon > 0$ there exist closed rectangles $R_1, \ldots, R_N$, such that

$$A \cap B \subset \bigcup_{i=1}^{N} R_i \quad \text{and} \quad \sum_{i=1}^{N} |R_i| < \varepsilon.$$

Then using $M = \|f\|_\infty < \infty$ and Corollary 8.6

$$\Big| \int_{A\cap B} f \Big| \le M|A \cap B| \le M\Big|\bigcup_{i=1}^{N} R_i\Big| \le M \sum_{i=1}^{N} |R_i| < M\varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, the assertion is proved. ∎

Remark 8.8 (1) *The proof of Corollary 8.7 shows that an integral over a set of Jordan measure zero vanishes. In particular, $|A| = 0$ for every set of Jordan measure zero A.*

(2) *Changing the integrand on a set of Jordan measure zero does not change the value of the integral.*

(3) *Since for a Jordan measurable set A the boundary is a set of Jordan measure zero,*

$$\int_{\overline{A}} f\,dx = \int_A f\,dx = \int_{\mathring{A}} f\,dx\,.$$

The following theorems deal with the construction of Jordan measurable sets and consequently also with the construction of sets of Jordan measure zero as boundaries of a domain of integration.

Theorem 8.9 *Let the function f be Riemann integrable on the Jordan measurable set $A \subset \mathbb{R}^n$. Then the graph*

$$G(f) = \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in A\}$$

is a set of Jordan measure zero in $\mathbb{R}^{n+1}$.

Proof Choose a closed rectangle $R \supset A$ and $\varepsilon > 0$. Since f_A is integrable on R, there exists, see Lemma 7.3, a partition $P = \{S\}$ of R, such that

$$U_R(P, f) - L_R(P, f) = \sum_{S\in P} (\sup_S f - \inf_S f)|S| < \varepsilon.$$

Obviously the union of all cartesian products $S \times [\inf_S f, \sup_S f] \subset \mathbb{R}^{n+1}$ defines a covering of $G(f_A)$ consisting of finitely many closed rectangles. Hence $|G(f)| \leq |G(f_A)| < \varepsilon$. ■

Lemma 8.10 *Let $A \subset \mathbb{R}^n$ be a set of Jordan measure zero, let $q \geq n$ and let $g : A \to \mathbb{R}^q$ be Lipschitz continuous. Then $g(A)$ is a set of Jordan measure zero in $\mathbb{R}^q$. [In general, for $q < n$ this assertion is wrong.]*

Proof First note that for every $\varepsilon > 0$ there exists a covering of A by finitely many compact *cubes* C_k, $1 \leq k \leq N$, *of equal side length* with the property $\sum_{k=1}^N |C_k| < \varepsilon$. To find this covering we start with a covering of A by finitely many closed rectangles R_i, $1 \leq i \leq M$, such that $\sum_{i=1}^M |R_i| < \varepsilon \cdot 2^{-n}$. Let $\delta > 0$ denote the smallest side length of all R_i. Then for every R_j there exist finitely many cube $C_{k,j}$ of side length δ such that $R_j \subset \bigcup_k C_{k,j}$ and $\sum_k |C_{k,j}| < 2^n |R_j|$. Now all cubes $(C_{k,j})_{k,j}$ cover the set A and $\sum_{k,j} |C_{k,j}| < \varepsilon$.

Let $L \geq 0$ be a Lipschitz constant of g using the maximum norm in $\mathbb{R}^n$ and in $\mathbb{R}^q$. Given $\varepsilon > 0$ cover A by compact cubes (C_k), $k = 1, \ldots, N$, of side length $\delta < 1$ such that

$$\sum_{k=1}^{N} |C_k| = N\delta^n < \varepsilon\,.$$

If $A \cap C_k \neq \emptyset$, fix a point $x_k \in A \cap C_k$. Then every $x \in A \cap C_k$ satisfies $\|x - x_k\| \leq \delta$ yielding the estimate

$$\|g(x) - g(x_k)\| \leq L\delta\,.$$

Hence $g(A \cap C_k)$ is contained in a cube $Q_k \subset \mathbb{R}^q$ of side length $2L\delta$. Summarizing we get that

$$g(A) = \bigcup_k g(A \cap C_k) \subset \bigcup_k Q_k$$

and, since $\delta < 1$ and $q \geq n$,

$$\sum_{k=1}^{N} |Q_k| \leq N(2L\delta)^q = (2L)^q \cdot N\delta^n \cdot \delta^{q-n} < (2L)^q \varepsilon\,.$$

■

Lemma 8.11 *Let $G \subset \mathbb{R}^n$ be an open set, let $q \geq n$ and let $g : G \to \mathbb{R}^q$ be continuously differentiable. Then the range $g(A)$ of every compact set of Jordan measure zero $A \subset G$ is a set of Jordan measure zero in $\mathbb{R}^q$.*

Proof Since $A \subset G$ is compact, A may be covered by finitely many compact cubes $C_k \subset G$. By Lemma 8.10 $g(A \cap C_k)$ being the image of $A \cap C_k$ under the Lipschitz continuous function $g\big|_{C_k}$ is a set of Jordan measure zero. Then the identity $g(A) = \bigcup_k g(A \cap C_k)$ proves the assertion. ■

Theorem 8.12 *Let $g : G \to \mathbb{R}^n$ be an injective, continuously differentiable function on an open set $G \subset \mathbb{R}^n$ and let the functional matrix $Dg(x)$ be invertible for every $x \in G$. Then the image of every compact Jordan measurable set under g is compact and Jordan measurable.*

Proof By assumption and the Inverse Function Theorem the inverse $g^{-1} : g(G) \to \mathbb{R}^n$ is continuous; hence g maps open sets to open sets.

Let $A \subset G$ be compact and Jordan measurable. Then $g(A)$ is compact and $\partial g(A) \subset \overline{g(A)} = g(A) \subset g(G)$. Since ∂A and by Lemma 8.10 $g(\partial A)$ are sets of Jordan measure zero, it suffices due to Lemma 8.1 to show that

$$\partial g(A) \subset g(\partial A)$$

(actually $\partial g(A) = g(\partial A)$).

