Lecture Script

Analysis I

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Contents

Chapter I

Number Basics

1 The Real Numbers

What are the real numbers? Depending on your point of view this can be a difficult question. In the following, we describe the set $\mathbb{R}$ of real numbers by giving rules which allow us to 'calculate' with these numbers. This set of rules (or axioms) form *the axiom system* of the real numbers.

This system consists of the following:

- Field axioms
- Ordering axioms
- Completeness axiom

All statements about the real numbers can be derived exclusively from these axioms. We begin with the field axioms.

1.1. The Field Axioms.
There are two operations on the set $\mathbb{R}$, namely addition '+' and multiplication '$\cdot$':

Addition:	$\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $(x, y) \mapsto x + y$
Multiplication:	$\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $(x, y) \mapsto x \cdot y$

These satisfy the following field axioms:

Axioms of Addition

(A1) *Law of Commutativity*: for all $x, y \in \mathbb{R}$, $x + y = y + x$.

(A2) *Law of Associativity*: for all $x, y, z \in \mathbb{R}$, $(x + y) + z = x + (y + z)$.

(A3) *Existence of a Neutral Element*: There exists $0 \in \mathbb{R}$ such that $x + 0 = x$ for all $x \in \mathbb{R}$.

(A4) *Existence of an Inverse Element*: For every $x \in \mathbb{R}$ there exists a $-x \in \mathbb{R}$ such that $x + (-x) = 0$.

Axioms of Multiplication

(M1) *Law of Commutativity*: for all $x, y \in \mathbb{R}$, $x \cdot y = y \cdot x$.

(M2) *Law of Associativity*: for all $x, y, z \in \mathbb{R}$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(M3) *Existence of a Neutral Element*: There exists a $1 \in \mathbb{R}, 1 \neq 0$ such that $x \cdot 1 = x$ for all $x \in \mathbb{R}$.

(M4) *Existence of an Inverse Element*: for every $x \in \mathbb{R}$ with $x \neq 0$, there exists an $x^{-1} \in \mathbb{R}$ such that $x \cdot (x^{-1}) = 1$.

The *law of distributivity* shows how addition and multiplication interact.

(D) *Law of Distributivity*: for all $x, y, z \in \mathbb{R}$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

A set $\mathbb{K}$ of elements $a, b, \ldots$, together with the binary operations $a + b$ and $a \cdot b$ which satisfy the above axioms, is called a *field*. In the lecture *linear algebra*, fields and their axioms will be treated in greater detail. At this point, we only remark that the elements 0 and 1 are uniquely determined and that the statement $x \cdot y = 0$ implies that at least one of x and y is zero.

We introduce the following simplifying notations

$$xy := x \cdot y, \quad \frac{x}{y} := x \cdot y^{-1}, \quad x - y := x + (-y), \quad x^2 := x \cdot x, \quad 2x := x + x$$

1.2. The Ordering Axioms.
In $\mathbb{R}$, certain numbers have the distinguished property of being positive (written $x > 0$) such that we have:

(O1) For every $x \in \mathbb{R}$, exactly one of the following relations is true: $x = 0$, $x > 0$, $-x > 0$

(O2) $x > 0, \;\; y > 0 \Rightarrow x + y > 0$

(O3) If $x > 0, \;\; y > 0$, it follows that $x \cdot y > 0$

The second axiom states compatibility with addition, the third one compatibility with multiplication.

The following definition enables us to compare any two elements of $\mathbb{R}$:

1.3 Definition. Let $x, y \in \mathbb{R}$. We define

$$\begin{aligned} x > y \quad &:\Leftrightarrow \quad x - y > 0 \\ x \geq y \quad &:\Leftrightarrow \quad x - y > 0 \text{ or } x - y = 0. \end{aligned}$$

An element $x \in \mathbb{R}$ with $x > 0$ is called *positive (positiv)*.

For $x > y$ and $x \geq y$ one can also write $y < x$ respectively $y \leq x$. If $x < 0$, then x is called *negative (negativ)*.

1.4. Calculation rules. *Let $x, y, z, u, v \in \mathbb{R}$. Then the following statements hold:*
a) Exactly one of the following relations holds: $x = y$, $x < y$ or $x > y$ (Law of Trichotomy)
b) $x < y$ and $y < z \Rightarrow x < z$, (Transitivity)
c) $x < y$ and $u \leq v$ implies $x + u < y + v$ (Monotonicity of Addition)
d) $x < y \Rightarrow -x > -y$
e) $x < y, \; u > 0 \Rightarrow xu < yu$, (Monotonicity of Multiplication)
f) $x \neq 0 \Rightarrow x^2 > 0$, particularly $1 > 0$
g) $0 < x < y \Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$
h) $x < y \Rightarrow x < \frac{x+y}{2} < y$

Proof. a) Follows from Definition 1.3 and Ordering Axiom (O1).
b) By Definiton 1.3, $y - x > 0$ and $z - y > 0$. Ordering Axiom (O2) implies $\underbrace{(y - x) + (z - y)}_{=z-x} > 0 \quad \Rightarrow z > x \Rightarrow x < z$
c) – e) Exercises
f) Let $x > 0$, then $x \cdot x = x^2 > 0$ by (O2). If $x < 0$, then d) implies that $-x > 0$ and from there $(-x)(-x) = (-x)^2 > 0$ by (O3). That $(-x)(-x) = x^2$ follows from results in the solution of Tutorial 1. Namely, we get $(-x)(-x) = (-1)x \cdot (-1)x = (-1)^2x^2$. We get also $(-1)(-1) = -(1 \cdot (-1)) = -(-1) = 1$. So $(-x)(-x) = x^2$.
g) $x^{-1} = \underbrace{x}_{>0} \cdot \underbrace{(x^{-1})^2}_{>0} > 0$, analogously $y^{-1} > 0$. Therefore $x^{-1} \cdot y^{-1} > 0$. Given this, together with $0 < x < y$, it holds that:

$$y^{-1} = x \cdot (x^{-1}y^{-1}) < y(x^{-1}y^{-1}) = x^{-1}$$

h) Exercise

□

The field- and ordering axioms imply that in addition to $0, 1$, other numbers exist in $\mathbb{R}$. In fact, adding 0 resp. 1 to both sides of the inequality $0 < 1$, we get $0 + 0 = 0 < 1 + 0 = 1$, $1 < 1 + 1 = 2$, therefore $2 \neq 0, 2 \neq 1$.

1.5 Definition. (Absolute Value). Let $x \in \mathbb{R}$. We define the *absolute value (Absolutbetrag) of* x as

$$|x| := \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

1.6 Remark. For the absolute value, we have the following rules:

a) $|x| \geq 0 \;\forall\, x \in \mathbb{R}$ and $|x| = 0 \Leftrightarrow x = 0$.

b) $|-x| = |x|, \quad x \in \mathbb{R}$

c) $\big||x|\big| = |x|, \quad x \in \mathbb{R}$

d) $|x \cdot y| = |x| \cdot |y|, \quad x, y \in \mathbb{R}$

e) $|\frac{x}{y}| = \frac{|x|}{|y|}, \quad x \in \mathbb{R}, y \neq 0$

f) $|x + y| \leq |x| + |y|, \quad x, y \in \mathbb{R}$ (Triangle Inequality)

g) $||x| - |y|| \leq |x - y|, \quad x, y \in \mathbb{R}$ (Reverse Triangle Inequality)

Proof. a) b) c) d) e) g) as exercises. For f): Let $x, y \in \mathbb{R}$. Then $x \leq |x|$, $y \leq |y|$ and also $-x \leq |x|$ and $-y \leq |y|$. The monotonicity of addition 1.4. c) gives that $-x-y \leq |x|+|y|$ and also that $x+y \leq |x|+|y|$. Therefore, it holds that $|x+y| \leq |x|+|y|$.

□

We now consider the positive integers as a subset of $\mathbb{R}$. To this end, we have the following definition:

1.7 Definition. Let $M \subset \mathbb{R}$. Then M is called *inductive (induktiv)*, if the following hold:

a) $0 \in M$

b) $x \in M \Rightarrow x + 1 \in M$.

Obviously, the set $\mathbb{R}$ of real numbers is inductive. If we define $M := \{x \in \mathbb{R} : x \geq a\}$, then M is inductive if we have $a \leq 0$.

1.8. Theorem and Definition. *There exists a smallest inductive subset of* $\mathbb{R}$*; this is called the* set of nonnegative integers *and is denoted by* $\mathbb{N}_0$*.*

Proof. Let $M \subset \mathbb{R}$ be inductive. Set

$$\mathbb{N}_0 := \bigcap_{\substack{M \subset \mathbb{R} \\ M \text{ inductive}}} M;$$

in other words, $\mathbb{N}_0$ is the intersection of all inductive subsets of $\mathbb{R}$. Therefore, it holds that $0 \in \mathbb{N}_0$, since $0 \in M$ for all inductive sets $M \subset \mathbb{R}$.

$$\begin{aligned} \text{Additionally, let } x \in \mathbb{N}_0 \quad &\Rightarrow \quad x \in M \ \text{ for all inductive subsets } M \subset \mathbb{R} \\ &\Rightarrow \quad x+1 \in M \ \text{ for all inductive subsets } M \subset \mathbb{R} \\ &\Rightarrow \quad x+1 \in \mathbb{N}_0. \end{aligned}$$

Therefore, $\mathbb{N}_0$ is inductive and since $\mathbb{N}_0 \subset M$ for all inductive sets $M \subset \mathbb{R}$, then $\mathbb{N}_0$ is the smallest inductive subset of $\mathbb{R}$.

□

1.9 Corollary. (Induction). *Let* $N \subset \mathbb{N}_0$ *be a set with the following properties:*

a) $0 \in N$

b) $x \in N \Rightarrow x + 1 \in N$

Then $N = \mathbb{N}_0$*.*

The proof is obvious since $\mathbb{N}_0$ is the smallest inductive subset of $\mathbb{R}$.

This corollary enables us to consider the method of *proof by induction.*

1.10 Theorem. *For every* $n \in \mathbb{N}_0$ *let the proposition* $A(n)$ *be defined. If it holds that:*

a) $A(0)$ *is true (Induction Start).*

b) If $A(n)$ *is true, then* $A(n+1)$ *is also true (Induction Step).*

Then $A(n)$ *holds for all* $n \in \mathbb{N}_0$*.*

Proof. Set $N := \{n \in \mathbb{N}_0 : A(n) \text{ is true }\} \Rightarrow N \subset \mathbb{N}_0$ inductive $\overset{1.9}{\Rightarrow} N = \mathbb{N}_0$.

The assumption in b) that $A(n)$ is true is called the Induction Hypothesis.

1.11 Examples.

a) *The Bernoulli Inequality:*
Let $x > -1$ and $n \in \mathbb{N}_0$. Then

$$(1+x)^n \geq 1 + nx.$$

The proof is left as an exercise.

b) *Geometric Series:* Let $q \in \mathbb{R}$ with $q \neq 1$ and $n \in \mathbb{N}_0$. Then

$$q^0 + q^1 + q^2 \cdots + q^n = \frac{1-q^{n+1}}{1-q}.$$

Proof. Induction Start: $A(0)$ is true since $q^0 = 1 = \frac{1-q}{1-q} = 1$.
Induction Step (IS):
By assumption, $A(n)$ is true. Then:

$$\begin{aligned} \underbrace{q^0 + q^1 + \cdot + q^n} + q^{n+1} &= \frac{1-q^{n+1}}{1-q} + q^{n+1} = \frac{1-q^{n+1} + (1-q)q^{n+1}}{1-q} \\ &= \frac{1-q^{n+2}}{1-q} \end{aligned}$$

Therefore $A(n)$ holds for all $n \in \mathbb{N}_0$.

1.12 Theorem. (Properties of $\mathbb{N}_0$). *The following statements hold:*

a) $0, 1 \in \mathbb{N}_0$

b) $n \in \mathbb{N}_0 \Rightarrow n = 0$ *or* $n \geq 1$

c) $n, m \in \mathbb{N}_0 \Rightarrow n + m, n \cdot m \in \mathbb{N}_0$

d) $n, m \in \mathbb{N}_0, n \geq m \Rightarrow n - m \in \mathbb{N}_0$

e) Let $n \in \mathbb{N}_0$. *There does not exist an* $m \in \mathbb{N}_0$ *such that* $n < m < n+1$.

f) Every nonempty set M *of nonnegative integers contains a smallest element, i.e. let* $M \neq \emptyset, M \subset \mathbb{N}_0 \Rightarrow\ \exists\, m \in M$ *with* $m \leq n\ \forall\, n \in M$.

Proof. a) $0 \in \mathbb{N}_0$ by definition and $\mathbb{N}_0$ is inductive. Therefore $0 + 1 = 1 \in \mathbb{N}_0$.
b) Set $B := \{0\} \cup \{n \in \mathbb{N}_0 : n - 1 \in \mathbb{N}_0 \text{ and } n - 1 \geq 0\} \subset \mathbb{N}_0$. Then B is inductive. In fact, $0 \in B$. Additionally, let $n \in B$. We need to show that $n + 1 \in B$. If $n = 0$, then it follows that $n + 1 = 1 \in B$. If $n \neq 0$, then $0 \leq n - 1 \Rightarrow 0 < 1 \leq n = (n+1) - 1 \in \mathbb{N}_0$ and therefore $n + 1 \in B$. $\Rightarrow B = \mathbb{N}_0$ and therefore, the claim.
c)d)e)f) as exercises

□

1.13. A Variant of the Induction Principle.
If for some $n_0 \in \mathbb{N}_0$:

a) *$A(n_0)$ is true.*

b) $A(n_0), A(n_0+1), \dots A(n)$ being true $\Rightarrow A(n+1)$ is also true

Then $A(n)$ is true for all $n \geq n_0$.

Thus one can show, for example, that $2^n > n^2$ if $n \geq 5$.

