6 Curves and Line Integrals

Definition A *parameterized path* (*arc, curve*) in $\mathbb{R}^n$ is a continuous map

$$\gamma : I \to \mathbb{R}^n, \quad t \mapsto \gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$$

on a compact interval $I \subset \mathbb{R}$. The path γ is called *differentiable* or *continuously differentiable* iff all components $\gamma_i(t),\ 1 \le i \le n$, are differentiable or continuously differentiable, resp.

If γ is continuously differentiable, in short, γ is a C^1-path, then for $t \in I$

$$\gamma'(t) = (\gamma_1'(t), \dots, \gamma_n'(t)) \in \mathbb{R}^n$$

is called the *tangential vector* of the curve γ at the parameter value t. If $\gamma'(t) \neq 0$, then γ is called *regular* at the parameter value t. In this case, the normalized vector

$$\frac{\gamma'(t)}{|\gamma'(t)|}$$

of Euclidean length 1 is called the *tangential unit vector*. The path γ is called regular iff $\gamma'(t) \neq 0$ for all $t \in I$.

Example 6.1

(1) The curve $\gamma : [0, 2\pi] \to \mathbb{R}^2,\ \gamma(t) = (\cos t, \sin t)$ describes the unit circle $\partial\, U_1(0) = \{x \in \mathbb{R}^2 \mid |x| = 1\}$ in the mathematically positive sense (counterclockwise orientation). For every parameter value t the curve is regular and

$$\gamma'(t) = (-\sin t, \cos t)$$

is the tangential unit vector.

(2) Let $\tilde{\gamma} :\ [0, 2\pi] \to \mathbb{R}^2$ be defined by $\tilde{\gamma}(t) = (\cos t, -\sin t)$. Again $\tilde{\gamma}$ describes the unit circle, but in the mathematically negative sense (clockwise orientation). Although $\gamma(I) = \tilde{\gamma}(I)$, the regular curve $\tilde{\gamma}$ is considered as a curve different from γ.

(3) Let $\hat{\gamma} = [0, \sqrt{2\pi}] \to \mathbb{R}^2$ be defined by $\hat{\gamma}(t) = (\cos t^2, \sin t^2)$. Obviously $\hat{\gamma}$ describes the unit circle, $\hat{\gamma}([0, \sqrt{2\pi}]) = \partial\, U_1(0)$. However, $\hat{\gamma}'(0) = 0$ implying that the curve is *not* regular at the parameter value $t = 0$. In particular the curve $\hat{\gamma}$ is considered as a curve different from the curve γ.

Definition A *polygonal line* in $\mathbb{R}^n$ is a (continuous) path $\gamma : I = [a, b] \to \mathbb{R}^n$ together with a partition $P : a = t_0 < t_1 < \cdots < t_m = b$ of $[a, b]$ such that $\gamma\,|_{[t_{j-1}, t_j]}$ is affine linear. Then the *arc length* $s(\gamma)$ of γ is defined by

$$s(\gamma) = \sum_{j=1}^{m} |\gamma(t_j) - \gamma(t_{j-1})|.$$

Note that the real number $s(\gamma)$ does not change when considering a refinement P' of P.

Definition Let $\gamma : I \to \mathbb{R}^n$ be a path and let $P : t_0 < t_1 < \ldots < t_m$ with $t_j \in I,\ 0 \le j \le m$, be a partition of I. Then let

$$s(P,\gamma) := \sum_{j=1}^{m} |\gamma(t_j) - \gamma(t_{j-1})|$$

denote the arc length of the polygonal line interpolating $\gamma(t_0), \ldots, \gamma(t_m)$. The path γ is called *rectifiable* iff the set $\{s(P,\gamma) \mid P \text{ a partition of } I\}$ is bounded. In this case

$$s(\gamma) := \sup_P s(P,\gamma)$$

is called the *arc length* of γ.