Let $y \in \partial g(A) \subset g(G)$. Since $g(G)$ is open, there exist sequences $(y_k) \subset g(A)$ and $(y'_k) \subset g(G) \backslash g(A) = g(G \backslash A)$ converging to y as $k \to \infty$. The continuity of g^{-1} implies that the sequences $\big(g^{-1}(y_k)\big) \subset A$ and $g^{-1}(y'_k) \subset G \backslash A$ converge to $g^{-1}(y)$ as $k \to \infty$. Hence $g^{-1}(y) \in \partial A$ and consequently $y \in g(\partial A)$. ∎

Theorem 8.13 *Let $A \subset \mathbb{R}^n$ be Jordan measurable and let $f : A \to \mathbb{R}_+$ be Riemann integrable. Then the 'ordinate set'*

$$H(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}_+ : x \in A\,, \quad 0 \leq y \leq f(x)\}$$

is Jordan measurable and has the $(n + 1)$–dimensional volume

$$|H(f)| = \int_A f(x)\, dx\,.$$

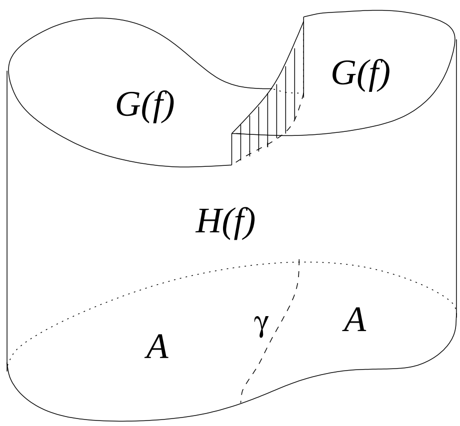

Fig. 8.2 The ordinate set of $f : A \subset \mathbb{R}^n \to \mathbb{R}_+$

Proof First we prove the Jordan measurability of $H(f)$ and define $M := \sup_A f$. If f is continuous at a point $x \in \overset{\circ}{A}$, then $(x, y) \in H(f)^\circ$ for all $0 < y < f(x)$. Due to this remark we get the inclusion

$$\begin{aligned} \partial H(f) \;\subset\; & (\partial A \times [0, M]) \cup (A \times \{0\}) \cup G(f) \cup \\ & \{x \in A : \; x \text{ is a point of discontinuity of } f\} \times [0, M] \,. \end{aligned}$$

Here $\partial A \times [0, M]$ is a set of Jordan measure zero in $\mathbb{R}^{n+1}$, since it is true for $\partial A \subset \mathbb{R}^n$. By Theorem 8.9 both $A \times \{0\}$ and $G(f)$ are sets of Jordan measure zero. Finally, by Theorem 7.8, the fourth set in the above inclusion is a set of Lebesgue measure zero. Hence $\partial H(f)$ is a set of Lebesgue measure zero and – due to its compactness – even a set of Jordan measure zero. This proves the measurability of $H(f)$.

To compute integral we choose a closed rectangle $R \supset A$. Since $H(f) \subset R \times [0, M]$, by definition and by Fubini's Theorem

$$|H(f)| = \int_{R\times[0,M]} \chi_{H(f)}(x, y)\, d(x, y) = \int_R \Big(\int_0^M \chi_{H(f)}(x, y) dy \Big) dx \,.$$

For every $x \in A$ the inner integral equals $\int_0^M \chi_{[0,f(x)]}(y)\, dy = f(x)$; however, it vanishes for $x \in R \backslash A$. Thus we conclude that $|H(f)| = \int_A f(x) dx$. ■

Definition A (two-dimensional) *projected domain with respect to the x–axis* is a set $A \subset \mathbb{R}^2$ of the form

$$A = \{(x, y) \in \mathbb{R}^2 : \; a \le x \le b, \; \varphi(x) \le y \le \psi(x)\} \,;$$

here $a < b$ and $\varphi, \psi \in C^0[a, b]$ such that $\varphi \le \psi$ are given.

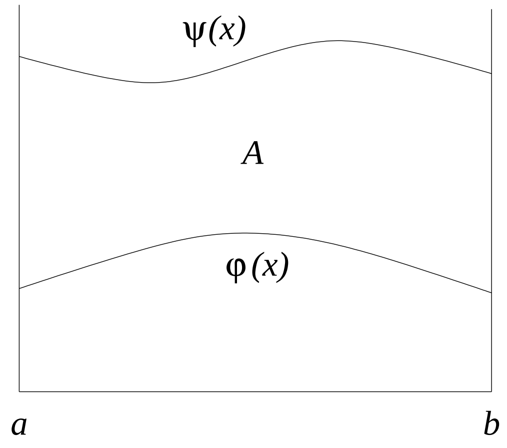

Fig. 8.3 A projected domain

Theorem 8.14 *Let $f : A \to \mathbb{R}$ be a continuous function on a projected domain $A = \{(x, y) : x \in [a, b],\ \varphi(x) \le y \le \psi(x)\}$. Then*

$$\int_A f(x, y)\, d(x, y) = \int_a^b \Big(\int_{\varphi(x)}^{\psi(x)} f(x, y) dy \Big)\, dx \,.$$

Proof As in the proof of Theorem 8.13 we conclude that A is Jordan measurable. Let $m = \min \varphi$, $M = \max \psi$ and let $R = [a, b] \times [m, M]$. Then Fubini's Theorem yields

$$\begin{aligned} \int_A f(x, y)\, d(x, y) &= \int_R f_A(x, y)\, d(x, y) \\ &= \int_a^b \Big(\int_m^M f_A(x, y) dy \Big)\, dx \\ &= \int_a^b \Big(\int_{\varphi(x)}^{\psi(x)} f(x, y) dy \Big)\, dx \,. \end{aligned}$$

■

9 The Change of Variable Formula

From Analysis I the Change of Variable Formula is well-known: Let $f : [a, b] \to \mathbb{R}$ be continuous and let $g : [\alpha, \beta] \to \mathbb{R}$ be a continuous differentiable bijection of the interval $[\alpha, \beta]$ onto the interval $[a, b]$ such that $g' > 0$. Then

$$\int_{[a,b]} f(x)\, dx = \int_{[\alpha,\beta]} f(g(t)) g'(t)\, dt\, .$$

Here the different "infinitesimal increments" dx and $g'(t)\, dt$ can be interpreted as follows: From $x = g(t)$ we get $\frac{dx}{dt} = g'(t)$, hence "$dx = g'(t)\, dt$", i.e., the interval $[t\, ,\ t + dt]$ of length dt is mapped by g onto the interval

$$g([t\, ,\ t + dt]) = [g(t),\ g(t + dt)] \doteq [g(t)\, ,\ g(t) + g'(t)\, dt] = [x\, ,\ x + dx]$$

of length $dx = g'(t)\, dt$. If $g' < 0$ and consequently $g(\alpha) = b\, ,\ g(\beta) = a$, the above formula stil holds in the more general form

$$\int_{g([\alpha,\beta])} f(x)\, dx = \int_{[\alpha,\beta]} f(g(t)) |g'(t)|\, dt\, ;$$

here $[\alpha, \beta]$ and $g([\alpha, \beta])$ denote the intervals with end points α and β or with a and b, resp., independently of the orientation $\alpha < \beta$ or $\beta < \alpha$ and $a < b$ or $b < a$.