1.14. Examples of Induction. We now consider *recursive definitions*:

a) *Powers*: For $x \in \mathbb{R}$ set

$$\begin{aligned} x^0 &:= 1 \\ x^{n+1} &:= x \cdot x^n, \quad n \in \mathbb{N}_0 \end{aligned}$$

b) *Factorials*:

$$\begin{aligned} 0! &:= 1 \\ (n+1)! &:= (n+1) \cdot n!, \quad n \in \mathbb{N}_0 \end{aligned}$$

c) *Finite Series and Products*:
Let $a_j \in \mathbb{R}$ for $j \in \mathbb{N}_0$. We set

$$\sum_{j=0}^{0} a_j := a_0, \quad \sum_{j=0}^{n+1} a_j := a_{n+1} + \sum_{j=0}^{n} a_j, \quad n \in \mathbb{N}_0$$

$$\prod_{j=0}^{0} a_j := a_0, \quad \prod_{j=0}^{n+1} a_j := a_{n+1} \cdot \prod_{j=0}^{n} a_j, \quad n \in \mathbb{N}_0.$$

Analogously, we define

$$\sum_{j=l}^{n} a_j \quad \text{and} \quad \prod_{j=l}^{n} a_j, \qquad n \geq l$$

d) *Binomial coefficients*:
For $a \in \mathbb{R},\ n \in \mathbb{N}_0$, set

$$\binom{a}{0} := 1, \quad \binom{a}{n+1} := \frac{a-n}{n+1}\binom{a}{n}$$

The following statements can be proved by induction:

i) Let $a \in \mathbb{R}$ and $n, m \in \mathbb{N}_0 \Rightarrow a^n \cdot a^m = a^{n+m}$

ii) For $n, k \in \mathbb{N}_0$ with $0 \leq k \leq n$, we have $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

iii) For $n, k \in \mathbb{N}_0$, we have $\binom{n}{k} = \begin{cases} \binom{n}{n-k} & \text{falls} \quad k \leq n \\ 0 & \text{falls} \quad k > n \end{cases}$

iv) For $n, k \in \mathbb{N}_0$, we have $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ "Pascal's Triangle"

1.15 Theorem. (Binomial Theorem). *Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}.$$

Proof. Induction Start: For $n = 0$, it holds that:

$$1 = (a+b)^0 = \sum_{j=0}^{0} \binom{0}{j} a^0 b^0 = 1.$$

Induction Step: Let the statement from the theorem hold for some $n \in \mathbb{N}_0$: Then

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b) \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} \\
&= \sum_{j=0}^{n} \binom{n}{j} a^{j+1} b^{n-j} + \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j+1} \\
&= \sum_{j=1}^{n+1} \binom{n}{j-1} a^j b^{n-(j-1)} + \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j+1} \\
&= \sum_{j=1}^{n} \underbrace{[\binom{n}{j-1} + \binom{n}{j}]}_{\overset{1.14d)iv)}{=} \binom{n+1}{j}} a^j b^{n-j+1} + \underbrace{\binom{n}{0}}_{=1} a^0 b^{n+1} + \underbrace{\binom{n}{n}}_{=1} a^{n+1} b^0 \\
&= \sum_{j=0}^{n+1} \binom{n+1}{j} a^j b^{n-j+1},
\end{aligned}$$

i.e. the statement from the theorem also holds for $n+1$.

$\square$

1.16 Definition. a) A set $M \subset \mathbb{R}$ is said to be *bounded from above (nach oben beschränkt)*, if there exists an $s \in \mathbb{R}$ such that

$$m \leq s \text{ for all } m \in M.$$

If this is the case, s is called an *upper bound (obere Schranke)* of M.
b) An upper bound s_0 is called the *least upper bound (kleinste obere Schranke)* or the *supremum* of $M \subset \mathbb{R}$, if for every upper bound s of M,

$$s_0 \leq s.$$

Remark. a) If s_0, s_0' are both least upper bounds of M, it follows that $s_0 \leq s_0', s_0' \leq s_0$, therefore $s_0 = s_0'$. Therefore the supremum is *uniquely* (*eindeutig*) determined.
b) The following axiom states that there exists a supremum in any nonempty upper bounded set of real numbers.

1.17. Completeness Axiom. Let $M \subset \mathbb{R}$ be a nonempty set with an upper bound. Then M has a supremum s_0. We define $\sup M := s_0$.

Now we have axiomatically introduced $\mathbb{R}$ as a set that is equipped with addition $+$, multiplication $\cdot$, and order $<$, and that satisfies the field-, order- and completeness axioms.

1.18 Definition. Let $\emptyset \neq M \subset \mathbb{R}$ and $s_0 = \sup M$. If $s_0 \in M$, then s_0 is called the *maximum* of M. We define $\max M := s_0$.

1.19 Examples. a) Let $M := \{x \in \mathbb{R}, x < 1\}$. Then $\sup M = 1 =: s_0$, although M has no maximum. $s_0 = 1$ is clearly an upper bound of M. Assume there exists an upper bound $s < 1$ of M. $\overset{1.6.h}{\Rightarrow} s < \frac{s+1}{2} < 1$ this contradicts the assumption that s is an upper bound of M. Additionally, $1 \notin M$, therefore $s_0 = 1$ is not a maximum.
b) Let $a \geq 0$ and $M := \{x \in \mathbb{R} : x^2 \leq a\}$. Then M is bounded from above, for example by $1 + \frac{a}{2}$. Furthermore, M is obviously nonempty, as $0 \in M$. Therefore, the completeness axiom implies that $s_0 := \sup M$ exists. Moreover, we have

$$s_0^2 = a.$$

Proof.

i) If $a = 0$, then we have $s_0 = 0$. In the following, we therefore assume that $a > 0$.

ii) We first prove $s_0^2 \geq a$: We assume the statement is false. Then $a - s_0^2 > 0$, therefore $\varepsilon := \frac{a-s_0^2}{2s_0+1} > 0$.

Furthermore we have $\varepsilon < 1$, because $\varepsilon \geq 1$ would imply that

$$a - s_0^2 \geq 2s_0 + 1 \Leftrightarrow a \geq s_0^2 + 2s_0 + 1 = (s_0+1)^2.$$

This would imply $s_0 + 1 \in M$ and hence $s_0 + 1 \leq \sup M = s_0$. Contradiction! Therefore

$$(s_0+\varepsilon)^2 = s_0^2 + 2s_0\varepsilon + \varepsilon^2 < s_0^2 + (2s_0+1)\varepsilon = s_0^2 + a - s_0^2 = a.$$

Hence $s_0 + \varepsilon \in M$ and consequently $s_0 + \varepsilon \leq s_0$, contradicting the definition of s_0. Therefore, $s_0^2 \geq a$.

iii) Now we prove $s_0^2 \leq a$: Assume that the statement is false. Then, $s_0^2 - a > 0$. Define $\delta := \frac{s_0^2-a}{2s_0} > 0$. Then $s := s_0 - \delta = \frac{2s_0^2-s_0^2+a}{2s_0} = \frac{s_0^2+a}{2s_0} > 0$ and $s^2 = s_0^2 - 2s_0\delta + \delta^2 = s_0^2 - s_0^2 + a + \delta^2 = a + \delta^2 > a$. Therefore $s^2 > a \geq x^2$ for all $x \in M$ and $s > x$ for all $x \in M$. Hence $s^2 < s_0^2$ is an upper bound of M in contradiction to the minimality of s_0.

Statements ii) and iii) imply $s_0^2 = a$.

c) *Corollary.* For every real number $a > 0$, there exists exactly one real number $w > 0$ with $w^2 = a$. The number w is called the *square root (Wurzel)* of a and is denoted by $w = \sqrt{a}$.

1.20 Definition. a) A set $M \subset \mathbb{R}$ is said to be *bounded from below* (*nach unten beschränkt*), if there exists an $r \in \mathbb{R}$ such that

$$r \leq m \text{ for all } m \in M.$$

In this case, r is called a *lower bound* (*untere Schranke*) of M.
b) A lower bound r_0 is called the *greatest lower bound* (*größte untere Schranke*) or the *infimum*, if for all lower bounds r of M,

$$r \leq r_0.$$

We define $\inf M := r_0$.
c) If $r_0 \in M$, then r_0 is called the *minimum* of M, and we define $\min M := r_0$.
d) If $M \subset \mathbb{R}$ is bounded from above and below, then M is called *bounded* (*beschränkt*).

1.21 Lemma. *Let $M \subset \mathbb{R}$ and $-M := \{-m : m \in M\}$. Then the following statements hold:*

a) M is bounded from below $\Leftrightarrow -M$ is bounded from above.

b) Every nonempty set M that is bounded from below has an infimum. The infimum is uniquely determined.

c) $M \neq \emptyset$ is bounded from below $\Rightarrow \inf M = -\sup(-M)$.

Proof. Exercise

1.22 Theorem. (Characterization Theorem for Suprema). *Let $\emptyset \neq M \subset \mathbb{R}$ be an upper bounded set and $s_0 \in \mathbb{R}$. Then:*

$\sup M = s_0 \Leftrightarrow$ *For all $m \in M$ we have $m \leq s_0$, and moreover, to each $\varepsilon > 0$ there exists an $m_1 \in M$ such that $m_1 > s_0 - \varepsilon$.*

Proof. $\Rightarrow$: Let $s_0 = \sup M$. Then $m \leq s_0$ for all $m \in M$. Assume there exists $\varepsilon > 0$ such that for all $m_1 \in M$ we have $m_1 \leq s_0 - \varepsilon$. Then $s := s_0 - \varepsilon$ is an upper bound. Contradiction!
$\Leftarrow$: Let s_0 be an upper bound of M. Assume there exists $s \in \mathbb{R}$ such that $s < s_0$ and $m \leq s \ \forall\, m \in M$. Set $\varepsilon := s_0 - s > 0$. Then $s = s_0 - \varepsilon$ and $m \leq s_0 - \varepsilon \ \forall\, m \in M$. Contradiction!

□

To conclude this section, we define the *natural numbers* $\mathbb{N}$ and the *integers* $\mathbb{Z}$ as

$$\mathbb{N} := \mathbb{N}_0 \backslash \{0\} \quad \text{and} \quad \mathbb{Z} := \mathbb{N}_0 \cup \{-n : n \in \mathbb{N}\}.$$

The set $\mathbb{Q}$ of *rational numbers* is then given as

$$\mathbb{Q} := \{p/q : p, q \in \mathbb{Z}, q \neq 0\};$$

Furthermore, we call the elements of $\mathbb{R} \setminus \mathbb{Q}$ *irrational numbers* .

1.23 Corollary. *a) $\mathbb{N}_0$ is not bounded from above.*

b) Archimedes' Principle:
$\forall\, a > 0,\ b \in \mathbb{R}\ \exists\, n \in \mathbb{N}_0$ *such that* $n \cdot a > b$.

c) "Classical Method of Deduction" in Analysis:
If $0 \leq a < \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a = 0$. (Recall $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$.)

Proof. a) Assume $\mathbb{N}_0$ is bounded from above. Then there exists an $s_0 = \sup \mathbb{N}_0$ by the Completeness Axiom. The Characterization Theorem of sup (see Theorem 1.22) with $\varepsilon = 1$ implies that there exists an $n \in \mathbb{N}_0$ with $n > s_0 - 1$. $\Rightarrow n + 1 > s_0$ in contradiction to the definition of s_0.
b) Assume $n \cdot a \leq b$ for all $n \in \mathbb{N}_0$. Then $\mathbb{N}_0$ is bounded from above by $\frac{b}{a}$. Contradiction to a)!
c) Assume $a > 0$. Then $n \cdot a < 1$ for all $n \in \mathbb{N}_0$ in contradiction to b).

$\square$

2 The Complex Numbers

In this chapter we give an axiomatic introduction to the field of complex numbers and begin with the following definition:

2.1 Definition. On $\mathbb{R}^2 := \{(a,b) : a,b \in \mathbb{R}\}$ we define addition and multiplication as follows:

$$\begin{array}{lll} \textit{Addition} & \oplus : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : & (a,b) \oplus (c,d) := (a+c, b+d) \\ \textit{Multiplication} & \odot : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : & (a,b) \odot (c,d) := (ac-bd, ad+bc) \end{array}$$

Then for $x = (a,b), y = (c,d)$ and $z = (e,f) \in \mathbb{R}^2$, $\oplus$ and $\odot$ fulfill the field axioms from § 1, where

$$\begin{array}{lcll} 0_\oplus & = & (0,0) & \textit{additive neutral element} \quad \oplus \\ 1_\odot & = & (1,0) & \textit{multiplicative neutral element} \quad \odot \\ -(a,b) & = & (-a,-b) & \textit{additive inverse element} \quad \oplus \\ (a,b)^{-1} & = & (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}) & \textit{multiplicative inverse element} \quad \odot \\ & & & \text{if}(a,b) \neq 0_\oplus = (0,0) \end{array}$$

For the proof of this fact we refer to linear algebra. $\mathbb{R}^2$ equipped with $\oplus$ and $\odot$ is therefore a field, which we call the field of *complex numbers*, denoted by $\mathbb{C}$.

For $(a,0) \in \mathbb{C}$ we have

$$\begin{array}{rcl} (a,0) \oplus (b,0) & = & (a+b, 0), \\ (a,0) \odot (b,0) & = & (a \cdot b, 0), \end{array}$$

i.e. if one identifies $a \in \mathbb{R}$ with $(a,0) \in \mathbb{C}$, then $\mathbb{R}$ is a subfield of $\mathbb{C}$.

2.2 Definition. We define $i := (0,1) \in \mathbb{C}$. The number $i \in \mathbb{C}$ is called the *imaginary unit (imaginäre Einheit).*

Then by definition of $\odot$:

$$i^2 = (0,1) \odot (0,1) = (-1,0) = -1.$$

i.e. i is a solution to the equation $x^2 + 1 = 0$.

2.3 Remark. The field $\mathbb{C}$ cannot be ordered, i.e. there cannot exist a relation "$<$", such that in $\mathbb{C}$ the ordering axioms from Chapter 1 hold. For if it were the case that such an ordering existed, then in the same way as for $\mathbb{R}$ we would be able to prove that $x^2 > 0$ for all $x \in \mathbb{C}$ s.t. $x \neq 0$. Thus we would get $-1 = i^2 > 0$. But we can prove that $-1 < 0$, so this is a contradiction.

2.4 Remark. Let $z = (a, b) \in \mathbb{C}$ with $a, b \in \mathbb{R}$. Then

$$(a, b) = \underbrace{(a, 0)}_{=a} \oplus \underbrace{(0, 1)}_{=i} \odot \underbrace{(b, 0)}_{=b}.$$

If we identify a with $(a, 0)$ as above, then we get that

$$\mathbb{C} \ni (a, b) = z = a + i \cdot b.$$

The real number a is called the *real part* (Realteil) of $z = a + ib$ and is denoted by $Re(z) = a$. The number b is called the *imaginary part* (*Imaginärteil*) of $z = a + ib$. We set $Im(z) = b$.

2.5 Definition. (Conjugation and Absolute Value).

a) Let $a, b \in \mathbb{R}$ and $z = a + ib \in \mathbb{C}$. The complex number

$$\overline{z} := a - ib$$

is called the *complex conjugate* of z.

b) The *absolute value* $|z|$ of z is defined as $|z| := \sqrt{z\overline{z}} = \sqrt{a^2 + b^2} \geq 0$.

For $z \in \mathbb{R}$ the definition coincides with that of Section 1.

2.6 Lemma. (Calculation Rules for Complex Numbers). *For complex numbers $z, w \in \mathbb{C}$, we have the following calculation rules:*

a) $Re(z + w) = Re(z) + Re(w), \quad Im(z + w) = Im(z) + Im(w)$

b) $\overline{z + w} = \overline{z} + \overline{w}, \quad \overline{z \cdot w} = \overline{z} \cdot \overline{w}$

c) $z \cdot \overline{z} = |z|^2$

d) $z = 0 \Leftrightarrow |z| = 0 \Leftrightarrow Re(z) = 0 = Im(z)$

e) $|z| = |\overline{z}|$

f) $|z + w| \leq |z| + |w|$

Proof. Exercise

Chapter II

Convergence of Sequences and Series

Many of the basic theorems about infinite sequences and series, that we will examine in the following, are due to Augustin-Louis Cauchy (1789–1857), one of the greatest french mathematicians of his time. Already as a twelve year old pupil, he stood out because of his talent, why Lagrange said about him

> Vous voyez ce petit jeune homme, eh bien! il nous remplacera tous tant que nous sommes de géomètres.

and advised Cauchy's father

> Don't let this child touch a mathematical book before his seventeenth year. If you do not hurry up to give him a solid literary education, his inclination will carry him away.