Example 6.2 Let $\gamma : I = [a,b] \to \mathbb{R}^n$ be Lipschitz continuous, i.e., there exists an $L > 0$ such that

$$|\gamma(t) - \gamma(t')| \le L\,|t - t'| \quad \text{for all} \quad t, t' \in I.$$

Then γ is rectifiable and $s(\gamma) \le L(b-a)$. To prove this result consider an arbitrary partition $P : t_0 < \ldots < t_m$ in $[a,b]$. Then

$$s(P,\gamma) = \sum_{j=1}^{m} |\gamma(t_j) - \gamma(t_{j-1})| \le L \sum_{j=1}^{m} |t_j - t_{j-1}| = L(b-a).$$

Theorem 6.3 *Let $\gamma : [a,b] \to \mathbb{R}^n$ be a continuously differentiable path. Then γ is rectifiable and*

$$s(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

Corollary 6.4 *Let the path $\gamma : [a,b] \to \mathbb{R}^2$ be defined as the graph of a continuously differentiable function $f : [a,b] \to \mathbb{R}$, i.e.,*

$$\gamma(t) = \big(t, f(t)\big), \quad t \in [a,b].$$

Then γ is rectifiable and

$$s(\gamma) = \int_a^b \sqrt{1 + f'(t)^2}\, dt.$$

Example 6.5 Parameterize the unit circle by $\gamma : [0, 2\pi] \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$, cf. Example 6.1 (1), and consider the arc $\gamma|_{[0,x]}$. Then by Theorem 6.3

$$s(\gamma|_{[0,x]}) = \int_0^x |(-\sin t, \cos t)|\, dt = \int_0^x 1\, dt = x.$$

Hence, for every $x \in (0, 2\pi]$, the arc length of the arc parameterized by $\gamma(t), t \in [0, x]$, equals x. This result explains the name radian measure (Bogenmaß) of the 'angle' x. In particular, $s(\gamma) = 2\pi$ is the arc length of the unit circle.

Proof of Theorem 6.3

Claim 1 γ is rectifiable and $s(\gamma) \le \int_a^b |\gamma'(t)|dt$.

Proof For an arbitrary partition $P : a = t_0 < \ldots < t_m \le b$ of $[a, b]$

$$\begin{aligned} s(P, \gamma) &= \sum_{j=1}^m |\gamma(t_j) - \gamma(t_{j-1})| \\ &= \sum_{j=1}^m \left| \int_{t_{j-1}}^{t_j} \gamma'(t)dt \right| \\ &\le \sum_{j=1}^m \int_{t_{j-1}}^{t_j} |\gamma'(t)|dt \\ &= \int_a^b |\gamma'(t)|dt \end{aligned}$$

where we used the 'triangle inequality' $|\int_{t_{j-1}}^{t_j} \gamma'(t)| \le \int_{t_{j-1}}^{t_j} |\gamma'(t)|dt$. This inequality may be proved by using the triangle inequality on $\mathbb{R}^n$ and by approximating $\int_{t_{j-1}}^{t_j} \gamma'(t)dt$ by Riemann sums. (■)

Claim 2 For every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$\left| \int_a^b |\gamma'(t)|dt - s(P, \gamma) \right| < \varepsilon.$$

Proof The map $t \mapsto |\gamma'(t)|$ is uniformly continuous on $[a, b]$. Hence for every $\varepsilon > 0$ we find a $\delta > 0$ such that for every partition $P : a = t_0 < t_1 < \ldots < t_m = b$ satisfying $\max_j (t_j - t_{j-1}) < \delta$ the inequality

$$\left| \int_a^b |\gamma'(t)|dt - \sum_{j=1}^m |\gamma'(t_j)|(t_j - t_{j-1}) \right| < \varepsilon$$

holds. Moreover, there exists a $\delta_1 > 0$ such that

$$|\gamma'(t) - \gamma'(\tau)| < \varepsilon \quad \text{for all} \quad t, \tau \in [a, b], \ |t - \tau| < \delta_1.$$

Hence

$$\begin{aligned} \left|\gamma'(t_j) - \frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}}\right| &= \left|\frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \left(\gamma'(t_j) - \gamma'(\tau)\right) d\tau\right| \\ &\leq \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} |\gamma'(t_j) - \gamma'(\tau)| d\tau \\ &< \varepsilon \end{aligned}$$

provided that $\delta < \delta_1$. This estimate allows to replace $\sum |\gamma'(t_j)| (t_j - t_{j-1})$ by $\sum |\gamma(t_j) - \gamma(t_{j-1})|$, since

$$\begin{aligned} &\left|\sum_{j=1}^{m} |\gamma'(t_j)| (t_j - t_{j-1}) - \sum_{j=1}^{m} |\gamma(t_j) - \gamma(t_{j-1})|\right| \\ &\leq \sum_{j=1}^{m} \left|\gamma'(t_j) - \frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}}\right| (t_j - t_{j-1}) \\ &\leq \varepsilon \sum_{j=1}^{m} (t_j - t_{j-1}) = \varepsilon(b - a). \end{aligned}$$