Now the question of the n-dimensional analogue of the infinitesimal volume $dx = dx_1 \cdot \ldots \cdot dx_n$, when applying a C^1-mapping $x = g(t)$, arises.

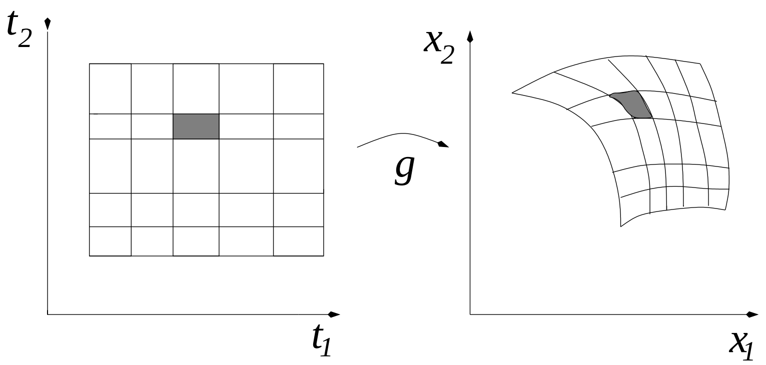

Fig. 9.1 Transformation of a partition

Let $a_1, \ldots, a_n \in \mathbb{R}^n$ be linearly independent vectors, and let $V(a_1, \ldots, a_n)$ be the n-dimensional "volume" of the parallepiped P spanned by $a_1, \ldots, a_n$, i.e.,

$$P(a_1, \ldots, a_n) = \left\{ \sum_{i=1}^{n} t_i\, a_i : 0 \le t_i \le 1\, ,\ i = 1, \ldots, n \right\}\, .$$

Obviously, we get the following properties:

$$
\begin{aligned}
V(a_1, \ldots, \lambda\, a_i, \ldots, a_n) &= \lambda\, V(a_1, \ldots, a_n) \ \text{ for } \lambda \geq 0 && (V1)\\
V(a_1, \ldots, a_i + a_i', \ldots, a_n) &= V(a_1, \ldots, a_i, \ldots, a_n)\\
&\quad + V(a_1, \ldots, a_i', \ldots, a_n) && (V2)\\
V(a_1, \ldots, a_i, \ldots, a_i, \ldots, a_n) &= 0 && (V3)\\
V(e_1, \ldots, e_n) &= 1 && (V4)
\end{aligned}
$$

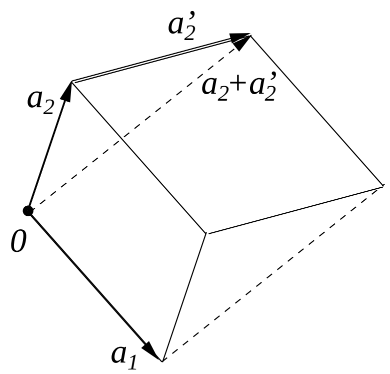

Fig. 9.2 Parallepipeds

$(V3)$ implies that a degenerate parallepiped for which two spanning vectors coincide has volume equal to 0. From $(V2)$ with $a_i' = -a_i$ and from $V(a_1, \ldots, 0, \ldots, a_n) = 0$, cf. $(V1)$ with $\lambda = 0$, we get that $(V1)$ even holds when $\lambda < 0$. Hence the mapping V takes on also negative values and *doesn't* coincide with the volume $|P(a_1, \ldots, a_n)|$ considered in Sections 7 and 8. On the other hand, $(V1)$ (for all $\lambda \in \mathbb{R}$) and $(V2)$ imply that

$$V : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}$$

is linear in every component, i.e., it is *multilinear.*

From *Linear Algebra* it is well-known that there exists a unique multilinear mapping satisfying $(V1)$ - $(V3)$ and normalized by $(V4)$, namely the *determinant*

$$\det : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}\,.$$

Hence $V \equiv \det$.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping (matrix) with column vectors

$$a_i := Ae_i\,,\ 1 \leq i \leq n\,.$$

Obviously A maps the unit cube $P(e_1, \ldots, e_n)$ onto the parallepiped

$$A\left(P(e_1, \ldots, e_n)\right) = P(Ae_1, \ldots, Ae_n) = P(a_1, \ldots, a_n)\,.$$

Then the determinant of the linear mapping A is defined by

$$\det A := V(a_1, \ldots, a_n)\,.$$

Hence the nonnegative number $|\det A|$ equals the volume of the parallepiped $P(a_1, \ldots a_n)$.

Let A be an invertible linear mapping ($\det A \neq 0$), and let

$$V'(b_1, \ldots, b_n) := \frac{1}{\det A}\, V(Ab_1, \ldots,\ Ab_n)\,.$$

Evidently, V' satisfies the axioms $(V1)$ - $(V4)$. Due to the uniqueness assertion above $V' \equiv \det = V$. Thus

$$V(Ab_1, \ldots, Ab_n) = \det A \cdot V(b_1, \ldots b_n)\,;$$

the "volume" of the parallepiped $P(Ab_1, \ldots, Ab_n) = A\,P(b_1, \ldots, b_n)$ equals the "volume" of $P(b_1, \ldots, b_n)$ multiplied by $\det A$. We conclude:

> $|\det A|$ is the scaling term for the n-dimensional volume under the linear mapping A.

For further calculations with determinants we cite the following results from Linear Algebra.