In 1816, Cauchy was appointed a position as professor at the Ecole Polytechnique in Paris and his three textbooks *Cours d'Analyse, Résumé des leçons sur le calcul infinitésimal, Leçons sur le calcul différentiel* are said to have introduced the formal rigour in modern analysis. The systematic way, in which the theory of infinite series is developed in *Cours d'Analyse* is still exemplary today.

The infinitely small quantities, that were used by Cauchy, were replaced by precise and clear expressions involving inequalities by Karl Weierstrass (1815–1897). Thereby, a standardised choice of variable names proved very useful. ε is used as an arbitrarily small positive number (probably derived from the french erreur), and δ is the number that corresponds to ε.

From 1864 on, Weierstrass taught at the university of Berlin. In his lectures, he treats the convergence of sequences and series and, more generally, the infinitesimal calculus in 'Weierstrassian rigour' and thus became the father of 'epsilonics' which is standard today in any lecture about analysis.

1 Convergence of Sequences

We begin this chapter, which is very important for this analysis class and for the further development of analysis, by some remarks on functions and their properties.

1.1. Introduction.

a) Let X, Y be two sets. A *function* or a *mapping* $f : X \to Y$ is a rule, which assigns to every $x \in X$ one *unique* element $y \in Y$. We write

$$f : X \to Y, x \mapsto f(x).$$

b) The set $\operatorname{graph}(f) := \{(x, f(x)) : x \in X\} \subset X \times Y$ is called the *graph* of f.

c) Two functions $f, g : X \to Y$ are equal, if $f(x) = g(x)$ for all $x \in X$.

d) The set $\operatorname{Fun}(X, Y)$ is defined to be the set of all functions $f : X \to Y$.

e) Let $f : X \to Y$ be a function. Then X is called the *domain* of f and $f(X)$ is called the *range* of f. Further we say:

   f is called *injective*, if $x_1, x_2 \in X, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

   f is called *surjective*, if $f(X) = Y$.

   f is called *bijective*, if f is injective and surjective.

f) If $Y \subset \mathbb{R}$ ($Y \subset \mathbb{C}$) holds, then f is called a real-valued (complex-valued) function.

Let M be a set. We call a mapping $f : \mathbb{N} \to M$, which assigns an element a_n of M to each $n \in \mathbb{N}$, a *sequence in M*. If we let $a_n := f(n)$ for all $n \in \mathbb{N}$, we write $(a_n)_{n\in\mathbb{N}}$. If we have $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, then $(a_n)_{n\in\mathbb{N}}$ is called a *real sequence*; analogously, if we have $a_n \in \mathbb{C}$, $n \in \mathbb{N}$, then $(a_n)_{n\in\mathbb{N}}$ is called a *complex sequence*. Occasionally it is convenient to start a sequence with a_0. In this case, the sequence is a mapping $\mathbb{N}_0 \to M$ and we write $(a_n)_{n\in\mathbb{N}_0}$

1.2 Definition. A complex sequence $(a_n)_{n\in\mathbb{N}_0}$ *converges* to $a \in \mathbb{C}$, if

$$(\forall\, \varepsilon > 0)\ (\exists\, N_0 \in \mathbb{N})\ (\forall\, n \geq N_0) \quad |a - a_n| < \varepsilon\,.$$

The number a is called the *limit value* or just *limit* of the sequence $(a_n)_{n\in\mathbb{N}}$ and we write

$$\lim_{n\to\infty} a_n = a \quad \text{or} \quad a_n \overset{n\to\infty}{\longrightarrow} a.$$

If there exists an $a \in \mathbb{C}$ with $\lim a_n = a$, then $(a_n)_{n\in\mathbb{N}_0}$ is called a *convergent sequence*, otherwise a *divergent sequence*. If $(a_n)_{n\in\mathbb{N}_0}$ converges to 0, then $(a_n)_{n\in\mathbb{N}_0}$ is called a *null sequence*.

1.3 Remarks. a) The limit is uniquely determined, i.e.

$$\left.\begin{array}{l} a_n \to a^* \\ a_n \to a_* \end{array}\right\} \Rightarrow a^* = a_*$$

Proof. Let $\varepsilon > 0$ be arbitrarily chosen. Then $n_0^1, n_0^2 \in \mathbb{N}_0$ exist, with:

$$\begin{aligned} |a_n - a^*| &< \frac{\varepsilon}{2} \quad \forall\, n \geq n_0^1 \\ |a_n - a_*| &< \frac{\varepsilon}{2} \quad \forall\, n \geq n_0^2 \end{aligned}$$

Here $a^* - a_* = a^* - a_n + a_n - a_*$ implies

$$0 \leq |a^* - a_*| \leq |a^* - a_n| + |a_n - a_*| < \varepsilon \;\forall\, n \geq \max\{n_0^1, n_0^2\},$$

i.e. $|a^* - a_*| = 0 \Leftrightarrow a^* = a_*$ due to the classical conclusion method of Analysis (Chapter I, 1.23).
b) If a_n is defined only for $n \geq N$, then we denote $(a_N, a_{N+1}, \cdots)$ as a sequence too, and write $(a_n)_{n \geq N}$.

1.4 Examples.

a) For $a \in \mathbb{C}$, the constant sequence $(a, a, \cdots)$ converges to a.

b) The sequence $(\frac{1}{n})_{n\geq 1}$ is a null sequence. We prove this as follows: Let $\varepsilon > 0$ be arbitrary. By the Archimedean Property I 1.23 there exists $n_0 \in \mathbb{N}_0$ with $n_0 \cdot \varepsilon > 1$. Thus:

$$|0 - \frac{1}{n}| \leq \frac{1}{n_0} < \varepsilon, \quad \forall\, n \geq n_0$$

c) The sequence $(\frac{n}{n+1})_{n\in\mathbb{N}}$ converges to 1.
Again choose $\varepsilon > 0$ arbitrarily. By the Archimedean Property I 1.23 there exists $n_0 \in \mathbb{N}_0$ with $n_0 \cdot \varepsilon > 1$. Thus:

$$|1 - \frac{n}{n+1}| = |\frac{1}{n+1}| < \frac{1}{n_0} < \varepsilon, \quad \forall\, n \geq n_0.$$

d) Let $a_n := \sum\limits_{j=1}^{n} \frac{1}{j(j+1)}$ for $n \geq 1$.
Since $\frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1}$, it follows that $a_n = 1 - \frac{1}{n+1}$; hence $a_n \to 1$ for $n \to \infty$.

e) Let $a_n = (-1)^n$. Then $(a_n)_{n\in\mathbb{N}_0}$ diverges: Assume for the moment that (a_n) converges to $a \in \mathbb{C}$. Then there exists $n_0 \in \mathbb{N}$ with $|a - a_n| < \frac{1}{2} \;\forall\, n \geq n_0$. Thus

$$2 = |a_{n+1} - a_n| \leq |a_{n+1} - a| + |a - a_n| < \frac{1}{2} + \frac{1}{2} = 1.$$

Here we get a contradiction, which means that $(a_n)_{n\in\mathbb{N}}$ is divergent.

1.5 Definition. A sequence $(a_n)_{n\geq 1} \subset \mathbb{C}$ is called *bounded*, if there exists a constant $M > 0$ with

$$|a_n| \leq M \quad \forall\, n \in \mathbb{N}.$$

1.6 Theorem. *Every convergent sequence $(a_n)_{n\geq 0}$ is bounded.*

Proof. Let $\lim\limits_{n\to\infty} a_n = a \in \mathbb{C}$. By hypothesis, in particular for $\varepsilon = 1$ there exists $n_0 \geq 1$ with $|a - a_n| < 1$ for all $n \geq n_0$. Thus for $n \geq n_0$ we have:

$$|a_n| \leq |a_n - a| + |a| \leq 1 + |a|.$$

Hence

$$|a_n| \leq \max \underbrace{\{|a_0|, |a_1|, \cdots |a_{n_0-1}|, 1 + |a|\}}_{\text{finitely many}} =: M \quad \forall\, n \in \mathbb{N}_0.$$

$\square$

1.7 Examples. a) The sequence $((-1)^n)_{n\in\mathbb{N}}$ is bounded, but not convergent.

b) For $q \in \mathbb{C}$ let $a_n := q^n$. Then:

i) if $|q| > 1$, then (a_n) is not bounded, thus divergent.

ii) if $|q| < 1$, then (a_n) is a null sequence.

1.8 Lemma. (Calculation rules for convergent sequences). *Let $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ be two convergent sequences with $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$. Then the following statements hold:*

a) $(a_n + b_n) \overset{n\to\infty}{\longrightarrow} a + b$

b) $(a_n \cdot b_n) \overset{n\to\infty}{\longrightarrow} ab$

c) If $b \neq 0$, then there exists $n_0 \in \mathbb{N}_0$ with $b_n \neq 0\ \forall\, n \geq n_0$ and $\frac{a_n}{b_n} \overset{n\to\infty,\, n\geq n_0}{\longrightarrow} \frac{a}{b}$.

Proof. a) Let $\varepsilon > 0$ be arbitrary. Then there exist $n_1, n_2 \in \mathbb{N}_0$ with

$$\begin{aligned} |a - a_n| &< \frac{\varepsilon}{2}, \quad \forall\, n \geq n_1, \\ |b - b_n| &< \frac{\varepsilon}{2}, \quad \forall\, n \geq n_2. \end{aligned}$$

For $n_0 := \max\{n_1, n_2\}$ holds:

$$|a+b-(a_n+b_n)| \leq \underbrace{|a-a_n|}_{<\frac{\varepsilon}{2}} + \underbrace{|b-b_n|}_{<\frac{\varepsilon}{2}} < \varepsilon \quad \forall\, n \geq n_0,$$

thus the claim.
b) Exercise.
c) Exercise.

□

The following example illustrates the above calculation rules for convergent sequences. For $n \geq 2$, we let

$$a_n = \frac{3n^2-2n+1}{-n^2+n} = \frac{3-\frac{2}{n}+\frac{1}{n^2}}{-1+\frac{1}{n}}.$$

Now the above Lemma 1.8 implies that $\lim_{n\to\infty} a_n = -3$.

An important approach to determine whether a given sequence converges is to estimate its terms by the terms of a convergent sequence. For that, we have to assure that convergence and order are compatible. This is the statement of the following lemma.

1.9 Lemma. (Compatibility of convergence and order). *Let $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ be two real and convergent sequences with $\lim a_n = a$ and $\lim b_n = b$. If a number $n_0 \in \mathbb{N}$ exists with $a_n \leq b_n$ for all $n \geq n_0$, then $a \leq b$ holds.*

Proof. Assume, that $a > b$. Then $\varepsilon := \frac{a-b}{2} > 0$ and hence, by hypotheses, there exists $n_0 \in \mathbb{N}$ with

$$\begin{aligned} a - a_n &\leq |a-a_n| < \varepsilon \quad \forall\, n \geq n_0, \\ b_n - b &\leq |b-b_n| < \varepsilon \quad \forall\, n \geq n_0. \end{aligned}$$

Thus

$$b_n < b + \varepsilon = \frac{2b}{2} + \frac{a-b}{2} = \frac{a+b}{2} = a - \varepsilon < a_n \quad \forall\, n \geq n_0.$$

Contradiction!

□

1.10 Corollary. (Sandwich Theorem). *Let $(a_n)_n, (b_n)_n$ and $(c_n)_n$ be real sequences, for which $\lim a_n = a$ and $\lim b_n = a$. Let also $n_0 \in \mathbb{N}$ exist with*

$$a_n \leq c_n \leq b_n,\ \forall\, n \geq n_0.$$

Then $\lim\limits_{n\to\infty} c_n = a$.

Proof. Exercise.

Criteria which imply the convergence of a sequence without explicit knowledge about the limit are especially important. For this, we introduce the following notions.

1.11 Definition. A real sequence $(a_n)_{n\in\mathbb{N}}$ is called
a) (*monotone*) *increasing*, if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$
b) *strictly* (*monotone*) *increasing*, if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$
c) (*monotone*) *decreasing*, if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$
d) *strictly* (*monotone*) *decreasing*, if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.
If one of the cases i)-iv) holds, then (a_n) is simply called *monotone*.

1.12 Theorem. *Every bounded and monotone real sequence $(a_n)_{n\geq 1}$ converges.*

a) If (a_n) is increasing, then $a_n \to \sup\{a_n, n \in \mathbb{N}\}$.

b) If (a_n) is decreasing, then $a_n \to \inf\{a_n, n \in \mathbb{N}\}$.

Proof. a) The hypothesis implies that $s := \sup\{a_n : n \in \mathbb{N}\}$ exists. Let $\varepsilon > 0$ be given. The characterisation of the supremum from Theorem I 1.22 implies that there exists $n_0 \in \mathbb{N}$ with

$$s - \varepsilon < a_{n_0} \leq a_n \leq s \quad \forall\, n \geq n_0.$$

Hence, $-\varepsilon < a_n - s \leq 0 \quad \forall\, n \geq n_0$ and thus $|a_n - s| < \varepsilon\ \forall\, n \geq n_0$.
b) Exercise

□

We now apply the theorem above to define the *root function*.

1.13 Theorem. *Let $a > 0$ and $k \in \mathbb{N}$ with $k \geq 2$. Then there exists one and only one real number $w > 0$ with $w^k = a$. In this case we write $\sqrt[k]{a} := a^{1/k} := w$.*

Proof. We begin with the *existence* of the number w. For this we define the sequence (a_j) recursively via

$$a_0 := a + 1, \quad a_{j+1} := a_j\Big(1 + \frac{a - a_j^k}{k \cdot a_j^k}\Big), \quad j \in \mathbb{N}_0.$$

Then we claim:

a) $a_j > 0 \quad \forall\, j \in \mathbb{N}_0$

b) $a_j^k \geq a \quad \forall\, j \in \mathbb{N}_0$

c) $a_{j+1} \leq a_j \quad \forall\, j \in \mathbb{N}_0$, i.e. (a_j) is monotone decreasing.

We prove this by induction.

$n = 0$: It is trivial that $a_0 > 0$, $a_0^k \geq a$ and $a_1 \leq a_0$.

Induction step: Let $n \in \mathbb{N}_0$ be such that $a_n > 0$, $a_n^k \geq a$ and $a_{n+1} \leq a_n$. Then $ka_n^k + a - a_n^k > 0$, hence $a_{n+1} > 0$. By the Bernoulli inequality we have

$$a_{n+1}^k = a_n^k\Big(1 + \frac{a - a_n^k}{ka_n^k}\Big)^k \geq a_n^k\Big(1 + \frac{k(a - a_n^k)}{ka_n^k}\Big) = a,$$

i.e. $a_{n+1}^k \geq a$. Finally, $a_{n+2} \leq a_{n+1}$ since $a - a_{n+1}^k \leq 0$. Thus the properties a), b) and c) hold.