Summarizing the previous inequalities we get that

$$\left|\int_a^b |\gamma'(t)| dt - s(P, \gamma)\right| \leq \varepsilon(1 + b - a)$$

using the triangle inequality. (■) ■

In Example 6.1 we introduced three curves $\gamma, \tilde{\gamma}$ and $\hat{\gamma}$ with the same image (or trace), i.e., $\gamma([0, 2\pi]) = \tilde{\gamma}([0, 2\pi]) = \hat{\gamma}([0, \sqrt{2\pi}]) = \partial U_1(0)$, which nevertheless had to be considered as three different curves. On the other hand,

$$\delta : [0, \pi] \to \mathbb{R}^2, \quad \delta(t) = (\cos 2t, \sin 2t),$$

defines a regular curve with $\delta([0, \pi]) = \gamma([0, 2\pi])$ and with the same orientation. Since δ and γ are related to each other by the invertible, continuously differentiable function $t \mapsto 2t$, the curves γ and δ will be identified.

Definition 6.6 Let $\varphi : [\alpha, \beta] \to [a, b]$ be an invertible continuously differentiable function such that $\varphi^{-1} : [a, b] \to [\alpha, \beta]$ is continuously differentiable as well. Then φ is called a C^1-*diffeomorphism of parameters.* Furthermore, φ is called an *orientation-preserving parameter transformation* iff φ is strictly increasing; φ is called *orientation reversing* iff φ is strictly decreasing.

Theorem 6.7 *Let $\gamma : [a,b] \to \mathbb{R}^n$ be a continuously differentiable path, let $\varphi : [\alpha,\beta] \to [a,b]$ be a C^1-diffeomorphism of parameters and let $\delta = \gamma \circ \varphi : [\alpha,\beta] \to \mathbb{R}^n$.*

(1) *If γ is regular, then δ is regular as well. In this case the tangential vectors $\delta'(\tau)$ and $\gamma'(\varphi(\tau))$ are parallel at each $\tau \in [\alpha,\beta]$. These tangential vectors point into the same (opposite) direction iff φ is an orientation-preserving (-reversing) parameter transformation.*

(2) *$s(\gamma) = s(\delta)$, i.e., the arc length is invariant with respect to C^1-diffeomorphisms of parameters.*

Proof

(1) Since φ and φ^{-1} are continuously differentiable on $[\alpha,\beta]$ and $[a,b]$, resp., and since $\varphi^{-1}(\varphi(\tau)) = \tau$ implying $1 = (\varphi^{-1})' \cdot \varphi'$ we conclude that either $\varphi' > 0$ (orientation-preserving) or $\varphi' < 0$ (orientation-reversing) on $[\alpha,\beta]$. Then the identity

$$\delta'(\tau) = \gamma'(\varphi(\tau))\ \varphi'(\tau)$$

proves (1).

(2) Using the Change of Variable Formula we get when $\varphi' > 0$

$$\begin{aligned} s(\delta) &= \int_\alpha^\beta |\delta'(\tau)|\, d\tau = \int_\alpha^\beta |\gamma'(\varphi(\tau))|\ |\varphi'(\tau)|\, d\tau \\ &= \int_\alpha^\beta |\gamma'(\varphi(\tau))|\ \varphi'(\tau)\ d\tau = \int_a^b |\gamma'(t)|\ dt. \end{aligned}$$

When $\varphi' < 0$ we proceed analogously. ■

Among the set of all orientation-preserving parameter transformations for a given regular C^1-path $\gamma : [a,b] \to \mathbb{R}^n$ we choose that one 'related to arc length'. Let

$$\psi(t) = \int_a^t |\gamma'(t)|\ dt, \quad t \in [a,b].$$

Since $\psi'(t) = |\gamma'(t)| > 0,\ \psi(a) = 0$ and

$$\psi(b) = \int_a^b |\gamma'(t)|\ dt = s(\gamma),$$

the inverse $\varphi = \psi^{-1}$ is a C^1-function from $[0, s(\gamma)]$ to $[a,b]$. Then

$$\delta = \gamma \circ \varphi : [0, s(\gamma)] \to \mathbb{R}^n$$

is another parametrization of the path γ such that

$$\delta'(\tau) = \gamma'(\varphi(\tau)) \cdot \varphi'(\tau) = \frac{\gamma'(\varphi(\tau))}{\psi'(\varphi(\tau))} = \frac{\gamma'(t)}{|\gamma'(t)|}.$$

Hence δ is a new parametrization of γ with the same orientation satisfying

$$s(\delta|_{[0,\tau]}) = \int_0^\tau |\delta'(\tilde{\tau})| d\tilde{\tau} = \tau \quad \text{for all} \quad \tau \in [0, s(\gamma)].$$

Therefore δ is called *the parametrization of the path γ with respect to arc length.* We note that working with paths parameterized with respect to arc length may simplify computations and formulae a lot.