Expansion Theorem of Laplace *For a matrix* $A = (a_{ij}) \in \mathbb{R}^{n,n}$ *and for* $i, j \in \{1, \ldots, n\}$ *let* $A_{ij} \in \mathbb{R}^{n-1\,,\,n-1}$ *denote the submatrix of* A *when leaving out the* i*th row and the* j*th column. Then for fixed* $i \in \{1, \ldots, n\}$

$$\boxed{\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}}$$

(Expansion according to the i*th row) and for fixed* $j \in \{1, \ldots, n\}$

$$\boxed{\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}}$$

(Expansion according to the j*th column.)*

Example By mathematical induction on n it is easily proved that

$$\det \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix} = a_{11} \det \begin{pmatrix} a_{22} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} = \ldots = a_{11} a_{22} \cdot \ldots \cdot a_{nn}\,.$$

Theorem 9.1 *(Change of Variable Formula) Let $G \subset \mathbb{R}^n$ be open, let $g : G \to \mathbb{R}^n$ be continuously differentiable, injective and assume that either* $\det Dg(t) > 0$ *on G or that* $\det Dg(t) < 0$ *on G. Furthermore, let T be a compact Jordan measurable subset of G and let $f : g(T) \to \mathbb{R}$ be continuous. Then*

$$\boxed{\int_{g(T)} f(x)\,dx = \int_T f(g(t))\,|\det Dg(t)\,|\,dt\ .}$$

Thus, formally, the change of variables $x = g(t)$ yields $dx = |\det Dg(t)|dt$.

Proof By Theorem 8.12 $g(T)$ is Jordan measurable. Since $g \in C^1(G)$ and consequently the mapping $f(g(t))\,|\det Dg(t)\,|$ is continuous w.r.t. t, both Riemann integrals are well-defined.

We will show in three steps that it suffices to prove the Change of Variable Formula in a much simpler situation.

Claim 1 *It suffices to prove the Change of Variable Formula for compact rectangles $S \subset T$.*

Proof Assume that the Change of Variable Formula holds for arbitrary compact rectangles $S \subset T$. Since T is a compact subset of the open set G, $d_0 := \text{dist}\,(T, G^c) > 0$. Then

$$G_0 = \left\{x \in \mathbb{R}^n : \text{dist}\,(x, T) < \frac{d_0}{2}\right\}$$

is open and $\overline{G}_0$ is a compact subset of G. Consequently,

$$L := \max_{t \in \overline{G}_0} \|Dg(t)\| < \infty\,.$$

Since ∂T is compact, for any $\varepsilon > 0$ there exist compact cubes C_i, $1 \le i \le N$, of equal side length $\delta < d_0/2$ such that

$$\partial T \subset \bigcup_{i=1}^N C_i\,, \quad \sum_{i=1}^N |C_i| < \frac{\varepsilon}{3^n}\,.$$

Now consider a partition of $\mathbb{R}^n$ consisting of cubes S of side length δ and define

$$P = \{S : S \cap T \neq \emptyset\}$$

and its subsets

$$P_0 = \{S \in P : S \cap \partial T = \emptyset\}\,, \quad P_1 = \{S \in P : S \cap \partial T \neq \emptyset\}\,.$$

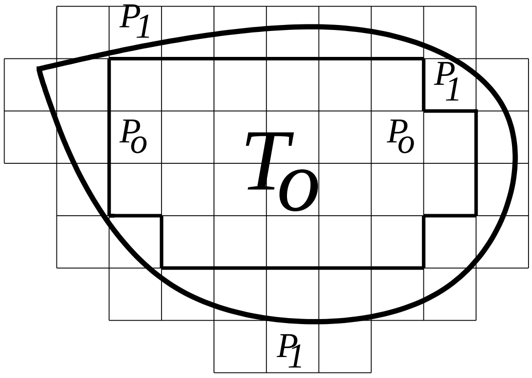

Fig. 9.3 The partition P

We will use the Change of Variable Formula on

$$T_0 = \bigcup_{S \in P_0} S$$

whereas the integrals on $S \in P_1$ will be estimated by ε.

If $S \in P_1$, the properties $S \cap \partial T \neq \emptyset$ and $\partial T \subset \bigcup_{i=1}^{N} C_i$ imply that there exists an $i \in \{1, \ldots, N\}$ such that $S \cap C_i \neq \emptyset$. For this i,

$$\sum_{S \in P_1,\, S \cap C_i \neq \emptyset} |S| \leq (3\delta)^n = 3^n |C_i| \,.$$

Hence

$$\sum_{S \in P_1} |S| \leq 3^n \sum_{i=1}^{N} |C_i| < \varepsilon$$

and, since $T \setminus T_0 \subset \bigcup_{S \in P_1} S$,

$$|T \setminus T_0| < \varepsilon \,.$$

Since $S \in P$ is even contained in $\overline{G}_0$, the function g is Lipschitz continuous on S with Lipschitz constant L. Now, as in the proof of Lemma 8.10, for every $S \in P$

$$|g(S)| \leq (2L)^n |S| \,.$$

Furthermore the injectivity of g implies that

$$g(T) \setminus g(T_0) = g(T \setminus T_0) \subset g(\bigcup_{S \in P_1} S) = \bigcup_{S \in P_1} g(S) \,.$$

Thus we get the estimate

$$|g(T) \setminus g(T_0)| \leq \sum_{S \in P_1} |g(S)| \leq (2L)^n \sum_{S \in P_1} |S| \leq (2L)^n \varepsilon \,.$$

In the following let

$$\varphi(t) := f(g(t))|\det Dg(t)|\,, \quad M := \max(\|\varphi\|_{\infty,T}\,,\ \|f\|_{\infty,g(T)}).$$

Since the rectangles $S \in P_0$ as well as their images $g(S)$ are non-overlapping (note that $g(S) \cap g(S') = g(S \cap S') \subset g(\partial S \cap \partial S') \subset \partial g(S) \cap \partial g(S')$), Corollary 8.7 yields

$$\int_{g(T_0)} f(x)\,dx = \int_{T_0} \varphi(t)\,dt\,.$$

Moreover, we may use the estimates

$$\left|\int_{T\setminus T_0} \varphi(t)\,dt\right| \le M|T \setminus T_0| < M\varepsilon$$

and

$$\left|\int_{g(T)\setminus g(T_0)} f(x)\,dx\right| \le M|g(T) \setminus g(T_0)| \le M(2L)^n\varepsilon\,.$$

Summarizing we conclude that

$$\begin{aligned} &|\textstyle\int_{g(T)} f(x)\,dx - \int_T \varphi(t)\,dt| \\ &\le |\textstyle\int_{g(T)\setminus g(T_0)} f(x)\,dx| + |\int_{T\setminus T_0} \varphi(t)\,dt| \\ &\le M(1+(2L)^n)\varepsilon\,. \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, we get that $\int_{g(T)} f\,dx = \int_T \varphi\,dt$. Hence Claim 1 is proved. $(\square)$

The next steps are part of a lengthy mathematical induction on the dimension n. The beginning of the induction (IB) $n = 1$ is given by the one-dimensional Change of Variable Formula from Analysis I. In order to apply the induction hypothesis (IH), i.e. the validity of the Change of Variable Formula in $\mathbb{R}^{n-1}$, we need a local factorization of g into simpler functions.