So $(a_j)_{j\in\mathbb{N}_0}$ is a bounded and monotone decreasing sequence. Therefore, Theorem 1.12 implies that

$$w := \lim_{j\to\infty} a_j = \inf\{a_j : j \in \mathbb{N}\}.$$

Further $\lim\limits_{j\to\infty} a_{j+1} = w$ and $(\lim\limits_{j\to\infty} a_j)^k \underbrace{=}_{1.8b)} \lim\limits_{j\to\infty} a_j^k \geq a > 0$. In addition

$$w \leftarrow a_{j+1} = a_j\Big(1 + \frac{a - a_j^k}{k \cdot a_j^k}\Big) \to w \cdot \Big(1 + \frac{a - w^k}{k \cdot w^k}\Big),$$

thus $w = w(1 + \frac{a-w^k}{k\cdot w^k})$, and so $a = w^k$.
Uniqueness. Let $u, v > 0$ with $u^k = w = v^k$ and $u \neq v$. Without loss of generality, let $u < v$. Then $w = u^k < v^k = w$. Contradiction!

$\square$

At this point, we want to give a geometric interpretation of the sequence $(a_j)_{j\in\mathbb{N}_0}$. This sequence is the foundation for a method to calculate approximations of the root of a given number - cf. also the ‘Newton method’. Consider the tangent of the function $f(x) = x^k - a$ at the point $x = a_j$. This tangent intersects the x-axis in the point $x = a_{j+1}$. We furthermore remark that this method converges for every initial value $a_1 > 0$ and that there exists a constant $M > 0$ with $|\sqrt[k]{a} - a_{j+1}| \leq M|\sqrt[k]{a} - a_j|^2$, $j \in \mathbb{N}$. We therefore speak of *quadratic convergence* of the method.

1.14 Remark. Starting from the n-th root $\sqrt[n]{a}$ of a real number $a \geq 0$, for $p, q \in \mathbb{N}$ we define more generally

$$a^{p/q} := (a^{1/q})^p = (a^p)^{1/q},$$

and for $a > 0$

$$a^{-p/q} := (a^{-1})^{p/q}.$$

If we furthermore define $a^0 = 1$, we obtain by induction the following calculation rules

$$a^{p+q} = a^p a^q, \quad a^{pq} = (a^p)^q, \quad a^p b^p = (ab)^p$$

for $a > 0, b > 0$ and $p, q \in \mathbb{Q}$. The general power a^x for $a > 0$ and $x \in \mathbb{R}$ will be defined later via the exponential function; therefore we do not elaborate the above elementary examination of the powers with rational exponent any further.

Now, we come to another application of Theorem 1.12

1.15 Theorem. (The number e). *Let $(a_n)_{n\geq 1}$ be the sequence defined by*

$$a_n := \left(1 + \frac{1}{n}\right)^n, \quad n \geq 1.$$

Then $(a_n)_{n\geq 1}$ converges. The limit, called Euler's number, is denoted by e and satisfies

$$2 \leq \lim_{n\to\infty} a_n = e \leq 3.$$

Proof. By Theorem 1.12 and Lemma 1.9 it is enough to show that

a) (a_n) is increasing and

b) $2 \leq a_n \leq 3$ for all $n \geq 1$.

Proof of a). For $n \geq 2$ holds:

$$\begin{aligned}
\frac{a_n}{a_{n-1}} &= \frac{\left(\frac{n+1}{n}\right)^n}{\left(\frac{n}{n-1}\right)^{n-1}} = \left(\frac{\frac{n+1}{n}}{\frac{n}{n-1}}\right)^n \cdot \frac{n}{n-1} = \left(\frac{n^2-1}{n^2}\right)^n \cdot \frac{n}{n-1} \\
&= \left(1 - \frac{1}{n^2}\right)^n \cdot \frac{n}{n-1} \overset{\text{Bernoulli}}{\geq} (1 - \frac{1}{n})\frac{n}{n-1} = 1.
\end{aligned}$$

Thus $a_n \geq a_{n-1}$ holds.
Proof of b). The statement a) implies that $a_1 = 2 \leq a_n$. Further:

$$a_n = (1 + \frac{1}{n})^n \underset{\text{I 1.15}}{\overset{\text{Binom. Thm}}{=}} \sum_{j=0}^{n} \binom{n}{j} \frac{1}{n^j} = 2 + \sum_{j=2}^{n} \binom{n}{j} \frac{1}{n^j}.$$

For $2 \leq j \leq n$ we also have

$$\binom{n}{j}\frac{1}{n^j} = \frac{n!}{j!(n-j)!}\frac{1}{n^j} = \frac{1 \cdot 2 \cdots n}{1 \cdots (n-j)\underbrace{n \cdots n}_{\text{j times}}}\frac{1}{j!} \leq \frac{1}{j!} \leq \frac{1}{2^{j-1}}$$

and hence

$$a_n \leq 1 + \sum_{j=1}^{n} \frac{1}{2^{j-1}} = 1 + \sum_{j=0}^{n-1} \frac{1}{2^j} \underset{\text{I 1.11 b)}}{\overset{\text{geom. series}}{=}} 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} < 3.$$

□

To conclude this section, we consider some further important limits. The proofs of the limits d) and e) are very instructive exercises which require a good understanding of the convergence concept.

1.16 Examples.

a) For $s \in \mathbb{Q}$, $s > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n^s} = 0.$$

Given $\varepsilon > 0$, we choose $N_0 \in \mathbb{N}$ with $N_0 \geq \varepsilon^{-1/s}$. Then we have $\frac{1}{n^s} < \varepsilon$ for all $n > N_0$.

b) For $a > 0$, we have

$$\lim_{n\to\infty} \sqrt[n]{a} = 1.$$

We first consider the case $a \geq 1$. If we set $b_n := \sqrt[n]{a} - 1$, the Bernoulli inequality implies $a = (1 + b_n)^n \geq 1 + nb_n$. This implies in particular that $b_n < \frac{a}{n}$, and if we choose $N_0 > \frac{a}{\varepsilon}$, we have

$$|\sqrt[n]{a} - 1| = b_n < \varepsilon, \quad n > N_0.$$

If $a < 1$, then we have $a^{-1} > 1$ and the proposition follows from 1.8 c) and the above:

$$\lim_{n\to\infty} \sqrt[n]{a} = \big(\lim_{n\to\infty} \sqrt[n]{a^{-1}}\big)^{-1} = 1.$$

c) We have

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

For $b_n := \sqrt[n]{n} - 1 \geq 0$, the binomial theorem implies

$$n = (1 + b_n)^n \geq 1 + \frac{n(n-1)}{2} b_n^2, \qquad \text{hence} \qquad n - 1 \geq \frac{n(n-1)}{2} b_n^2.$$

Therefore, $b_n^2 \leq \frac{2}{n}$ for all $n \in \mathbb{N}$, and if we choose for given $\varepsilon > 0$ an $N_0 \in \mathbb{N}$ such that $N_0 \geq \frac{2}{\varepsilon^2}$, then we have

$$|\sqrt[n]{n} - 1| = b_n < \varepsilon, \quad n > N_0.$$

d) For $a \in \mathbb{C}$ with $|a| > 1$ and $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{n^k}{a^n} = 0,$$

i.e. for a with $|a| > 1$, the function $n \mapsto a^n$ grows faster than *any* power $n \mapsto n^k$. In this situation, we observe two contrary effects: the numerator n^k exceeds any bound, while the term $\frac{1}{a^n}$ tends to zero. At first sight, it is not evident which tendency outweighs the other.

e) For $a \in \mathbb{C}$, we have

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0,$$

i.e. the factorial function $n \mapsto n!$ grows faster than *any* of the functions $n \mapsto a^n$.

2 Bolzano-Weierstrass Theorem

In the previous chapter we have observed that all convergent sequences are bounded. In the following, we will examine the converse situation, i.e. we consider bounded sequences and ask if there exist convergent subsequences. If we consider for example the sequence $(a_n)_{n\in\mathbb{N}} = (-1)^n$, the above question is easy to answer: there are at least two convergent subsequences, namely $(a_{2n})_{n\in\mathbb{N}}$ and $(a_{2n+1})_{n\in\mathbb{N}}$. The following theorem of Bolzano-Weierstraß gives an affirmative answer to this question in a general context.

We begin this section with the formal definition of a subsequence of a given sequence.

2.1 Definition. Let (a_n) be a sequence and $\varphi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function (i.e. $\varphi(n+1) > \varphi(n)\ \forall\ n \in \mathbb{N}$). Then $\left(a_{\varphi(k)}\right)_{k\in\mathbb{N}}$ is called a *subsequence* of (a_n). If we put $\varphi(k) := n_k$, we write $(a_{n_k})_{k\in\mathbb{N}}$.

Example. Let $a_n := (-1)^n$. Take $\varphi(n) = 2n$, then $a_{2n} = 1\ \forall\ n \in \mathbb{N}_0$. If we choose $\varphi(n) = 2n+1$, then $a_{2n+1} = -1\ \forall\ n \in \mathbb{N}_0$.

2.2 Lemma. *Let $(a_n)_{n\in\mathbb{N}_0}$ be a real sequence. Then $(a_n)_{n\in\mathbb{N}_0}$ has a monotone subsequence.*

Proof. Let $(a_n)_{n\in\mathbb{N}_0}$ be a sequence of real numbers. Consider

$$A := \{k \in \mathbb{N}_0 : a_k \geq a_m \text{ for all } m > k\}.$$

Now either A has only finitely many elements, or else it has infinitely many.

Case 1: Suppose A has infinitely many elements. Then define $n_0 := \min A$ and

$$n_{i+1} := \min(A \backslash (\bigcup_{j=0}^{i} \{n_j\}))\,, \quad i \in \mathbb{N}_0\,.$$

Then $(a_{n_k})_{k\in\mathbb{N}_0}$ is decreasing.

Case 2: Suppose A has finitely many elements. If $A \neq \emptyset$ we let $n_0 := \max A + 1$. If $A = \emptyset$ we let $n_0 := 0$. Then since $n_0 \notin A$ there exist $m > n_0$ such that $a_m > a_{n_0}$. Let n_1 be the least such m. Since also $n_1 \notin A$ we can continue, and in general define an increasing subsequence $(a_{n_k})_{k\in\mathbb{N}_0}$ by

$$n_{i+1} := \min\{k : k > n_i \text{ and } a_k > a_{n_i}\}.$$

$\square$

2.3 Theorem. (Bolzano-Weierstrass, 1st Version) *Every bounded sequence $(a_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$ has a convergent subsequence.*

Proof. a) Let $(a_n) \subset \mathbb{R}$. Then the claim follows from Lemma 2.2 and Theorem 1.12.

b) Let $(a_n) \subset \mathbb{C}$. Then $(\operatorname{Re} a_n)_{n\in\mathbb{N}_0}$ is a real and bounded sequence. According to a) it possesses a convergent subsequence $\big(\operatorname{Re} a_{\varphi_1(k)}\big)_{n\in\mathbb{N}}$. Further, $\big(\operatorname{Im} a_{\varphi_1(k)}\big)_{n\in\mathbb{N}}$ is a real and bounded sequence. Again from a), there exists a convergent subsequence $(\operatorname{Im} a_{\varphi_2(\varphi_1(k))})_{k\in\mathbb{N}}$. We put $\varphi = \varphi_2 \circ \varphi_1$. Then φ is strictly increasing and $\big(a_{\varphi(k)}\big)_{k\in\mathbb{N}}$ is a convergent subsequence of $(a_n)_{n\in\mathbb{N}}$.

□

In order to formulate another version of the Bolzano-Weierstrass theorem, we consider next the following definition of a cluster point.

2.4 Definition. (Cluster point). A number $a \in \mathbb{C}$ is called a *cluster point* of a sequence $(a_n)_n \subset \mathbb{C}$, if for each $\varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ such that $|a - a_n| < \varepsilon$.

2.5 Examples. a) Let $a_n = (\frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \dots)$. Then $a = 0$ is a cluster point of (a_n); but the sequence (a_n) is divergent.

b) The sequence $(a_n) = (i^n) = (1, i, -1, -i, 1, i, -1, \dots)$ has 4 cluster points, namely $1, i, -1, -i$ and 4 convergent subsequences.

c) Let $a_n = n$ for all $n \in \mathbb{N}$. Then (a_n) does not have cluster points and does not have a convergent subsequence.

2.6 Remarks. In the following let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

a) For $a \in \mathbb{K}$ and $\varepsilon > 0$ set $U_\varepsilon(a) := \{z \in \mathbb{K} : |a - z| < \varepsilon\}$. Then $U_\varepsilon(a)$ is called an *ε-neighborhood* of a.

b) a is the limit of the sequence $(a_n)_n \subset \mathbb{K} \Leftrightarrow$ for each $\varepsilon > 0$, $U_\varepsilon(a)$ contains almost all a_n, i.e. $U_\varepsilon(a)$ contains a_n for all but finitely many n.

c) a is a cluster point of the sequence $(a_n) \subset \mathbb{K} \Leftrightarrow$ for each $\varepsilon > 0$, $U_\varepsilon(a)$ contains infinitely many members of the sequence $(a_n)_n$.

2.7 Lemma. *Let (a_n) be a sequence in $\mathbb{K}$ and $a \in \mathbb{C}$. Then a is a cluster point of (a_n) if and only if there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}_0}$ of $(a_n)_n$ with $\lim\limits_{k\to\infty} a_{n_k} = a$.*

Proof. " $\Longrightarrow$ ": Remark 2.6 c) implies that for each $\varepsilon > 0$ infinitely many sequence members a_n lie in $U_\varepsilon(a)$. Let $n_0 := 0$ and for all $k \geq 1$ choose an $n_k > n_{k-1}$ with $a_{n_k} \in U_{\frac{1}{k}}(a)$. Then $(n_k)_{n\in\mathbb{N}}$ is strictly increasing and $|a - a_{n_k}| < \frac{1}{k}, k \geq 1$, i.e.

$$\lim_{k\to\infty} a_{n_k} = a.$$

" $\Longleftarrow$ ": Let $a := \lim\limits_{k\to\infty} a_{n_k}$. Then for each $\varepsilon > 0$ we have that $U_\varepsilon(a)$ contains almost all members a_{n_k} of $(a_{n_k})_k$, and thus infinitely many of $(a_n)_n$, cf. Remark 2.6 b), c).

$\square$

2.8 Theorem. (Bolzano-Weierstrass, 2nd Version). *Every real and bounded sequence $(a_n)_{n\in\mathbb{N}}$ has a cluster point. Further, the set of cluster points of $(a_n)_{n\in\mathbb{N}}$ has a minimum r and a maximum s.*

Remarks. In the situation above we set

(Limit Inferior) $\liminf\limits_{n\to\infty} a_n := \underline{\lim}\, a_n := r \quad (= \lim\limits_{n\to\infty} (\inf\{a_k : k \geq n\})$, see (T5.2))

(Limit Superior) $\limsup\limits_{n\to\infty} a_n := \overline{\lim}\, a_n := s \quad (= \lim\limits_{n\to\infty} (\sup\{a_k : k \geq n\})$, see (T5.2)).