Next we try to integrate vector fields, i.e. mappings F from $\mathbb{R}^n$ to $\mathbb{R}^n$, along paths γ in $\mathbb{R}^n$ such that only the tangential part of F along γ will yield a contribution to that 'line integral'. This kind of integral is important e.g. in physics since the translation of a particle along γ in a force field F requires or yields energy iff $F(\gamma(t)) \cdot \gamma'(t) \neq 0$. However, line integrals are also important in analysis when looking for 'antiderivatives' of vector fields.

Definition 6.8 (1) Let $U \subset \mathbb{R}^n$ be open. Then a mapping $f : U \to \mathbb{R}^n$ is called a *vector field.*

(2) Let $f : U \to \mathbb{R}^n$ be a continuous vector field and let $\gamma : [a, b] \to U$ be a rectifiable path. Given a partition $P : a = t_0 < t_1 < \ldots < t_m = b$ of $[a, b]$ and 'intermediate points' ζ_j on the arc $\gamma([t_{j-1}, t_j])$, $1 \leq j \leq m$, i.e., $\zeta_j = \gamma(\tau_j)$ with $\tau_j \in [t_{j-1}, t_j]$,

$$R(P, f, \gamma) := \sum_{j=1}^{m} f(\zeta_j) \cdot \big(\gamma(t_j) - \gamma(t_{j-1})\big)$$

is called a *Riemann sum* of f along γ (with intermediate points ζ_j).

Theorem 6.9 *Let $f : U \to \mathbb{R}^n$ be a continuous vector field and let $\gamma : [a, b] \to U$ be a rectifiable path. Then there exists a real number $I(f, \gamma)$ such that the Riemann sums $R(P, f, \gamma)$ converge to $I(f, \gamma)$ when the mesh size of the partition P converges to 0.*

To be more precise, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Delta(P) := \max_{1 \leq j \leq m} |t_j - t_{j-1}| < \delta$$

implies

$$|R(P, f, \gamma) - I(f, \gamma)| < \varepsilon.$$

For $I(f,\gamma)$, *the* line integral of f along γ, *we will write*

$$\int_\gamma f(x)\cdot dx$$

(also $\int_\gamma (f_1 d\gamma_1 + \ldots + f_n d\gamma_n)$ *or* $\int_\gamma (f_1 dx_1 + \ldots + f_n dx_n)$*).*

Proof Given partitions P and P' of $[a,b]$ with $\Delta(P), \Delta(P') < \delta$ define the refinement $P'' = P \cup P' = \{t_j\}$ satisfying again $\Delta(P'') < \delta$. Then

$$\begin{aligned} |R(P,f,\gamma) - R(P',f,\gamma)| &= \left| \sum_j \left(f(\zeta_j) - f(\zeta_j')\right) \cdot \left(\gamma(t_j) - \gamma(t_{j-1})\right) \right| \\ &\leq \sum_j |f(\zeta_j) - f(\zeta_j')| \, |\gamma(t_j) - \gamma(t_{j-1})| \end{aligned}$$

where ζ_j and ζ_j' are intermediate points with respect to the partitions P and P', resp. However, ζ_j and ζ_j' will not necessarily lie on the arc $\gamma([t_{j-1}, t_j])$ related to P''. Nevertheless, there are parameters $\tau_j, \tau_j' \in [a,b]$ such that

$$\zeta_j = \gamma(\tau_j), \quad \zeta_j' = \gamma(\tau_j') \quad \text{and} \quad |\tau_j - \tau_j'| \leq 2\delta.$$

Now let $\varepsilon > 0$ be given. Since $f \circ \gamma = [a,b] \to \mathbb{R}^n$ is uniformly continuous, there exists $\delta > 0$ such that

$$|f\circ\gamma(\tau) - f\circ\gamma(\tau')| < \frac{\varepsilon}{s(\gamma)} \quad \text{for all} \quad \tau, \tau' \in [a,b], \ |\tau - \tau'| < 2\delta.$$

Hence

$$|R(P,f,\gamma) - R(P',f,\gamma)| \leq \frac{\varepsilon}{s(\gamma)} \sum_{j-1}^{m} |\gamma(t_j) - \gamma(t_{j-1})| \leq \varepsilon \qquad (6.1)$$

provided that $\Delta(P) < \delta$ and $\Delta(P') < \delta$.