Lemma 9.2 *(Lemma of Factorization) Let $G \subset \mathbb{R}^n$ be open, $n \ge 2$, and let $g : G \to \mathbb{R}^n$ be a C^1-function with $\det Dg(t) \ne 0$ for all $t \in G$. Then for every $t_0 \in G$ there exist an open neighborhood $U \subset G$ of t_0 and injective C^1-functions h, ψ with the following properties:*

$$g = h \circ \psi\,,$$

$\psi(U) \subset \mathbb{R}^n$ is open, and choosing a suitable enumeration of indices, for $t = (t_1, \ldots, t_n)$, $y = (y_1, \ldots, y_n)$

$$\psi(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_{n-1}(t) \\ t_n \end{pmatrix}, \quad h(y) = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ h_n(y) \end{pmatrix}.$$

Postponing the proof of this Lemma we first of all consider mappings of the type h.

Claim 2 *The Change of Variable Formula holds for injective C^1–mappings h of the type $h(y) = (y_1, \ldots, y_{n-1}, h_n(y))^T$ with* $\det Dh(y) \neq 0$.

Proof By Claim 1 it suffices to prove this claim for rectangles $R = [a_1, b_1] \times \ldots \times [a_n, b_n]$ only. Due to the structure of h

$$Dh(y) = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ \partial_1 h_n(y) & \cdots & \partial_{n-1} h_n(y) & \partial_n h_n(y) \end{pmatrix},$$

such that the Expansion Theorem of Laplace (expansion according to the nth column) yields

$$\partial_n h_n(y) = \det Dh(y) \neq 0 \quad \text{for all } y \in R.$$

Assume without loss of generality that $\partial_n h_n(y) > 0$ on R implying that for every fixed $y' = (y_1, \ldots, y_{n-1}) \in R' := [a_1, b_1] \times \ldots \times [a_{n-1}, b_{n-1}]$ the mapping

$$h_n(y', \cdot) : [a_n, b_n] \to \mathbb{R}\,, \quad y_n \mapsto h_n(y', y_n),$$

is strictly increasing. In particular,

$$h(R) = \{(y', y_n) : y' \in R'\,,\ h_n(y', a_n) \leq y_n \leq h_n(y', b_n)\}$$

is a projected domain, cf. Section 8. Then Fubini's Theorem and a trivial generalization of Theorem 8.14 to $\mathbb{R}^n$ prove for a continuous function f that

$$\begin{aligned} &\int_R f(h(y))\, |\det Dh(y)|\, dy \\ &= \int_R f(h(y))\, \partial_n h_n(y)\, dy \\ &= \int_{R'} \Big(\int_{a_n}^{b_n} f(y', h_n(y))\, \partial_n h_n(y', y_n)\, dy_n \Big)\, dy' \\ &= \int_{R'} \Big(\int_{h_n(y', a_n)}^{h_n(y', b_n)} f(y', s)\, ds \Big)\, dy' \\ &= \int_{h(R)} f(x)\, dx\,. \end{aligned}$$

(□)

Claim 3 *Assume that the Change of Variable Formula holds in $\mathbb{R}^{n-1}$. Then it holds in $\mathbb{R}^n$ as well.*

Proof By Claim 1 it suffices to prove the Change of Variable Formula for rectangles $R \subset \mathbb{R}^n$ only. By Lemma 9.2, for every $t \in R$, there exists an open, rectangular neighborhood U_t of t such that $g|_{U_t}$ can suitably be factorized. Since R is compact, R may be covered by finitely many open sets U_{t_j} , $1 \leq j \in N$. Partitioning R we may assume that

$$R = \bigcup_{i=1}^{M} R_i$$

with compact non-overlapping rectangles R_i such that each R_i is contained in some U_{t_j}. Since g possesses a factorization on every U_{t_j} it suffices to consider the case $g = h \circ \psi$ on a compact rectangle R with $R \subset U$, U an open rectangle. Note that Claim 2 yields

$$\int_{g(R)} f(x)\, dx = \int_{h(\psi(R))} f(x)\, dx = \int_{\psi(R)} F(y)\, dy$$

where

$$F(y) = f(h(y))\, |\det Dh(y)| \ .$$

Now we write $R = R' \times R_n$ with $R' \subset \mathbb{R}^{n-1}$, $R_n = [a_n, b_n] \subset \mathbb{R}$, and $t = (t', t_n)$, $y = (y', y_n)$. Furthermore, due to the special form of ψ, we may define for fixed $t_n \in R_n$ the function $\gamma_{t_n} : R' \to \mathbb{R}^{n-1}$ by

$$\gamma_{t_n}(t') = (g_1(t', t_n), \ldots, g_{n-1}(t', t_n))^T$$

and try to apply the Change of Variable Formula to $\gamma_{t_n}(\cdot)$.

Since $R = R' \times R_n \subset U =: U' \times U_n$, the function $\gamma_{t_n}(\cdot)$ is a C^1–mapping on $U' \subset \mathbb{R}^{n-1}$ for every $t_n \in R_n$. Its Jacobi determinant $\det D\gamma_{t_n}(\cdot)$ will be computed as follows: The product rule of differentiation applied to $g = h \circ \psi$ yields $Dg(t) = Dh(y) \cdot D\psi(t)$ for $y = \psi(t)$ and the product rule for determinants shows that $\det Dg(t) = \det Dh(y) \cdot \det D\psi(t)$. Since $\det Dg(t) \neq 0$, also $\det D\psi(t) \neq 0$ for all $t \in U$. Finally, by the Expansion Theorem of Laplace,

$$D\psi(t) = \left(\begin{array}{ccc|c} \partial_1 g_1 & \cdots & \partial_{n-1} g_1 & \partial_n g_1 \\ \vdots & & \vdots & \vdots \\ \partial_1 g_{n-1} & \cdots & \partial_{n-1} g_{n-1} & \partial_n g_{n-1} \\ \hline 0 & \cdots & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} & & & * \\ & D\gamma_{t_n}(t') & & \vdots \\ & & & * \\ \hline 0 & \cdots & 0 & 1 \end{array}\right)$$

leads to the formula

$$\det D\gamma_{t_n}(t') = \det D\psi(t) \ .$$

In particular, $\det D\gamma_{t_n}(t') \neq 0$ on U'. Hence, even $\det D\gamma_{t_n}(t') > 0$ in every $t' \in U'$ (or < 0 for all $t' \in U'$). Actually, $\gamma_{t_n}(\cdot)$ is injective on U' since ψ is injective on U. We conclude that γ_{t_n} satisfies the assumptions of Theorem 9.1 on $U' \subset \mathbb{R}^{n-1}$ and that the induction hypothesis may be applied.