Proof. Let $H := \{h \in \mathbb{R} : h \text{ is a cluster point of } (a_n)\}$. Then

$$\inf a_n \leq h \leq \sup a_n \text{ for all } h \in H.$$

Further $H \neq \emptyset$, due to the first version of the Bolzano-Weierstrass Theorem 2.3. In addition, the Completeness Axiom implies that $s := \sup H$ exists. We still have to show that $s \in H$ holds. To this end, let $\varepsilon > 0$. By the characterisation of the supremum given in Theorem I,1.22, there exists $a \in H$ with

$$a \leq s < a + \frac{\varepsilon}{2},$$

thus $|s - a| < \frac{\varepsilon}{2}$. For $x \in U_{\varepsilon/2}(a)$ we then have

$$|s - x| \leq |s - a| + |a - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

i.e. $U_{\varepsilon/2}(a) \subset U_\varepsilon(s)$. Now $U_{\varepsilon/2}(a)$ contains infinitely many a_n, and so does $U_\varepsilon(s)$. Remark 2.6 implies that $s \in H$. The proof for the limit inferior follows the same pattern.

$\square$

2.9 Examples. a) We again consider the sequence $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$. Clearly, we have $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$.

b) Consider the sequence

$$1\,, \frac{1}{2}\,, \frac{2}{2}\,, \frac{3}{2}\,, \frac{1}{3}\,, \frac{2}{3}\,, \frac{3}{3}\,, \frac{4}{3}\,, \frac{5}{3}\,, \frac{1}{4}\,, \dots\,, \frac{7}{4}\,, \frac{1}{5}\,, \dots\,, \frac{9}{5}\,, \dots$$

which is formally defined as

$$a_n = \frac{j}{k+1} \quad \text{for } n = k^2 + j, \quad j = 1, 2, \dots, 2k+1,\ k \in \mathbb{N}_0.$$

Then every rational number q with $0 < q < 2$ is contained in this sequence (even infinitely many times) and we have $\limsup_{n\to\infty} a_n = 2$ and $\liminf_{n\to\infty} a_n = 0$. Moreover, every x with $0 \le x \le 2$ is a cluster point of this sequence. In particular, the sequence has infinitely many cluster points.

So far we have studied the convergence of a sequence only for the case in which the limit was explicitly known. An exception to this is only Theorem 1.12. We consider now the so called "inner" criterion.

2.10 Definition. A sequence $(a_n) \subset \mathbb{K}$ is called *Cauchy sequence* if for each $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \quad \text{for all} \quad n, m \ge n_0.$$

The significance of this criterion is that it provides a necessary and sufficient condition for the convergence of $(a_n)_{n\in\mathbb{N}}$ without involving the limit a itself.

2.11 Theorem (Cauchy criterion for sequences). *Let $(a_n) \subset \mathbb{K}$ be a sequence. Then (a_n) is convergent, if and only if (a_n) is a Cauchy sequence.*

Proof. " $\Longrightarrow$ ": Let $a = \lim a_n$ and $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ with $|a - a_n| < \frac{\varepsilon}{2}$ for all $n > n_0$, thus

$$|a_n - a_m| \le |a_n - a| + |a - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad n, m \ge n_0.$$

" $\Longleftarrow$ ": Let (a_n) be a Cauchy sequence. We divide the proof in 3 steps:
a) The sequence (a_n) is bounded: For $\varepsilon = 1$ there exists an m_0 such that

$$|a_n| - |a_{m_0}| \le |a_n - a_{m_0}| < 1 \text{ for all } n \ge m_0.$$

Thus $|a_n| \le 1 + |a_{m_0}|$ for $n \ge m_0$. Hence $|a_n| \le \max\{|a_0|, |a_1| \dots, |a_{m_0-1}|, |1 + a_{m_0}|\}, n \in \mathbb{N}$, and (a_n) is a bounded sequence.

b) The first version of the Bolzano-Weierstrass theorem implies that (a_n) has a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} a_{n_k} = a$.

c) Let $\varepsilon > 0$. The hypothesis implies that $m_1 \in \mathbb{N}$ exists with $|a_n - a_m| < \frac{\varepsilon}{2}$ for all $n, m \geq m_1$. Step b) implies further that $|a - a_{n_k}| < \frac{\varepsilon}{2}, \quad n_k > m_1$. Thus

$$|a_n - a| \leq \underbrace{|a_n - a_{n_k}|}_{<\frac{\varepsilon}{2}} + \underbrace{|a_{n_k} - a|}_{<\frac{\varepsilon}{2}} < \varepsilon \text{ for all } n \geq m_1\,,$$

i.e. $a_n \overset{n\to\infty}{\longrightarrow} a$.

$\square$

2.12 Remarks. a) The property that every Cauchy sequence converges in $\mathbb{K}$ is also called the *completeness* of $\mathbb{K}$.

b) The set of rational numbers $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\}$ is not complete.

c) We have

Completeness axiom	$\Longleftrightarrow$	Archimedean property and completeness of $\mathbb{R}$
	$\Longleftrightarrow$	Archimedean property and theorem of Bolzano-Weierstrass in $\mathbb{R}$

d) For $q \in \mathbb{C}$, $q \neq 1$ with $|q| = 1$ set $a_n := q^n$. Then

$$|a_{n+1} - a_n| = |q|^n|q - 1| = |q - 1| > 0, \quad \text{for all} \quad n \geq 1,$$

i.e. (a_n) is not a Cauchy sequence. Thus (a_n) is divergent. Therefore, for $q \in \mathbb{C}, q \neq 1, |q| = 1$:

$$(q^n)_{n\in\mathbb{N}} \text{ is convergent} \iff |q| < 1 \text{ or } q = 1.$$

We introduce the following *notation*: for $a, b \in \mathbb{R}$ with $a \leq b$ let

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$

2.13 Theorem. (Banach fixed point theorem). *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a, b] \to [a, b]$ be a mapping. Assume that there exists q, $0 < q < 1$ such that for all $x, y \in [a, b]$ we have*

$$|f(x) - f(y)| \leq q\,|x - y|. \tag{2.1}$$

Then there exists a unique $r \in [a, b]$ with $f(r) = r$. This means that r is the unique fixed point of f.

2.14 Remark. A mapping which satisfies the condition (2.1) is called a *strict* contraction.

Proof. For $x_0 \in [a,b]$ and $n \in \mathbb{N}_0$, define

$$x_{n+1} := f(x_n).$$

Then the following statements hold:
a) The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent: We show first the inequality

$$A(m): \qquad |x_m - x_{m-1}| \le q^{m-1}|x_1 - x_0|, \quad m \ge 1$$

via induction. The basis step $m = 1$ is clear. Let $m \in \mathbb{N}$, such that $A(m)$ is true. Then

$$\begin{aligned} |x_{m+1} - x_m| &= |f(x_m) - f(x_{m-1})| \\ &\overset{\text{contr.}}{\le} q|x_m - x_{m-1}| \\ &\overset{\text{I.H.}}{\le} qq^{m-1}|x_1 - x_0| = q^m|x_1 - x_0|. \end{aligned}$$

Thus $A(m+1)$ is true, and so the inequality holds for all $m \ge 1$.
Next we estimate $|x_m - x_n|$ for $m > n$:

$$\begin{aligned} |x_m - x_n| &\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &\le (q^{m-1} + q^{m-2} + \cdots + q^n)|x_1 - x_0| \\ &\overset{\text{geom. series}}{=} q^n \frac{1 - q^{m-n}}{1-q}|x_1 - x_0| \le \frac{q^n}{1-q}|x_1 - x_0|. \end{aligned}$$

From Remark 2.12 d) it follows that $\lim\limits_{n\to\infty} q^n = 0$, because $0 < q < 1$. Thus (x_n) is a Cauchy sequence and Theorem 2.11 implies that (x_n) converges. We set $r := \lim\limits_{n\to\infty} x_n$.

b) We show $f(r) = r$: Let $\varepsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}_0$ with $|r - x_n| < \frac{\varepsilon}{2}$ for all $n \ge n_0$, thus

$$\begin{aligned} |f(r) - r| &\le |f(r) - x_{n_0+1}| + |x_{n_0+1} - r| \\ &\overset{\text{Def.}}{=} |f(r) - f(x_{n_0})| + |x_{n_0+1} - r| \\ &\le q|r - x_{n_0}| + |x_{n_0+1} - r| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The classical method of deduction in analysis from chapter I implies that $f(r) = r$.

c) We next show that the fixed point r is uniquely determined.
Assume that there exists an $r' \in [a,b]$ with $f(r') = r'$. Then

$$|r - r'| = |f(r) - f(r')| \le q|r - r'|.$$

From this it follows that $(1-q)|r - r'| = 0$, which implies that $|r - r'| = 0$ and, therefore, that $r = r'$.

$\square$

2.15 Remarks.

a) The proof above is constructive, i.e. we build the fixed point r as $r = \lim_{n\to\infty} f^n(x_0)$ with $f^n = f \circ f \cdots \circ f$.

b) The following error estimates hold:

$$\begin{aligned} |r - x_n| &\leq \frac{q^n}{1-q}|x_1 - x_0| \quad \textit{a-priori-estimate} \\ |r - x_n| &\leq \frac{q}{1-q}|x_n - x_{n-1}| \quad \textit{a-posteriori-estimate} \end{aligned}$$

c) The Banach fixed point theorem also holds if [a,b] is replaced by $\mathbb{R}$ or by $M := \{a \in \mathbb{R} : a \leq x\}$.

d) A generalization of this theorem to mappings on so-called complete metric spaces will be very important in the lecture " Ordinary Differential Equations".

To conclude this section, we discuss the concepts of *infinite limits*.

Often, it is convenient to write $\lim a_n = \infty$ if the terms of a sequence become large for large n, although strictly speaking, the sequence is *divergent*, and of course ∞ is not its limit, as it is not even a number. Nevertheless, this notation is often very natural and convenient and, therefore, we now make precise what we mean if we use it.

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. We write

$$\lim a_n = \infty \ (-\infty),$$

if for arbitrary, fixed $K > 0$ there exists an $N_0 \in \mathbb{N}$ with $a_n \geq K$ $(a_n \leq -K)$ for all $n \geq N_0$. Furthermore, we write

$$\limsup a_n = \infty \qquad (\liminf a_n = -\infty),$$

if for each $K > 0$ there exists an $N_0 \in \mathbb{N}$ with $a_{N_0} \geq K$ $(a_{N_0} \leq -K)$.
For a complex sequence $(c_n)_{n\in\mathbb{N}}$, we write

$$\lim c_n = \infty,$$

if for each $K > 0$ there exists an $N_0 \in \mathbb{N}$ with $|c_n| \geq K$ for all $n \geq N_0$.

3 Infinite Series

Let (a_n) be a sequence in $\mathbb{K}$, where again $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In this section we analyze the question, how the notation $\sum_{n=0}^{\infty} a_n$ has to be understood and under which conditions one can speak of a convergent/divergent infinite series.

3.1 Definition. a) Let $(a_n)_{n\in\mathbb{N}_0}$ be a sequence in $\mathbb{K}$. By the *(infinite) series with terms* a_n, notation

$$a_0 + a_1 + a_2 + \dots \quad \text{or} \quad \sum_{j=0}^{\infty} a_j,$$

we mean the *sequence of partial sums* $(s_n)_{n\in\mathbb{N}_0}$,

$$s_n := \sum_{j=0}^{n} a_j\,, \qquad s_n \text{ being the } n\text{-th partial sum of the series.}$$

That is, we use the symbol $\sum_{j=0}^{\infty} a_j$ to denote the sequence $(s_n)_{n\in\mathbb{N}_0}$.

b) If the sequence $(s_n)_{n\in\mathbb{N}_0}$ converges to $s \in \mathbb{K}$, then the series $\sum_{j=0}^{\infty} a_j$ is called *convergent*. In this case we use the symbol $\sum_{j=0}^{\infty} a_j$ also to denote the limit s of the sequence of partial sums. That is, we set $\displaystyle\sum_{j=0}^{\infty} a_j := s$.

Otherwise, the series is called *divergent*.

3.2 Examples. .

a) *Geometric series.* If $q \in \mathbb{C}$ with $|q| < 1$, then $\displaystyle\sum_{j=0}^{\infty} q^j = \frac{1}{1-q}$.

For consider $s_n = \sum_{j=0}^{n} q^j \overset{I,1.11\text{ b)}}{=} \frac{1-q^{n+1}}{1-q} \underset{1.7\,b)}{\overset{n\to\infty}{\longrightarrow}} \dfrac{1}{1-q}$.

b) *Harmonic series.* The series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{diverges.}$$

For consider for $n \ge 1$ the difference $\displaystyle s_{2n} - s_n = \sum_{j=n+1}^{2n} \frac{1}{j} \ge n \cdot \frac{1}{2n} = \frac{1}{2}\,,$

i.e., $(s_n)_{n\in\mathbb{N}}$ is not a Cauchy sequence, thus does not converge!

c) The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

In fact $\frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1}$, and

$$s_n = \sum_{j=1}^{n} (\frac{1}{j} - \frac{1}{j+1}) = 1 - \frac{1}{n+1} \overset{n\to\infty}{\longrightarrow} 1.$$

Sums of type $\sum_{j=0}^{n}(c_j - c_{j+1})$ are called telescoping sums.

The Cauchy Criterion 2.11 is an *inner* criterion for the convergence of sequences. The following lemma gives an analogue criterion for series.

3.3 Lemma. (Cauchy's Convergence Criterion). *The series $\sum_{j=0}^{\infty} a_j$ converges, if and only if for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ with*

$$|\sum_{j=n}^{m} a_j| < \varepsilon \quad \textit{for all} \quad n, m \geq N_0.$$

Proof. Since $|\sum_{j=n}^{m} a_j| = |s_m - s_{n-1}|$, the claim follows from Cauchy's criterion for sequences, Theorem 2.11.

□

If we set $n = m$ in the above lemma, we see that the summands of a convergent series always form a null sequence. We write down this important fact in the following corollary.

3.4 Corollary. *Assume that $\sum_{j=0}^{\infty} a_j$ converges. Then $\lim_{j\to\infty} a_j = 0$.*

Proof. Choose $n = m$ in the above Lemma 3.3.

□

We note, that the example of the harmonic series shows that the converse of Corollary 3.4 does not hold.

3.5 Remark. Let (a_j) be a sequence with non-negative elements, i.e. $a_j \geq 0$ for all $j \in \mathbb{N}_0$. Then $\sum_{j=1}^{\infty} a_j$ converges, if and only if the sequence of partial sums $(s_n)_{n\in\mathbb{N}_0}$ is bounded.

Proof. Assume that $\sum_{j=0}^{\infty} a_j$ converges, i.e. the sequence of the partial sums $(s_n)_{n\in\mathbb{N}}$ converges. Theorem 1.6 implies now that $(s_n)_{n\in\mathbb{N}}$ is bounded.
Conversely, $(\sum_{j=0}^{n} a_j)_{n\in\mathbb{N}_0}$ is increasing, since $a_j \geq 0$. By assumption $(s_n)_{n\in\mathbb{N}}$ is bounded, which according to Theorem 1.12 means that $(s_n)_{n\in\mathbb{N}}$ converges.

□

3.6 Example. We consider the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ and show in the following that we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

where the number e was defined as $e = \lim\limits_{n\to\infty} (1+\frac{1}{n})^n$ already in 1.15.