This 'Cauchy property' will yield the existence of the number $I(f,\gamma) = \int_\gamma f(x)\cdot dx$ as follows. Consider a sequence of partitions $(P_n)_n$ of $[a,b]$ such that $\Delta(P_n) \to 0$ as $n \to \infty$. By (6.1) the sequence $(R(P_n,f,\gamma))_n$ is a Cauchy sequence; consequently there exists $I(f,\gamma) \in \mathbb{R}$ such that

$$R(P_n,f,\gamma) \to I(f,\gamma) \quad \text{as} \quad n \to \infty.$$

Actually, $I(f,\gamma)$ is independent of the sequence of partitions. Given partitions $(P_n), (P_n')$ with $\Delta(P_n) \to 0, \Delta(P_n') \to 0$ the sequence of partitions (P_n'') defined by $P_1, P_1', P_2, P_2', P_3, P_3', \ldots$ again yields a convergent sequence of Riemann sums $(R(P_n'', f, \gamma))$. Its unique limit equals the limits of the convergent subsequences $(R(P_n,f,\gamma))$ and $(R(P_n',f,\gamma))$. This argument completes the proof. ∎

Theorem 6.10 *Let $U \subset \mathbb{R}^n$ be open, let $\gamma : [a,b] \to U$ be a continuously differentiable path and let $f : U \to \mathbb{R}^n$ be a continuous vector field. Then*

$$\int_\gamma f(x) \cdot dx = \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt.$$

Proof Formally we argue that

$$\begin{aligned} \int_\gamma f(x) \cdot dx &\sim \sum_j f(\gamma(t_j))\ (\gamma(t_j) - \gamma(t_{t-1})) \\ &\sim \sum_j f(\gamma(t_j)) \cdot \gamma'(t_j)(t_j - t_{t-1}) \\ &\sim \int_a^b f(\gamma(t)) \cdot \gamma'(t)\ dt. \end{aligned}$$

To be more precise, let $P = \{t_j\}$ be a partition of $[a,b]$ and choose the intermediate points $\zeta_j = \gamma(t_j)$. Then

$$\begin{aligned} R(P,f,\gamma) &= \sum_{j=1}^m f(\gamma(t_j)) \cdot (\gamma(t_j) - \gamma(t_{t-1})) \\ &= \sum_{j=1}^m f(\gamma(t_j)) \cdot \gamma'(t_j)(t_j - t_{j-1}) \\ &\quad + \sum_{j=1}^m f(\gamma(t_j)) \cdot \int_{t_{j-1}}^{t_j} (\gamma'(s) - \gamma'(t_j))\, ds \\ &=: R_1(P,f,\gamma) + R_2(P,f,\gamma). \end{aligned}$$

Here $R(P,f,\gamma)$ converges to $\int_\gamma f(x) \cdot dx$ and $R_1(P,f,\gamma)$ converges to $\int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$ (being a Riemann sum of that integral) as the mesh size of P goes to 0. Furthermore, the boundedness of $|f \circ \gamma|$ on $[a,b]$ by a constant $M \geq 0$ and the uniform continuity of γ' on $[a,b]$, i.e., $|\gamma'(s) - \gamma'(\tilde{s})| < \varepsilon$ for all $s, s' \in [a,b]$, $|s - s'| < \delta$, yield the estimate

$$|R_2(P,f,\gamma)| \leq \sum_{j=1}^m M(t_j - t_{j-1})\, \varepsilon = \varepsilon\, M(b-a)$$

provided that the mesh size of P is less than δ. Hence $R_2(P,f,\gamma) \to 0$ as $\Delta(P) \to 0$ completing the proof. ■

Corollary 6.11 *Let $U \subset \mathbb{R}^n$ be open, let $f, g : U \to \mathbb{R}^n$ be continuous vector fields and let $\gamma : [a,b] \to U$ be rectifiable.*

(1)

$$\begin{aligned}\int_\gamma (f+g)(x)\cdot dx &= \int_\gamma f(x)\cdot dx + \int_\gamma g(x)\cdot dx \\ \int_\gamma (c\,f)(x)\cdot dx &= c\int_\gamma f(x)\cdot dx \quad \textit{for all} \quad c\in\mathbb{R}\end{aligned}$$