Using $y = \psi(t) = (\gamma_{t_n}(t'), t_n)$, $t' \in R'$, $t_n \in R_n$, a twofold application of Fubini's Theorem and the induction hypothesis show that

$$\begin{aligned}
&\int_{\psi(R)} F(y)\, dy \\
&= \int_{R_n} \left(\int_{\gamma_{t_n}(R')} F(y', t_n)\, dy' \right) dt_n \\
&= \int_{R_n} \left(\int_{R'} F(\gamma_{t_n}(t'), t_n)) \, |\det D\gamma_{t_n}(t')|\, dt' \right) dt_n \\
&= \int_{R} F(\psi(t)) \, |\det D\psi(t)|\, dt \, .
\end{aligned}$$

Since $f(h \circ \psi(t)) = f(g(t))$ and

$$|\det Dh(\psi(t))| \, |\det D\psi(t)| = |\det Dg(t)| \, ,$$

the assertion

$$\int_{g(R)} f(x)\, dx = \int_{R} f(g(t)) \, |\det Dg(t)|\, dt$$

is proved thus completing the induction step (IS) from $n-1$ to n. (■)

Finally we prove *Lemma 9.2.* At $t_0 \in G$ expand Jacobi's determinant of g according to the nth row to get

$$0 \neq \det Dg(t_0) = \det \begin{pmatrix} \partial_1 g_1 & \cdots & \partial_n g_1 \\ \vdots & & \vdots \\ \partial_1 g_n & \cdots & \partial_n g_n \end{pmatrix} = \sum_{k=1}^{n} (-1)^{n+k} \partial_k g_n(t_0) G_k(t_0)$$

with subdeterminant $G_k(t_0)$. In the above sum at least one term $G_k(t_0) \neq 0$. Therefore, asume that

$$0 \neq G_n(t_0) = \det \begin{pmatrix} \partial_1 g_1 & \cdots & \partial_{n-1} g_1 \\ \vdots & & \vdots \\ \partial_1 g_{n-1} & \cdots & \partial_{n-1} g_{n-1} \end{pmatrix} (t_0) \, .$$

By the Inverse Mapping Theorem there exists an open neighborhood $\widetilde{U}$ of t_0 on which g is injective.

As in Lemma 9.2 define

$$\psi(t) = (g_1(t), \ldots, g_{n-1}(t), t_n)^T \, .$$

Since

$$\det D\psi(t_0) = G_n(t_0) \neq 0 \, ,$$

the Inverse Mapping Theorem yields an open neighborhood $U \subset \widetilde{U}$ of t_0 such that $\psi|_U$ is a bijection from U onto an open neighborhood V of $\psi(t_0)$; its inverse $\varphi = (\psi|_U)^{-1} : V \to U$ is a C^1–function as well. Now define on V the C^1–function

$$h(y) := \big(y_1, \ldots, y_{n-1},\, g_n(\varphi_1(y), \ldots, \varphi_{n-1}(y),\, y_n)\big)^T .$$

By construction, for $t \in U$,

$$h(\psi(t)) = \big(g_1, \ldots, g_{n-1}\,,\, g_n(\varphi_1 \circ \psi, \ldots, \varphi_{n-1} \circ \psi,\, t_n)\big)(t) = g(t)\,;$$

moreover, h is injective on $V = \psi(U)$, since ψ and g are injective. Hence the functions h and ψ define the desired factorization of g on U. (■)

Now Theorem 9.1 is completely proved. ■

Appications of the Change of Variable Formula

Example (*Polar coordinates in* $\mathbb{R}^2$)

1. Writing $x \in \mathbb{R}^2$ in polar coordinates, i.e.,

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = g(r, \varphi) := \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

we get that

$$Dg(r, \varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

and consequently

$$\det Dg(r, \varphi) = r\cos^2\varphi + r\sin^2\varphi = r\,.$$

Let

$$G = \{(r, \varphi) : r > 0,\ 0 < \varphi < 2\pi\}$$

and let $T \subset G$ be a compact, Jordan measurable subset. Then for every continuous function $f : A = g(T) \to \mathbb{R}$

$$\int_A f(x)\,dx = \int_T f(r\cos\varphi,\ r\sin\varphi)\,r\,d(r, \varphi)\,.$$

Very often T is a set of the type $T = [r_1, r_2] \times [\varphi_1, \varphi_2]$ such that Fubini's Theorem yields

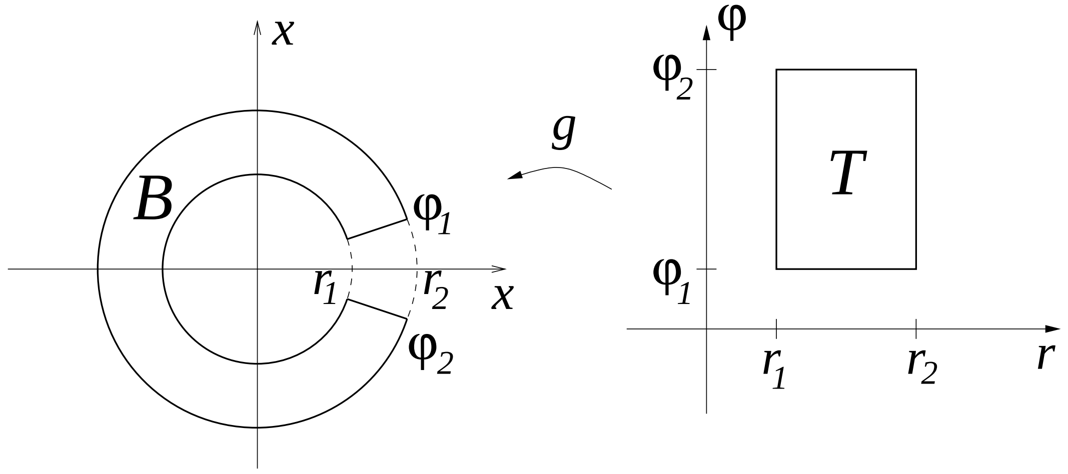

Fig. 9.4 Transformation to polar coordinates

$$\int_A f(x)\,dx = \int_{r_1}^{r_2} r \left(\int_{\varphi_1}^{\varphi_2} f(r\cos\varphi, r\sin\varphi)\,d\varphi \right) dr\,.$$

If f is even radially symmetric, i.e., $f(x) = \widetilde{f}(r)$ with a continuous function $\widetilde{f}$, the inner integral is elementary, and we get

$$\int_A f(x)\,dx = (\varphi_2 - \varphi_1) \int_{r_1}^{r_2} r\widetilde{f}(r)\,dr\,.$$

2. However, usually T is a compact subset of the closed strip $\{(r,\varphi) : r \geq 0\,,\ 0 \leq \varphi \leq 2\pi\}$ on which g fails to be injective (note that $g(r,0) = g(r,2\pi)$); furthermore, $\det Dg = 0$ for $r = 0$. Since in this situation Theorem 9.1 cannot be applied directly, a further limit procedure is necessary.