Let $a_n := (1+\frac{1}{n})^n$ for $n \in \mathbb{N}$. The proof of Theorem 1.15 implies, that $a_n \leq \sum_{j=0}^{n} \frac{1}{j!} \leq 3$ for all $n \geq 1$. Thus $(\sum_{j=0}^{n} \frac{1}{j!})_{n\in\mathbb{N}}$ is bounded and Remark 3.5 implies that $\sum_{j=0}^{\infty} \frac{1}{j!}$ converges. Let $e' := \sum_{j=0}^{\infty} \frac{1}{j!}$ be the limit of the series. Then by Lemma 1.9 we have that $\lim a_n = e \leq \sum_{j=0}^{\infty} \frac{1}{j!} = e'$. Thus $e \leq e'$.
We now show the inverse inequality: $e \geq \sum_{j=0}^{m} \frac{1}{j!}$ for each fixed $m \in \mathbb{N}$. Indeed for $n > m \geq 1$

$$a_n \overset{\text{Bin.Thm.}}{=} \sum_{j=0}^{n} \binom{n}{j} \frac{1}{n^j} \geq \sum_{j=0}^{m} \binom{n}{j} \frac{1}{n^j} = \sum_{j=0}^{m} \frac{1}{j!} \underbrace{\overbrace{\underbrace{\frac{n}{n}}_{=1} \underbrace{\frac{n-1}{n}}_{\to 1} \cdots \underbrace{\frac{n-j+1}{n}}_{\to 1}}^{\text{j-factors}}}_{\to 1 (n\to\infty)} .$$

Thus according to Lemma 1.9, $\lim_{n\to\infty} a_n = e \geq \sum_{j=0}^{m} \frac{1}{j!}$ uniformly in m, thus $e \geq \lim_{m\to\infty} \sum_{j=0}^{m} \frac{1}{j!} = e'$. Hence, summarizing, we have $e' = e$.

□

To obtain estimates for Euler's number e, consider

$$d_{n,k} := s_{n+k} - s_n, \quad k, n \in \mathbb{N}, \quad \text{where } s_n = \sum_{j=0}^{n} \frac{1}{j!} .$$

We have for arbitrary $n, k \in \mathbb{N}$ that

$$\frac{1}{(n+1)!} \leq d_{n,k} \leq \frac{s_k - 1}{(n+1)!} ,$$

which yields for $k \to \infty$

$$\frac{1}{(n+1)!} \leq e - s_n \leq \frac{e-1}{(n+1)!} . \tag{3.1}$$

In addition to giving us $2,66 < e < 2,8$ for $n = 2$, the above estimate is the basis for the following proof of the irrationality of e.

3.7 Theorem. *Euler's number e is irrational.*

Proof. Assume that e is rational. Then we can write e in the form $e = p/q$ with $p, q \in \mathbb{N}$. Take the above estimate for $n = q$ and multiply the inequality with $q!$. Then you get

$$0 < \frac{1}{q+1} \leq p(q-1)! - q!s_q < \frac{2}{q+1} \leq 1,$$

and therefore

$$0 < p(q-1)! - q!s_q < 1.$$

This is impossible, because $p(q-1)! - q!s_q \in \mathbb{Z}$.

$\square$

In the following, we examine the convergence of series with alternating signs in the summands. We begin with Dirichlet's criterion.

3.8 Theorem. (Dirichlet's Convergence Criterion). *Let $a_n \in \mathbb{C}$ for all $n \geq 1$ be such that the partial sums $(s_n)_{n\in\mathbb{N}} = (\sum_{j=1}^{n} a_j)_{n\in\mathbb{N}}$ are bounded. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing null sequence. Then $\sum_{j=1}^{\infty} \varepsilon_j a_j$ converges.*

An important consequence is the so-called Leibniz-Criterion.

3.9 Corollary. (Leibniz Criterion). *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing (hence real) null sequence. Then $\sum_{j=1}^{\infty}(-1)^j \varepsilon_j$ converges.*

3.10. Examples and Remarks. a) The series

$$\sum_{j=0}^{\infty}(-1)^j \frac{1}{j+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

converges and is called *alternating harmonic series*. We show in Analysis II, that the limit value of the series $\sum_{j=0}^{\infty}(-1)^j \frac{1}{j+1}$ is equal to $\log 2$.

b) A series of the form $\sum_{j=0}^{\infty}(-1)^j a_j$ with $a_j \geq 0$ for all $j \in \mathbb{N}_0$ is called *alternating*.

Proof of Theorem 3.8 For $m, n \in \mathbb{N}$ with $m \geq n$ set

$$\sigma_{n,m} := \sum_{j=n}^{m} \varepsilon_j a_j.$$

The assumption says that $\lim_{j\to\infty} \varepsilon_j = 0$; thus according to Lemma 3.3 (Cauchy Criterion) it is enough to show that there exists a constant $M > 0$ with

$$|\sigma_{n,m}| \leq M\varepsilon_n \quad \text{for all} \quad m, n \geq 1.$$

We first transform $\sigma_{n,m}$ by *Abel's summation by parts*: Set $s_n := \sum_{j=1}^{n} a_j$ and $s_0 = 0$ to obtain for $m \geq n \geq 1$

$$\begin{aligned}
\sigma_{n,m} &= \sum_{j=n}^{m} \varepsilon_j a_j = \sum_{j=n}^{m} \varepsilon_j (s_j - s_{j-1}) = \sum_{j=n}^{m} \varepsilon_j s_j - \sum_{j=n}^{m} \varepsilon_j \, s_{j-1} \\
&= \sum_{j=n}^{m} \varepsilon_j s_j - \sum_{j=n-1}^{m-1} \varepsilon_{j+1} s_j = \sum_{j=n}^{m-1} (\varepsilon_j - \varepsilon_{j+1}) s_j + \varepsilon_m s_m - \varepsilon_n s_{n-1}.
\end{aligned}$$

With $C := \sup\{|s_n|, n \in \mathbb{N}\}$ we get from the above that (recall $(\varepsilon_j)_j$ is decreasing)

$$\begin{aligned}
|\sigma_{n,m}| &\leq \sum_{j=n}^{m-1} \underbrace{(\varepsilon_j - \varepsilon_{j+1})}_{\geq 0} |s_j| + \varepsilon_m |s_m| + \varepsilon_n |s_{n-1}| \\
&\leq \sum_{j=n}^{m-1} (\varepsilon_j - \varepsilon_{j+1}) C + \varepsilon_m C + \varepsilon_n C \\
&= (\varepsilon_n - \varepsilon_m) C + \varepsilon_m C + \varepsilon_n C = 2 \varepsilon_n C = \underbrace{2\,C}_{=:M} \varepsilon_n \,.
\end{aligned}$$

□

A very important concept in the topic of convergence of series is that of absolute convergence.

3.11 Definition. (Absolute Convergence). A series $\sum_{j=0}^{\infty} a_j$ is called *absolutely convergent*, if $\sum_{j=0}^{\infty} |a_j|$ converges.

3.12 Remark. Every series $\sum_{j=0}^{\infty} a_j$ which converges absolutely, converges.
In fact $|\sum_{j=n}^{m} a_j| \leq \sum_{j=n}^{m} |a_j|$ for all $m \geq n$. Thus, the claim follows from the Cauchy criterion for series, Lemma 3.3.

3.13 Theorem. (Comparison Test [Majorantenkriterium]). *Let $(a_j)_{j \in \mathbb{N}_0} \subset \mathbb{C}$ and $(b_j)_{j \in \mathbb{N}_0} \subset \mathbb{R}$ be two sequences such that $|a_j| \leq b_j$ for almost all $j \in \mathbb{N}$. If $\sum_{j=0}^{\infty} b_j$ converges, then $\sum_{j=0}^{\infty} a_j$ converges absolutely.*

In the situation above the series $\sum_{j=0}^{\infty} b_j$ is said to *dominate* or *majorise* $\sum_{j=0}^{\infty} a_j$.

Proof. Since $\sum_{j=n}^{m} |a_j| \leq \sum_{j=n}^{m} b_j$ for all $m \geq n$, the claim follows from the Cauchy criterion Lemma 3.3.

□

Example. In Example 3.2 c) we have shown that $\sum_{j=1}^{\infty} \frac{1}{j(j+1)}$ converges. Observing that $0 < \frac{1}{(j+1)^2} \leq \frac{1}{j(j+1)}$ for $j \geq 1$, it follows that $\sum_{j=1}^{\infty} \frac{1}{(j+1)^2}$ converges and hence so does $\sum_{j=1}^{\infty} \frac{1}{j^2} = 1 + \sum_{j=1}^{\infty} \frac{1}{(j+1)^2}$.

In particular, if we choose as dominating series the geometric series, we get the so-called Root Test.

3.14 Theorem. (Root Test [Wurzelkriterium]). *Let $(a_n)_n$ be a sequence in $\mathbb{C}$.*
a) Assume that there exists some q, $0 < q < 1$, with

$$\sqrt[j]{|a_j|} \leq q \quad \text{for almost all} \quad j \in \mathbb{N}.$$

Then $\sum_{j=0}^{\infty} a_j$ is absolutely convergent.
b) If we have $\sqrt[j]{|a_j|} \geq 1$ for infinitely many $j \in \mathbb{N}$, then $\sum a_j$ diverges .

Proof. a) By assumption, there exists $N_0 \in \mathbb{N}$ with $\sqrt[j]{|a_j|} \leq q$ for all $j \geq N_0$. Thus $|a_j| \leq q^j$ for all $j \geq N_0$, which implies that $\sum_{j=N_0}^{\infty} |a_j|$ is dominated by the geometric series $\sum_{j=1}^{\infty} q^j$. (Note: The finite sum $a_0 + \cdot + a_{N_0-1}$ is trivially convergent.)

b) The assumption says, that $\sqrt[j]{|a_j|} \geq 1$ for infinitely many $j \in \mathbb{N}$. Thus $|a_j| \geq 1$ for infinitely many $j \in \mathbb{N}$. In particular, the sequence $(a_j)_j$ is not a null sequence, which means that $\sum_{j=1}^{\infty} a_j$ diverges.

□

Example. The series $\sum_{j=0}^{\infty} \frac{j^l}{2^j}$ converges for each fixed $l \in \mathbb{N}$, because

$$\sqrt[n]{|a_n|} = \frac{\sqrt[n]{n^l}}{2} = \frac{(\sqrt[n]{n})^l}{2} \longrightarrow \frac{1}{2};$$

Thus $\sqrt[j]{|a_j|} \leq \frac{2}{3} = q < 1$ for almost all $j \in \mathbb{N}$.

Often it is easier to implement the following test.

3.15 Theorem. (Ratio Test [Quotientenkriterium]).
a) Let $a_j \neq 0$ for almost all $j \in \mathbb{N}$ and assume that there exists $0 < q < 1$ with

$$\left|\frac{a_{j+1}}{a_j}\right| \leq q \quad \text{for almost all } j \in \mathbb{N}.$$

Then $\sum_{j=0}^{\infty} a_j$ converges absolutely.
b) If $\left|\frac{a_j+1}{a_j}\right| \geq 1$ for almost all (not only for infinitely many) $j \in \mathbb{N}$, then $\sum_{j=0}^{\infty} a_j$ diverges.

Proof. a) By assumption, there exists $N_0 \in \mathbb{N}$ with $|\frac{a_{j+1}}{a_j}| \leq q$ for all $j \geq N_0$. Thus for all $n \geq N_0 + 1$

$$|\frac{a_n}{a_{N_0}}| \overset{(*)}{=} \prod_{j=N_0}^{n-1} |\frac{a_{j+1}}{a_j}| = (\frac{a_{N_0+1}}{a_{N_0}} \frac{a_{N_0+2}}{a_{N_0+1}} \cdots \frac{a_n}{a_{n-1}}) \overset{Ass.}{\leq} q^{n-N_0}.$$

Thus $|a_n| \leq |a_{N_0}| q^{n-N_0}$ for all $n \geq N_0 + 1$ and

$$\sum_{n=0}^{\infty} |a_n| \leq \sum_{n=0}^{N_0} |a_n| + \frac{|a_{N_0}|}{q^{N_0}} \sum_{n=0}^{\infty} q^n .$$

Now, the Comparison Test implies the claim.

b) The assumption and the relation (*) imply, that $|\frac{a_n}{a_{N_0}}| \geq 1$ for all $n \geq N_0 + 1$. Hence (a_n) is not a null sequence and $\sum a_j$ diverges.

□

3.16 Example. The *exponential series*

$$\sum_{j=0}^{\infty} \frac{z^j}{j!}$$

converges for all $z \in \mathbb{C}$. This is clear for $z = 0$. Furthermore, for $z \neq 0$ we have

$$|\frac{a_{j+1}}{a_j}| = \frac{|z^{j+1}|}{(j+1)!} \frac{j!}{|z^j|} = \frac{|z|}{j+1} \overset{j\to\infty}{\longrightarrow} 0$$

i.e. $|\frac{a_{j+1}}{a_j}| \leq \frac{1}{2}$ for almost all $j \in \mathbb{N}$.

Now consider variants of the above root and quotient tests, in which the existence of a number q with $0 < q < 1$ is replaced by a condition concerning the limit inferior or the limit superior, respectively.

3.17 Theorem. (Another formulation of the Root and Ratio Tests).

a) *If* $\overline{\lim}_{j\to\infty} \sqrt[j]{|a_j|} < 1$, *then* $\sum_{j=0}^{\infty} a_j$ *converges absolutely.*

b) *If* $\overline{\lim}_{j\to\infty} \sqrt[j]{|a_j|} > 1$, *then* $\sum_{j=0}^{\infty} a_j$ *diverges.*

c) *Let* $a_j \neq 0$ *for almost all* $j \in \mathbb{N}$ *and* $\overline{\lim}_{j\to\infty} |\frac{a_{j+1}}{a_j}| < 1$. *Then* $\sum_{j=0}^{\infty} a_j$ *converges absolutely.*

d) *If* $\underline{\lim}_{j\to\infty} |\frac{a_{j+1}}{a_j}| > 1$, *then* $\sum_{j=0}^{\infty} a_j$ *diverges.*

3.18 Remarks. a) If $\overline{\lim}_{j\to\infty}\sqrt[j]{|a_j|}=1$, then *no* conclusion for convergence can be made! Indeed, consider for example $a_j=\frac{1}{j}$ and $b_j=\frac{1}{j^2}$. Then by Example 1.16 c) and Remark 1.14, we have

$$\begin{aligned}\sqrt[j]{|a_j|} &= \sqrt[j]{\frac{1}{j}} = \frac{1}{\sqrt[j]{j}} \stackrel{j\to\infty}{\longrightarrow} 1 \quad \text{and}\\ \sqrt[j]{|b_j|} &= \sqrt[j]{\frac{1}{j}} = \frac{1}{\sqrt[j]{j^2}} \stackrel{j\to\infty}{\longrightarrow} 1,\end{aligned}$$

but $\sum_{j=1}^{\infty} a_j$ diverges, while $\sum_{j=1}^{\infty} b_j$ converges.

b) The Ratio Test is "weaker" than the Root Test, i.e.

$$\overline{\lim_{j\to\infty}}\sqrt[j]{|a_j|} \le \overline{\lim_{j\to\infty}}|\frac{a_{j+1}}{a_j}|.$$

We conclude this first section about convergence of series with Cauchy's *Condensation Test.*

3.19 Theorem. (Cauchy's Condensation Test). *Let (a_n) be a decreasing null sequence. Then:*

$$\sum_{j=0}^{\infty} a_j \text{ converges} \quad \Longleftrightarrow \quad \sum_{j=0}^{\infty} 2^j a_{2^j} \text{ converges.}$$

The above theorem says that we can completely read off the convergence behavior of a given sequence from the convergence behavior of the 'condensed' sequence which only has elements with indexes 2^j, and thus far less elements than the original series.