(2) *Let γ^- denote the path obtained from γ by reversing the orientation, i.e.,*

$$\gamma^- : [a,b]\to U,\ \gamma^-(t) = \gamma(a+b-t).$$

Then

$$\int_{\gamma^-} f(x)\cdot dx = -\int_\gamma f(x)\cdot dx.$$

(3) *Let $\delta : [b,c]\to U$ denote another rectifiable path such that $\gamma(b) = \delta(b)$, and let $\gamma\oplus\delta$ denote the concatenated path connecting $\gamma(a)$ via $\gamma(b) = \delta(b)$ with $\delta(c)$. Then*

$$\int_{\gamma\oplus\delta} f(x)\cdot dx = \int_\gamma f(x)\cdot dx + \int_\delta f(x)\cdot dx.$$

(4)

$$|\int_\gamma f(x)\cdot dx| \le \|f\|_{\infty,\gamma}\, s(\gamma)$$

where $\|f\|_{\infty,\gamma} = \sup\{|f(\gamma(t))| \mid t\in[a,b]\}$.

Proof The assertions (1) - (3) are trivial. To prove (4) consider a partition P of $[a,b]$. Then by the triangle inequality and the Cauchy-Schwarz inequality

$$\begin{aligned}|R(P,f,\gamma)| &= \left|\sum_{j=1}^m f(\gamma(t_j))\cdot\big(\gamma(t_j)-\gamma(t_{j-1})\big)\right| \\ &\le \sum_{j=1}^m \left|f\big(\gamma(t_j)\big)\right|\,|\gamma(t_j)-\gamma(t_{j-1})| \\ &\le \|f\|_{\infty,\gamma}\sum_{j=1}^m |\gamma(t_j)-\gamma(t_{j-1})|\end{aligned}$$

proving that $|R(P,f,\gamma)| \le \|f\|_{\infty,\gamma}\, s(\gamma)$. Now Theorem 6.9 completes the proof. ∎

Theorem 6.12 *Let $U\subset\mathbb{R}^n$ be open, let $\gamma : [a,b]\to U$ be a rectifiable path and let $f : U\to\mathbb{R}^n$ be a continuous vector field. Furthermore, consider an orientation-preserving parameter transformation $\varphi : [\alpha,\beta]\to[a,b]$ and define $\delta = \gamma\circ\varphi$. Then*

$$\int_\gamma f(x)\cdot dx = \int_\delta f(x)\cdot dx,$$

i.e., the value of the line integral $\int_\gamma f(x)\cdot dx$ *does* not *depend on the parametrization of the path.*

Proof For simplicity we only consider the case of C^1-paths γ. Then by Theorem 6.10 and the Change of Variable Formula (with $t=\varphi(s)$)

$$\begin{aligned}\int_\gamma f(x)\cdot dx &= \sum_{i=1}^n \int_a^b (f_i\circ\gamma)(t)\gamma_i'(t)\,dt\\ &= \sum_{i=1}^n \int_\alpha^\beta (f_i\circ\gamma\circ\varphi)(s)\underbrace{\gamma_i'(\varphi(s))\varphi'(s)}_{=(\gamma_i\circ\varphi)'(s)=\delta_i'(s)}\,ds\\ &= \int_\delta f(x)\cdot dx.\end{aligned}$$

When γ is only rectifiable, but not C^1, a straightforward argument will use the approximants $R(P,f,\gamma)$ and $R(\tilde{P},f,\delta)$ of $\int_\gamma f(x)\cdot dx$ and of $\int_\delta f(x)\cdot dx$ resp. with partitions P of $[a,b]$ and corresponding partitions $\tilde{P}$ of $[\alpha,\beta]$. ■

Although by Theorem 6.12 the value of the line integral $\int_\gamma f(x)\cdot dx$ does not change when working with a different parametrization of the path γ with the same orientation, the integral will generally change when considering two different paths connecting the same points in U. However, under certain conditions of physical and mathematical importance, the value of the line integral will depend only on the endpoints of the path but not on the actual shape of the path between the endpoints.

Definition 6.13 *An open set* $U\subset\mathbb{R}^n$ *is called a* domain *iff it is* pathwise connected, *i.e., for any two points* $x,y\in U$ *there exists a path* $\gamma:[a,b]\to U$ *with endpoints* $\gamma(a)=x,\gamma(b)=y$.

We note that any two points x,y in a domain $U\subset\mathbb{R}^n$ may be connected by a polygonal line and even by a continuously differentiable path. The proof which will be omitted is based on the compactness of $\gamma([a,b])$, the uniform continuity of γ and on some approximation procedures.