   Assume that $f \in C^0(\overline{B_R(0)})$ and that $T = [0,R] \times [0,2\pi]$. Defining $T_\varepsilon = [\varepsilon, R] \times [\varepsilon, 2\pi - \varepsilon]$ for $\varepsilon > 0$ we get that

$$\int_{g(T_\varepsilon)} f(x)\,dx = \int_{T_\varepsilon} f(r\cos\varphi, r\sin\varphi)\, r\, d(r,\varphi)\,.$$

   Since $|T \setminus T_\varepsilon| \leq 2(\pi + R)\varepsilon$ and $|g(T) \setminus g(T_\varepsilon)| \to 0$ for $\varepsilon \to 0$, we conclude, as $\varepsilon \to 0$, that

$$\int_{g(T)} f(x)\,dx = \int_T f(r\cos\varphi, r\sin\varphi)\, r\, d(r,\varphi)\,.$$

3. If f is not bounded and consequently not Riemann integrable, $\int_B f(x)\,dx$ must be intepreted as an improper Riemann integral.

   E.g., we consider

$$f(x) = |x|^{-\alpha} \text{ on } B = \overline{B_1(0)} \setminus \{0\}$$

   and define $B_\varepsilon = \{x \in \mathbb{R}^2 : \varepsilon \le |x| \le 1\}$. Then *per definitionem* let

$$\int_B f(x)\,dx := \lim_{\varepsilon \to 0} \int_{B_\varepsilon} f(x)\,dx\,,$$

   provided that the above limit exists. By (1), (2)

$$\int_{B_\varepsilon} |x|^{-\alpha}\,dx = \int_\varepsilon^1 \Big(\int_0^{2\pi} 1\,d\varphi\Big)\, r^{-\alpha}\, r\,dr$$

$$= \frac{2\pi}{2-\alpha}\, r^{2-\alpha}\Big|_\varepsilon^1 \longrightarrow \frac{2\pi}{2-\alpha}$$

   as $\varepsilon \to 0+$, if $\alpha < 2$. Thus

$$\int_B |x|^{-\alpha}\,dx = \begin{cases} \frac{2\pi}{2-\alpha} & , \ \alpha < 2 \\ \infty & , \ \alpha \ge 2\,. \end{cases}$$

   We conclude that there exist unbounded functions such as $|x|^{-\alpha}$ (for $1 \le \alpha < 2$) which are integrable in a neighborhood of the origin of $\mathbb{R}^2$ although they are not integrable on $\mathbb{R}^1$.

4. If B is unbounded and consequently not Jordan measurable, then $\int_B f(x)\,dx$ may exist as an improper Riemann integral.

   As an application we consider the integral $\int_{\mathbb{R}^2} e^{-|x|^2}\,dx$ and define the approximating sets $B_R = \overline{B_R(0)}$ and $Q_R = \{x \in \mathbb{R}^2 : \|x\|_\infty \le R\}$. By (1), (2) we get that

$$\int_{B_R} e^{-|x|^2}\,dx = \int_0^R 2\pi r e^{-r^2}\,dr = -\,\pi e^{-r^2}\Big|_0^R \longrightarrow \pi \quad \text{for} \quad R \to \infty\,,$$

   such that in the sense of an improper Riemann integral

$$\int_{\mathbb{R}^2} e^{-|x|^2}\,dx = \pi\,.$$

   On the other hand, since $Q_{R/\sqrt{2}} \subset B_r \subset Q_R$,

$$\int_{Q_{R/\sqrt{2}}} e^{-|x|^2}\,dx \le \int_{B_R} e^{-|x|^2}\,dx \le \int_{Q_R} e^{-|x|^2}\,dx\,.$$

Hence Fubini's Theorem yields

$$\pi = \lim_{R\to\infty} \int_{Q_R} e^{-|x|^2}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \Big(\int_{-R}^{R} e^{-x_1^2-x_2^2}\, dx_2 \Big)\, dx_1$$

$$= \lim_{R\to\infty} \Big(\int_{-R}^{R} e^{-s^2}\, ds \Big)^2 = \Big(\int_{-\infty}^{\infty} e^{-s^2}\, ds \Big)^2 .$$

By this means we prove the formula

$$\int_{-\infty}^{\infty} e^{-s^2}\, ds = \sqrt{\pi} \quad \text{(Gauß' error function)}$$

which is of the utmost importance in statistics (normal or Gaussian distribution) and in the theory of partial differential equations (heat equation).

Example (*Cylindrical Coordinates in* $\mathbb{R}^3$) In axisymmetric problems cylindrical coordinates

$$x_1 = r\cos\varphi\,,\; x_2 = r\sin\varphi\,,\; x_3 = z$$

will be used. For $x = g(r,\varphi,z) = (r\cos\varphi,\, r\sin\varphi,\, z)^T$

$$\det Dg = \det \begin{pmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = r > 0\,.$$

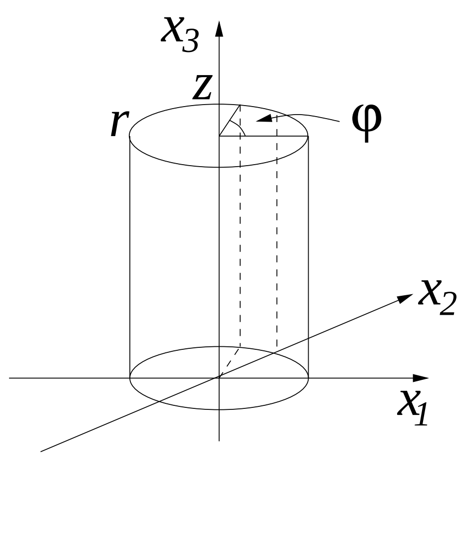

Fig. 9.5 Cylindrical coordinates

Let T be a compact subset of the domain

$$\{(r,\varphi,z) : r > 0,\ 0 < \varphi < 2\pi,\ z \in \mathbb{R}\}$$

and let $B = g(T)$. Then, for $f \in C^0(B)$,

$$\int_B f(x)\, dx = \int_T f(\cos\varphi, r\sin\varphi, z)\, r\, d(r,\varphi,z)\,.$$