Proof. Let $s_n := \sum_{j=0}^{n} a_j$ and $t_n := \sum_{j=0}^{n} 2^j a_{2^j}$.

"$\Longrightarrow$": For $n \ge 2^j$

$$\begin{aligned} s_n &\ge a_1 + a_2 + (a_3+a_4) + (a_5 + \cdots a_8) + \cdots + (a_{2^{j-1}+1} + \cdots + a_{2^j})\\ &\ge \frac{a_1}{2} + a_2 + 2a_4 + 4a_8 + \cdots + 2^{j-1}a_{2^j}\\ &= \frac{1}{2}(a_1 + 2a_2 + 4a_4 \cdots + 2^j a_{2^j}) = \frac{1}{2}t_j\end{aligned}$$

Let $\sum_{j=0}^{\infty} a_j =: s$. Then $t_j \le 2s$ for all j and according to Remark 3.5 $\sum_{j=0}^{\infty} 2^j a_{2^j}$ converges.

"$\Longleftarrow$": Let $n \le 2^{j+1}-1$. Then

$$\begin{aligned} s_n &\le a_0 + a_1 + (a_2+a_3) + (a_4 \cdots + a_7) + \cdots + (a_{2^j} + \cdots + a_{2^{j+1}-1})\\ &\le a_0 + a_1 + 2a_2 + 4a_4 + \cdots + 2^j a_{2^j} = a_0 + t_j\end{aligned}$$

Let $t = \sum_{j=0}^{\infty} 2^j a_{2^j}$. Then $s_n \leq a_0 + t$ for all $n \geq 0$ and Remark 3.5 implies that $\sum_{j=0}^{\infty} a_j$ converges.

$\square$

The above theorem implies, that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha},$$

with $\alpha \in \mathbb{Q}$, converges if and only if $\alpha > 1$. The corresponding condensed series

$$\sum_{j=0}^{\infty} 2^j 2^{-j\alpha} = \sum_{j=0}^{\infty} 2^{(1-\alpha)j} = \sum_{j=0}^{\infty} q^j \quad \text{with } q := 2^{1-\alpha}$$

is a geometric series and converges by 3.2 if and only if $q < 1$, or equivalently $\alpha > 1$. Bear in mind that at the moment, we have defined n^α only for $\alpha \in \mathbb{Q}$; we will define n^α for arbitrary $\alpha \in \mathbb{R}$ later.

The function given by the convergent series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1$$

(at the moment only for $s \in \mathbb{Q}$), is the famous *Riemann zeta function* . It is an important tool to study the distribution of prime numbers. In the lecture 'Analysis II', we will prove $\zeta(2) = \frac{\pi^2}{6}$. The still unsolved *Riemann hypothesis* states that all nontrivial roots of the zeta function have real part $\frac{1}{2}$.

4 Rearrangement and Products of Series

If we add finitely many real or complex numbers, the result does not depend on the order of the summands, i.e. any arbitrary rearrangement of the summands yields the same result. For infinite series, the situation is completely different. We will see in the following section that it is possible to change the value of a series by rearranging its terms and that one can even achieve divergence of a former convergent series this way. However, this at first sight quite surprising effect does not appear for absolutely convergent series. This is a reason why the concept of absolute convergence is so important. Of course, for a precise description of the situation we must first define the concept of rearrangement. We start with an example.

Consider the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2j-1} - \frac{1}{2j} + \ldots,$$

as well as a rearrangement of it, which is given by

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} + \ldots.$$

We denote the n-th partial sum of the original and the rearranged series by s_n and t_n, respectively, and we define $s := \lim_{n\to\infty} s_n$. Then we have

$$\begin{aligned}
s_2 &= \tfrac{1}{2} & 2t_3 &= 2 \cdot \tfrac{1}{4} = \tfrac{1}{2}\\
s_4 &= \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} & 2t_6 &= \tfrac{1}{2} + \underbrace{2\Big(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\Big)}_{\frac{1}{3}-\frac{1}{4}}\\
s_6 &= \tfrac{1}{2} + (\tfrac{1}{3} - \tfrac{1}{4}) + (\tfrac{1}{5} - \tfrac{1}{6}) & 2t_9 &= \tfrac{1}{2} + (\tfrac{1}{3} - \tfrac{1}{4}) + \underbrace{2\Big(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\Big)}_{\frac{1}{5}-\frac{1}{6}}
\end{aligned}$$

and because of $\frac{1}{2j-1} - \frac{1}{4j-2} - \frac{1}{4j} = \frac{1}{2}(\frac{1}{2j-1} - \frac{1}{2j})$, we can infer $2t_{3n} = s_{2n}$ for all $n \geq 1$. Because $(s_{2n})_{n\in\mathbb{N}}$ converges to s and the terms of the rearranged series converge to 0, there exists for each $\varepsilon > 0$ an $N_0 \in \mathbb{N}$ such that at the same time $|t_{3n} - \frac{s}{2}| < \frac{\varepsilon}{2}$, $|t_{3n+1} - t_{3n}| < \frac{\varepsilon}{2}$ and $|t_{3n+2} - t_{3n}| < \frac{\varepsilon}{2}$ for all $n \geq N_0$. This implies $|t_m - \frac{s}{2}| < \frac{\varepsilon}{2}$ for all $m > 3N_0 + 2$, which means that the rearranged sequence converges to $s/2$.

This example motivates the following definition.

4.1 Definition. Let $\sum_{n=0}^{\infty} a_n$ be a series of complex numbers and $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ a bijective mapping. Then $\sum_{n=0}^{\infty} a_{\varphi(n)}$ is called *rearrangement* of the series $\sum_{n=0}^{\infty} a_n$. Further the series $\sum_{n=0}^{\infty} a_n$ is called *unconditionally convergent*, if every rearrangement of the series $\sum_{n=0}^{\infty} a_n$ has the <u>same</u> limit value.

4.2 Theorem. *Every absolutely convergent series $\sum_{n=0}^{\infty} a_n$ is unconditionally convergent.*

Proof. Let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be a bijective mapping. Let

$$s_n := \sum_{j=0}^{n} a_j, \quad t_n := \sum_{j=0}^{n} a_{\varphi(j)}.$$

We show, that $(t_n)_n$ converges to s, where s denotes the limit value of the sequence s_n, i.e. $s := \lim_{n\to\infty} s_n$. According to the definition of the convergence of $\sum |a_n|$, for $\varepsilon > 0$ there exists an $N_0 \in \mathbb{N}$ with

$$\sum_{j=N_0}^{\infty} |a_j| < \frac{\varepsilon}{2}.$$

Hence

$$|s - \sum_{j=0}^{N_0-1} a_j| = |\sum_{j=N_0}^{\infty} a_j| \leq \sum_{j=N_0}^{\infty} |a_j| < \frac{\varepsilon}{2}.$$

Now choose N_1 so large, that $\{0, 1, 2 \cdots N_0 - 1\} \subset \{\varphi(0), \varphi(1) \cdots \varphi(N_1)\}$. Then for all $m \geq N_1$ holds

$$|\sum_{j=0}^{m} a_{\varphi(j)} - s| \leq |\sum_{j=0}^{m} a_{\varphi(j)} - \sum_{j=0}^{N_0-1} a_j| + \underbrace{|\sum_{j=0}^{N_0-1} a_j - s|}_{<\frac{\varepsilon}{2}} \leq \sum_{j=N_0}^{\infty} |a_j| + \frac{\varepsilon}{2} \leq \varepsilon.$$

Thus (t_m) converges to s.

□

The following result (due to Riemann) is quite surprising.

4.3 Theorem. *(*Riemann Rearrangement Theorem*). Let $\sum_{n=0}^{\infty} a_n$ be a convergent, but not absolutely convergent series of real numbers. Then there exists for every $b \in \mathbb{R}$ a rearrangement $\sum_{n=0}^{\infty} a_{\varphi(n)}$, which converges to b.*

The Riemann Rearrangement Theorem has the remarkable consequence, that you may only rearrange finitely may terms in a convergent series which is not absolutely convergent — otherwise the concept of convergent series does not make sense any more! On the other hand, the above Theorem 4.2 says that the value of an absolutely convergent series is invariant under rearrangement.

We do not prove the Riemann Rearrangement Theorem here and instead refer to the book of Mangold/Knopp.

In the following, we want to multiply two convergent series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. For this, we consider the product

$$(a_0 + a_1 + a_2 + \cdots)(b_0 + b_1 + \cdots).$$

Expanding gives terms of the following form, which have to be summed up.

$$\begin{array}{lllll} a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \cdots \\ a_1b_0 & a_1b_1 & a_2b_1 & \cdots & \\ a_2b_0 & a_2b_1 & a_2b_2 & \cdots & \end{array}$$

There is the question, in which order should the single terms be summed up. In particular, we ask when

$$(\sum_{j=0}^{\infty} a_j)(\sum_{j=0}^{\infty} b_j) = \sum_{j=0}^{\infty} p_j,$$

with $p_{\varphi(l,m)} = a_l b_m$ for $l, m \in \mathbb{N}_0$ and some bijection $\varphi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$. Possible orderings are

$$\begin{array}{ccccccc} 0 & & 1 & & 3 & & 6 \\ & \swarrow & & \swarrow & & \swarrow & \\ 2 & & 4 & & 7 & & \\ & \swarrow & & \swarrow & & & \\ 5 & & 8 & & & & \\ & \swarrow & & & & & \\ 9 & & & & & & \end{array} \quad \text{or} \quad \begin{array}{ccccccc} 0 & \rightarrow & 1 & & 4 & & 9 \\ & & \downarrow & & \downarrow & & \\ 3 & \leftarrow & 2 & & 5 & & \\ & & & & \downarrow & & \\ 8 & \leftarrow & 7 & \leftarrow & 6 & & \end{array}$$

We call the series $\sum_{j=0}^{\infty} p_j$ a *product series of* $\sum a_j$ *and* $\sum b_j$ if $p_{\varphi(l,m)} = a_l b_m$ for all $l, m \in \mathbb{N}_0$ and $\varphi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ is bijective.

4.4 Theorem. *Let* $\sum_{j=0}^{\infty} a_j$ *and* $\sum_{j=0}^{\infty} b_j$ *be two absolutely convergent series. Then all their product series converge to*

$$(\sum_{j=0}^{\infty} a_j) \cdot (\sum_{j=0}^{\infty} b_j)$$

.

Proof. Let $\sum_{j=0}^{\infty} p_j$ be an arbitrary product series of $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$. Then there exists for all $n \in \mathbb{N}$ an $m \in \mathbb{N}$ with

$$\sum_{j=0}^{n} |p_j| \leq \sum_{j=0}^{m} |a_j| \sum_{j=0}^{m} |b_j| \leq \sum_{j=0}^{\infty} |a_j| \sum_{j=0}^{\infty} |b_j|.$$

Now Remark 3.5 implies that $\sum_{j=0}^{\infty} |p_j|$ converges. Further, from Remark 3.12 it follows that also $\sum_{j=0}^{\infty} p_j$ converges, and Theorem 4.2 implies that the convergence is unconditional (i.e. independent from the chosen order). This means that *every* product series converges to the same $s \in \mathbb{C}$.

Use now the special product series given by

$$\begin{array}{ccccccc} a_0b_0 & & a_0b_1 & & a_0b_2 & \cdots \\ & & \downarrow & & \downarrow & \\ a_1b_0 & \leftarrow & a_1b_1 & & a_1b_2 & \cdots \\ & & & & \downarrow & \\ a_2b_0 & \leftarrow & a_2b_1 & \leftarrow & a_2b_2 & \cdots \end{array} \qquad \begin{array}{ccccccc} q_0 & & q_1 & & q_4 & \cdots \\ & & \downarrow & & \downarrow & \\ q_3 & \leftarrow & q_2 & & q_5 & \cdots \\ & & & & \downarrow & \\ q_8 & \leftarrow & q_7 & \leftarrow & q_6 & \cdots \end{array}$$

Then we have

$$q_0 + q_1 + \cdots q_{(n+1)^2-1} = \underbrace{(a_0 + \cdots + a_n)}_{\overset{n\to\infty}{\longrightarrow} \sum_{j=0}^{\infty} a_j} \underbrace{(b_0 + \cdots + b_n)}_{\overset{n\to\infty}{\longrightarrow} \sum_{j=0}^{\infty} b_j}.$$

From this, the claim follows.

□

If one chooses the following order for the summation

$$\begin{array}{ccccc} a_0b_0 & & a_0b_1 & & a_0b_2 \\ & \swarrow & & \swarrow & \\ a_1b_0 & & a_1b_1 & & \\ & \swarrow & & & \\ a_2b_0 & & & & \end{array} \quad \text{or} \quad \begin{array}{ccccc} p_0 & & p_1 & & p_3 \\ & \swarrow & & \swarrow & \\ p_2 & & p_4 & & \\ & \swarrow & & & \\ p_5 & & & & \end{array}$$

letting $c_0 := a_0b_0$, $c_1 := a_0b_1 + a_1b_0$ and generally

$$c_n := \sum_{j=0}^{n} a_j b_{n-j},$$

we obtain the following corollary.

4.5 Corollary. (Cauchy Product of Series). *Let $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$ be two absolutely convergent series and let*

$$c_n := \sum_{j=0}^{n} a_j b_{n-j}, \qquad n \in \mathbb{N}_0.$$

Then $\sum_{n=0}^{\infty} c_n$ converges absolutely and

$$(\sum_{j=0}^{\infty} a_j)(\sum_{j=0}^{\infty} b_j) = \sum_{n=0}^{\infty} c_n.$$

We remark that Corollary 4.5 in general does not hold for series which are only convergent, but not absolutely convergent.

In particular, let us return to the *exponential series* given by

$$\exp(z) := \sum_{j=0}^{\infty} \frac{z^j}{j!}, \quad z \in \mathbb{C}.$$

From the Ratio Test Theorem 3.15 and Example 3.16 it follows, that $\exp(z)$ is absolutely convergent for all $z \in \mathbb{C}$. Further, we can now show the important *functional equation* of the exponential series.