Definition 6.14 Let $U\subset\mathbb{R}^n$ be open and let $\varphi:U\to\mathbb{R}$ be differentiable. Then the vector field

$$f:U\to\mathbb{R}^n,\ f(x):=\nabla\varphi(x)$$

is called a **gradient field** and φ is called a **potential function** of f.

Obviously, the potential function φ defining the gradient field $f=\nabla\varphi$ is not unique, since $f=\nabla\varphi=\nabla(\varphi+c)$ for every constant $c\in\mathbb{R}$. However, by Theorem 4.16 , any other potential function ψ of f on a *domain* U differs from φ only by an additive constant c, since $\nabla(\varphi-\psi)=0$.

Theorem 6.15 *Let $U \subset \mathbb{R}^n$ be a domain and let $f : U \to \mathbb{R}^n$ be a gradient field with a continuously differentiable potential function $\varphi : U \to \mathbb{R}$.*

(1) *For any two points $x_0, x_1 \in U$ and any (piecewise) continuously differentiable path γ in U connecting x_0 with x_1*

$$\int_\gamma f(x) \cdot dx = \varphi(x_1) - \varphi(x_0).$$

Hence, the value of the line integral $\int_\gamma f(x) \cdot dx$ does not depend on γ itself, but only on the endpoints of γ: The line integral is independent of the path.

(2) *If γ is a* closed *and (piecewise) continuously differentiable path, i.e., the endpoints of γ coincide, then*

$$\int_\gamma f(x) \cdot dx = 0\,.$$

Very often this result on a line integral along a closed path γ is written as $\oint_\gamma f(x) \cdot dx = 0$.

Proof

(1) First let $\gamma : [a, b] \to U$ be continuously differentiable. Then by Theorem 6.10 and the chain rule

$$\begin{aligned} \int_\gamma f(x) \cdot dx &= \int_a^b (\nabla\varphi)(\gamma(t)) \cdot \gamma'(t)\,dt \\ &= \int_a^b \frac{d}{dt}(\varphi \circ \gamma)(t)\,dt \\ &= \varphi(\gamma(b)) - \varphi\big(\gamma(a)\big) \\ &= \varphi(x_1) - \varphi(x_0). \end{aligned}$$

If γ is only a piecewise C^1-function, the previous part may be applied to every C^1-part $\gamma\,|_{[t_{j-1},t_j]}$ of γ, $1 \le j \le m$. Then Corollary 6.11 (3) applied to the concatenation

$$\gamma = \gamma\,|_{[t_0,t_1]} \oplus \ldots \oplus \gamma\,|_{[t_{m-1},t_m]}$$

finishes the proof.

(2) is an easy consequence of (1) since $x_0 = x_1$. ■

Theorem 6.15 (1) may be considered as a generalization of the Fundamental Theorem of Calculus on $\mathbb{R}^1$ to the multidimensional case. But in contrast to the one-dimensional case in which every continuous function has an antiderivative, not every vector field will have a potential. Furthermore, the question how to find a potential of a given vector field arises.

Theorem 6.16 *Let $U \subset \mathbb{R}^n$ be a domain and let $f : U \to \mathbb{R}^n$ be a continuous vector field. If the line integral $\int_\gamma f(x) \cdot dx$ is independent of the path, then f is a gradient field.*
To be more precise, fix $a \in U$ and define $\varphi : U \to \mathbb{R}$ by

$$\varphi(x) = \int_a^x f(y) \cdot dy := \int_{\gamma_x} f(y) \cdot dx$$

where γ_x is an arbitrary (piecewise) continuously differentiable path in U connecting a with x. Then φ is continuously differentiable and

$$\nabla \varphi = f \quad in \quad U.$$

Proof First we note that φ is well-defined: Given two paths γ_x and γ'_x in U connecting a with x, the path independence of line integrals of f yields

$$\int_{\gamma_x} f(y) \cdot dy = \int_{\gamma'_x} f(y) \cdot dy.$$

To prove the differentiability of φ let $x \in U$ and choose $\varepsilon > 0$ such that the ball $U_\varepsilon(x)$ is contained in U. Then for all $h \in \mathbb{R}^n$ with $|h| < \varepsilon$ the straight line $[x, x+h]$ which may be parameterized by $\sigma(t) = x + th$, $t \in [0, 1]$, lies in U. Furthermore, let γ_x be a path in U from a to x and let $\gamma_x \oplus \sigma$ be the concatenated path from a to $x + h$ via x. Then by Theorem 6.10 and Corollary 6.11