If T is a box of the type $[r_1, r_2] \times [\varphi_1, \varphi_2] \times [z_1, z_2]$ with $0 < r_1 < r_2$, $0 < \varphi_1 < \varphi_2 < 2\pi$ (note that $r_1 = 0$ and $\varphi_1 = 0$, $\varphi_2 = 2\pi$ are allowed in the sense of improper Riemann integrals), Fubini's Theorem yields

$$\int_B f(x)\, dx = \int_{z_1}^{z_2} \left(\int_{r_1}^{r_2} \left(\int_{\varphi_1}^{\varphi_2} f(r\cos\varphi, r\sin\varphi, z)\, d\varphi \right) r\, dr \right) dz\,.$$

Example (*Polar or Spherical Coordinates in* $\mathbb{R}^3$) Every point $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ with $x_1^2 + x_2^2 + x_3^2 \neq 0$ can be written in the form

$$x_1 = r\cos\vartheta\cos\varphi, \quad x_2 = r\cos\vartheta\sin\varphi, \quad x_3 = r\sin\vartheta$$

with $r > 0$, $-\frac{\pi}{2} \leq \vartheta < \frac{\pi}{2}$, $0 \leq \varphi < 2\pi$. For the change of variables

$$g(r,\vartheta,\varphi) = (r\cos\vartheta\cos\varphi,\ r\cos\vartheta\sin\varphi,\ r\sin\vartheta)^T$$

we get Jacobi's determinant

$$\det Dg(r,\vartheta,\varphi) = \det \begin{pmatrix} \cos\vartheta\cos\varphi & -r\sin\vartheta\cos\varphi & -r\cos\vartheta\sin\varphi \\ \cos\vartheta\sin\varphi & -r\sin\vartheta\sin\varphi & r\cos\vartheta\cos\varphi \\ \sin\vartheta & r\cos\vartheta & 0 \end{pmatrix}$$

$$= -r^2\cos\vartheta < 0$$

(but $\det Dg = 0$, if $\vartheta = -\frac{\pi}{2}$).

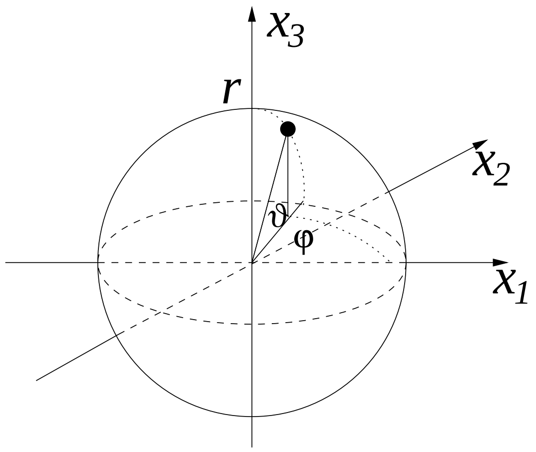

Fig. 9.6 Polar coordinates

Let T be a compact subset of $\{(r,\vartheta,\varphi) : r > 0\,,\ |\vartheta| < \frac{\pi}{2}\,,\ 0 < \varphi < 2\pi\}$ and let $B = g(T)$. Then, by the Change of Variable Formula, for $f \in C^0(B)$

$$\int_B f(x)\,dx = \int_T f(r\cos\vartheta\cos\varphi, r\cos\vartheta\sin\varphi, r\sin\vartheta)\,r^2\cos\vartheta\,d(r,\vartheta,\varphi)\,.$$

Again this result holds even when $r \geq 0\,,\ |\vartheta| \leq \frac{\pi}{2}$ and $0 \leq \varphi \leq 2\pi$. For $T = [r_1,r_2]\times[\vartheta_1,\vartheta_2]\times[\varphi_1,\varphi_2]$ with $0 \leq r_1 < r_2, -\frac{\pi}{2} \leq \vartheta_1 < \vartheta_2 \leq \frac{\pi}{2}$ and $0 \leq \varphi_1 < \varphi_2 \leq 2\pi$, Fubini's Theorem allows an iterated calculation of the integral on T.

Note that polar coordinates may also be used in the form

$$x_1 = r\sin\vartheta\cos\varphi, \quad x_2 = r\sin\vartheta\sin\varphi, \quad x_3 = r\cos\vartheta$$

where the angular variable $\vartheta \in (0,\pi)$ is measured between x and the e_3 - axis. In this case Jacobi's determinant equals $\det Dg(r,\vartheta,\varphi) = r^2\sin\vartheta \geq 0$.

Now we get for the volume of the three-dimensional ball $B_R(0)$

$$\begin{aligned}|B_R(0)| &= \int_0^R r^2\left(\int_0^\pi \sin\vartheta\,d\vartheta\right)\left(\int_0^{2\pi} d\varphi\right)\,dr\\ &= 2\cdot 2\pi\int_0^R r^2\,dr = \frac{4\pi}{3}R^3\,.\end{aligned}$$

Example (*Polar Coordinates in* $\mathbb{R}^n$) In this case we have to introduce – in addition to the Euclidean distance $r > 0$ – altogether $n-1$ angular variables

$$0 < \theta_1,\ldots,\theta_{n-2} < \pi\,, \quad 0 < \theta_{n-1} < 2\pi\,,$$

to write $x = (x_1,\ldots,x_n)^T$ in the form

$$\begin{aligned}x_1 &= r\cos\theta_1\\ x_2 &= r\sin\theta_1\cos\theta_2\\ x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3\\ &\vdots\\ x_{n-1} &= r\sin\theta_1\sin\theta_2\cdot\ldots\cdot\sin\theta_{n-2}\cos\theta_{n-1}\\ x_n &= r\sin\theta_1\sin\theta_2\cdot\ldots\cdot\sin\theta_{n-2}\sin\theta_{n-1}\,.\end{aligned}$$

The corresponding Jacobi determinant is

$$Dg(r,\theta_1,\ldots,\theta_{n-1}) = r^{n-1}\sin^{n-2}\theta_1\cdot\sin^{n-3}\theta_2\cdot\ldots\cdot\sin\theta_{n-2}\,.$$