$$\begin{aligned}
&\varphi(x+h) - \varphi(x) - f(x) \cdot h \\
&= \int_{\gamma_x \oplus \sigma} f(y) \cdot dy - \int_{\gamma_x} f(y) \cdot dy - f(x) \cdot h \\
&= \int_\sigma f(y) \cdot dy - f(x) \cdot h \\
&= \int_0^1 \big(f(x+th) - f(x)\big) \cdot h \, dt
\end{aligned}$$

inserting the parametrization of σ. Hence

$$|\varphi(x+h) - \varphi(x) - f(x) \cdot h| \le |h| \sup_{t \in [0,1]} |f(x+th) - f(x)|$$

where the sup-term converges to 0 as $h \to 0$ due to the continuity of f. This convergence proves the differentiability of φ at x and that $\nabla \varphi(x) = f(x)$. ∎

Since the path independence of line integrals cannot be checked in practice, we are looking for other necessary and sufficient criteria to prove that a given vector field is a gradient field.

Theorem 6.17 *Let $f : U \to \mathbb{R}^n$ be a C^1-gradient field on an open set $U \subset \mathbb{R}^n$. Then for all $1 \le j, k \le n$*

$$\partial_k f_j = \partial_j f_k \quad \text{in} \quad U.$$

In particular, in the two-dimensional case $\partial_1 f_2 = \partial_2 f_1$. In the three-dimensional case, the rotation *or* curl

$$\operatorname{rot} f(x) := \begin{pmatrix} \partial_2 f_3 - \partial_3 f_2 \\ \partial_3 f_1 - \partial_1 f_3 \\ \partial_1 f_2 - \partial_2 f_1 \end{pmatrix} (x) = 0.$$

Proof Since f is a C^1-vector field, its potential φ is of class C^2. Hence by Schwarz' theorem

$$\partial_k f_j = \partial_k(\partial_j \varphi) = \partial_j(\partial_k \varphi) = \partial_j f_k.$$

■

Definition 6.18 A domain $U \subset \mathbb{R}^n$ is called **star-shaped** iff there exists an $m \in U$ such that every segment

$$[m, x] \subset U \quad \text{for every} \quad x \in U.$$

Note that every *convex* domain U, i.e., $[x, y] \subset U$ for every $x, y \in U$, is star-shaped (with respect to every $m \in U$), but that a star-shaped domain is not necessarily convex.

Theorem 6.19 *Let $U \subset \mathbb{R}^n$ be star-shaped domain. A C^1-vectorfield $f : U \to \mathbb{R}^n$ is a gradient field iff*

$$\partial_j f_k = \partial_k f_j \quad \text{for all} \quad 1 \le j, k \le n.$$

Proof Without loss of generality assume that the 'center' m in the definition of the starshapedness equals 0. Then, using $\sigma_x(t) = tx$, $0 \le t \le 1$, $x \in U$, define

$$\begin{aligned} \varphi(x) &= \int_{\sigma_x} f(y) \cdot dy = \int_0^1 f(tx) \cdot x \, dt \\ &= \sum_{i=1}^n x_i \int_0^1 f_i(tx_1, \ldots, tx_n) \, dt. \end{aligned}$$

Since $f \in C^1$, we may differentiate φ with respect to x_k and get

$$\partial_k \varphi(x) = \int_0^1 f_k(tx) \, dt + \sum_{i=1}^n x_i \int_0^1 \partial_k f_i(tx) \, t \, dt$$

where by assumption $\partial_k f_i = \partial_i f_k$. Obviously $\frac{d}{dt}(t f_k(tx)) = f_k(tx) + tx \cdot \nabla f_k(tx)$, and consequently

$$\partial_k \varphi(x) = \int_0^1 \frac{d}{dt}(t f_k(tx)) \, dt = f_k(x) - 0 \, .$$

Now the theorem is proved. ■

The assumption in Theorem 6.19 that the domain U is star-shaped is not necessary and by far too strong. Actually it suffices to assume that U has 'no holes'. In that case the integral theorems of vector analysis, see Green's theorem $(n = 2)$ and Stokes' Theorem $(n = 3)$ below prove that the 'integrability conditions' $\partial_j f_k = \partial_k f_j$, $1 \leq j, k \leq n$, imply the path independence of line integrals of f and consequently the existence of a potential function of f.