4.6 Corollary. (Functional Equation of exp). *For $z, w \in \mathbb{C}$*

$$\exp(z)\exp(w) = \exp(z + w).$$

Proof. For $z, w \in \mathbb{C}$

$$\begin{aligned} \exp(z)\exp(w) &\overset{\text{Def.}}{=} \Big(\sum_{j=0}^{\infty} \frac{z^j}{j!}\Big) \Big(\sum_{j=0}^{\infty} \frac{w^j}{j!}\Big) \overset{\text{Cauchy Product 4.5}}{=} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{z^j}{j!} \frac{w^{n-j}}{(n-j)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} z^j w^{n-j} \overset{\text{Bin.Thm.}}{=} \sum_{n=0}^{\infty} \frac{1}{n!} (z+w)^n \\ &\overset{Def.}{=} \exp(z + w). \end{aligned}$$

$\square$

4.7 Corollary.

a) *For all $z \in \mathbb{C}$ holds:* $\exp(-z) = \frac{1}{\exp(z)}$. *In particular,* $\exp(z) \neq 0$ *for all $z \in \mathbb{C}$.*

b) *For all $x \in \mathbb{R}$ holds:* $\exp(x) > 0$.

c) *For all $q \in \mathbb{Z}$ holds:* $\exp(q) = e^q$.

d) *For all $q \in \mathbb{Q}$ holds:* $\exp(q) = e^q$.

The proof is left as an exercise. Finally, if we let

$$e^z := \exp(z), \quad z \in \mathbb{C},$$

the above proposition d) implies that this definition extends the original definition of e^q for rational exponents $q \in \mathbb{Q}$ (compare remark II.1.14) to arbitrary exponents $z \in \mathbb{C}$.

5 Power Series

Power series have a long tradition in analysis. If one expresses a function f in the form $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$, then this is called the *power series expansion of f centered at x_0*. The general theory of such expansions will be presented in the course on *Complex Analysis (Funktionentheorie)*. Only then, the full importance of power series will become visible.

Apart from these general properties, we are interested in power series because of the fact that their convergence behaviour can be described by the so-called *radius of convergence*. We start with the following definition.

5.1 Definition. Let $(a_n)_n \subset \mathbb{C}$ be a sequence of complex numbers and $z \in \mathbb{C}$. Then $\sum_{n=0}^{\infty} a_n z^n$ is called a *power series*.

In this section we will analyse the question, for which $z \in \mathbb{C}$ the series above converges.

5.2 Definition. Let $(a_n)_n \subset \mathbb{C}$. Then

$$\varrho := \frac{1}{\overline{\lim}\, \sqrt[n]{|a_n|}}$$

is called the *radius of convergence* of the series $\sum_{n=0}^{\infty} a_n z^n$ (we use the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$). This definition of the radius of convergence is also called *Cauchy Hadamard formula*.

We will in the following call the set

$$U_\varrho(0) := \{z \in \mathbb{C} : |z| < \varrho\}$$

the *disc of convergence* of the series $\sum_{n=0}^{\infty} a_n z^n$.

The following theorem is the main result of this section.

5.3 Theorem. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence ϱ. Then for $z \in \mathbb{C}$ we have:*

a) If $|z| < \varrho$, then $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent.

b) If $|z| > \varrho$, then $\sum_{n=0}^{\infty} a_n z^n$ diverges.

c) If $|z| = \varrho$, then in general no conclusion is possible.

Proof. The proof is an application of the root test. We have $\sqrt[n]{|a_n z^n|} = |z|\sqrt[n]{|a_n|}$. Since

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n z^n|} = |z|\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|} < 1 \Leftrightarrow |z| < \varrho\,,$$

the Root Test 3.14 implies the statement of the theorem, i.e. we have

$|z| < \varrho \Longrightarrow \sum_{n=0}^{\infty} a_n z^n$ converges absolutely.

$|z| > \varrho \Longrightarrow \sum_{n=0}^{\infty} a_n z^n$ diverges.

$|z| = \varrho \Longrightarrow$ no conclusion possible.

□

5.4 Remark. In addition to the root test, one can also use the ratio test to determine the radius of convergence. In particular, let $\sum_{n=0}^{\infty} a_n z^n$ be a power series for which

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| =: q.$$

exists. Then the power series $\sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence $\varrho = \frac{1}{q}$. To prove this we note that

$$\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| \to q|z|, \quad (n \to \infty)$$

holds. If $0 < q < \infty$, we choose $z_1, z_2 \in \mathbb{C}$ with $|z_1| < 1/q$ and $|z_2| > 1/q$, and the ratio test implies that the series $\sum_{n=0}^{\infty} a_n z_1^n$ converges absolutely and the series $\sum_{n=0}^{\infty} a_n z_2^n$ diverges. By Theorem 5.3, we therefore have $\varrho = 1/q$. The cases $q = 0$ and $q = \infty$ are proved similarly.

5.5 Examples.
a) The exponential series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has a radius of convergence of $\varrho = \infty$. Observe that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n!}{(n+1)!}\right| = \frac{1}{n+1} \to 0,$$

so that Remark 5.4 now implies $\varrho = \frac{1}{q} = \infty$.

b) The series $\sum_{n=0}^{\infty} n^n z^n$ has a radius of convergence $\varrho = 0$, because

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|} = \overline{\lim}\sqrt[n]{n^n} = \overline{\lim}\, n = \infty.$$

and hence $\varrho = \frac{1}{\infty} = 0$.

c) The series $\sum_{n=0}^{\infty} \frac{n!}{n^n} z^n$ has a radius of convergence $\varrho = e$. Proof as exercise.

Chapter III

Continuous Functions and the Basics of Topology

1 Continuous Functions

We begin this chapter by considering continuous functions and their properties. The notion of continuity that we use in the following is — like the notion of convergence — essentially due to Cauchy, who defined the continuity of a function in his *Cours d'Analyse (1821)* as follows:

> En d'autres termes, la fonction $f(x)$ restera continue par rapport à x entre les limites données, si, entre ces limites, un accroissement infiniment petit de la variable produit toujours un accroissement infiniment petit de la fonction elle-même.

Cauchy still used the then-common concept of 'infinitely small quantitiy' (quantité infiniment petite), however this was replaced over the years by the ε-δ formulation which is customary today. The latter was vitally influenced by Weierstraß.

We recall the definition of a function: Let X and Y be sets and $f : X \to Y$ a function, i.e. a rule that assigns a *unique (eindeutig)* element $y \in Y$ to every $x \in X$. The set graph $f := \{(x, f(x)), x \in X\} \subset X \times Y$ is called the graph of f.

We start with the definition of continuity of a function which builds on the concept of convergence.

1.1 Definition. (Continuity). A function $f : D \subset \mathbb{K} \to \mathbb{K}$ is called *continuous (stetig) at* $x_0 \in D$, if for every sequence $(x_n)_{n\geq 1} \subset D$ with $\lim\limits_{n\to\infty} x_n = x_0$, it holds that

$$\lim_{n\to\infty} f(x_n) = f(x_0).$$

In other words:

$$(x_n) \subset D,\ x_n \to x_0 \quad \Longrightarrow \quad f(x_n) \to f(x_0).$$

The function f is called *continuous in* D, if f is continuous at all points $x_0 \in D$.

The following theorem is a reformulation of the definition of continuity in ε-δ-language.

1.2 Theorem. (ε-δ criterion). *A function $f : D \subset \mathbb{K} \longrightarrow \mathbb{K}$ is continuous at $x_0 \in D$ if and only if*

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, x \in D, |x - x_0| < \delta) \quad |f(x) - f(x_0)| < \varepsilon.$$

Proof. $'' \Longrightarrow''$: We assume that the assertion is false. Then there exists an $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}$ there exists an $x_n \in D$ with

$$|x_0 - x_n| < 1/n \text{ and } |f(x_0) - f(x_n)| \geq \varepsilon_0.$$

Then $\lim\limits_{n\to\infty} x_n = x_0$ but $f(x_n) \not\to f(x_0)$ for $n \to \infty$. Contradiction!

$'' \Longleftarrow''$: By assumption, for every $\varepsilon > 0$, there exists a $\delta > 0$ with $|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon$. Let $x_n \to x_0$. Then there exists an $N_0 \in \mathbb{N}$ with $|x_n - x_0| < \delta$ for all $n \geq N_0$. Therefore, $|f(x_n) - f(x_0)| < \varepsilon$ for all $n \geq \mathbb{N}_0$, i.e. $\lim f(x_n) = f(x_0)$. $\square$

1.3 Examples. a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = ax + b$ with $a, b \in \mathbb{R}$. Then f is continuous, since $x_n \to x_0$ implies $f(x_n) = ax_n + b \to ax + b = f(x)$.

b) The absolute value $f : \mathbb{R} \to \mathbb{R}, x \mapsto |x|$ is a continuous function.

c) The *Heavyside function* $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

is continuous for all $x \in \mathbb{R} \setminus \{0\}$, but not continuous in 0.

d) The function f, given by

$$f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} 1, & x \geq 1, \\ \frac{1}{n}, & \frac{1}{n} \leq x < \frac{1}{n-1},\ n = 2, 3, \ldots \\ 0, & x \leq 0, \end{cases}$$

is continuous at 0, since we can choose e.g. $\delta = \varepsilon$ because of $|f(x) - f(0)| = |f(x)| \leq |x|$.

e) The *Dirichlet Function (Dirichletsche Sprungfunktion)*, given by

$$f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is discontinuous at all points $x \in \mathbb{R}$. Proof as an exercise.

f) Consider the function given by

$$f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q} \quad \text{with } q > 0 \text{ minimal} \\ 0, & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

Then f is continuous at all points $x_0 \in \mathbb{R}\backslash\mathbb{Q}$ but discontinuous at $x_0 \in \mathbb{Q}$! Proof as an exercise.

g) Let $f : D \subset \mathbb{R} \to \mathbb{R}$. Assume there exists an $L > 0$ with

$$|f(x) - f(y)| \leq L|x - y|, \quad x, y \in D.$$

Then f is continuous. Indeed, choose for $\varepsilon > 0$ a δ such that $\delta := \frac{\varepsilon}{L+1}$. A function that satisfies the above condition is called *Lipschitz continuous* and L is called the *Lipschitz constant* of f.

h) Every Lipschitz-continuous function f is continuous. The converse does not hold. Consider, for example, $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \sqrt{x}$. Then f is continuous, but not Lipschitz-continuous (Exercise).

i) Let the functions $f_1, \cdots f_4 : \mathbb{C} \to \mathbb{C}$ be given by

$$f_1(z) = |z|, \quad f_2(z) = \bar{z}, \quad f_3(z) = \operatorname{Re} z, \quad f_4(z) = \operatorname{Im} z.$$

Then the functions $f_1, \cdots f_4$ are Lipschitz-continuous with Lipschitz-constant 1, and therefore continuous.

The above definition of continuity via sequences allows us to apply our knowledge about convergent sequences to continuous functions. More precisely, we first define the sum, the product and the quotient of two functions. For this, let $f, g : D \subset \mathbb{K} \to \mathbb{K}$ be two functions, and $\alpha, \beta \in \mathbb{K}$. If we define

$$\begin{array}{rlrl} \alpha f + \beta g & : D \to \mathbb{K}, & (\alpha f + \beta g)(x) & := \alpha f(x) + \beta g(x) \\ f \cdot g & : D \to \mathbb{K}, & (f \cdot g)(x) & := f(x) \cdot g(x) \\ \frac{f}{g} & : \{x \in D : \; g(x) \neq 0\} \to \mathbb{K}, & (\frac{f}{g})(x) & := \frac{f(x)}{g(x)}, \end{array}$$

we have the following theorem.

1.4 Theorem. *Let $f, g : D \subset \mathbb{K} \to \mathbb{K}$ be continuous at $x_0 \in D$. Then the following statements hold:*

a) $\alpha f + \beta g : D \to \mathbb{K}$ is continuous at $x_0 \in D$ for all $\alpha, \beta \in \mathbb{K}$.

b) $f \cdot g : D \to \mathbb{K}$ is continuous at x_0.

c) If $g(x_0) \neq 0$, then there exists $\delta > 0$ with $g(x) \neq 0$ for $x \in U_\delta(x_0) \cap D$ and $\dfrac{f}{g} : U_\delta(x_0) \cap D \to \mathbb{K}$ is continuous at x_0.

Proof. The statements a) and b) follow from 1.1 and the calculation rules of convergent sequences.
c) By assumption $|g(x_0)| =: \gamma > 0$. Because g is continuous at x_0 it follows that there exists a $\delta > 0$ such that

$$|g(x_0)| - |g(x)| \leq |g(x_0) - g(x)| < \frac{\gamma}{2}, \qquad x \in U_\delta(x_0) \cap D.$$

Therefore, $|g(x)| > \frac{\gamma}{2}$ for $x \in U_\delta(x_0) \cap D$. The assertion then follows from the calculation rules of convergent sequences.

□

We now consider the composition of two functions $f : D_f \subset \mathbb{K} \to \mathbb{K}$ and $g : D_g \subset \mathbb{K} \to \mathbb{K}$ with $g(D_g) \subset D_f$. We define then $f \circ g : D_g \to \mathbb{K}$ as

$$(f \circ g)(x) := f(g(x)).$$

The following theorem says that the composition of two continuous functions is again continuous.

1.5 Theorem. *Let $f : D_f \subset \mathbb{K} \to \mathbb{K}$ and $g : D_g \subset \mathbb{K} \to \mathbb{K}$ be functions with $g(D_g) \subset D_f$. If g is continuous at $x_0 \in D_g$ and f is continuous at $g(x_0) \in D_f$, then $f \circ g$ is continuous at x_0.*

Proof. Let $(x_n) \subset D_g$ be a sequence in D_g with $\lim\limits_{n\to\infty} x_n = x_0$. By assumption g is continuous at x_0; therefore, $g(x_n) \to g(x_0)$. Then, because f is continuous at $g(x_0)$ it follows that $(f \circ g)(x_n) = f(g(x_n)) \to f(g(x)) = (f \circ g)(x)$, i.e. $f \circ g$ is continuous at x_0.

□

1.6 Examples. a) Polynomials, i.e. functions of the form

$$x \mapsto a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0, \qquad \text{with } a_j \in \mathbb{K}$$

for $j = 0, 1, 2, \ldots, n$, are continuous.
b) If p and q are polynomials, then the function $\dfrac{p}{q}$ given by

$$\frac{p}{q}(z) := \frac{p(z)}{q(z)} \text{ with } D_{\frac{p}{q}} = \{z \in \mathbb{K} : q(z) \neq 0\}$$

is also continuous. Such functions are called *rational functions (rationale Funktionen).*
c) The *sign function (Signumfunktion)* $\text{sign} : \mathbb{C}\backslash\{0\} \to \mathbb{C}$, $\text{sign}\,(z) := \frac{z}{|z|}$ is continuous.

Power series are the natural generalisation of polynomials. In the following theorem, we show that power series are continuous functions inside their disc of convergence.

1.7 Theorem. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $\varrho > 0$. Then $f : B_\varrho(0) := \{z \in \mathbb{C} : |z| < \varrho\} \to \mathbb{C}, z \mapsto \sum_{n=0}^{\infty} a_n z^n$ is a continuous function.*

Proof. Let $z_0 \in B_\varrho(0)$, $\varepsilon > 0$ and choose $r > 0$ such that $|z_0| < r < \varrho$. Theorem II, 5.3 implies that $\sum_{n=0}^{\infty} |a_n|\, r^n$ converges, i.e. that $N_0 \in \mathbb{N}$ exists with

$$\sum_{n=N_0+1}^{\infty} |a_n|\, r^n < \frac{\varepsilon}{4}.$$

Thus for $z \in \mathbb{C}$ with $|z| \leq r$

$$\begin{aligned} |f(z) - f(z_0)| &\leq |\sum_{n=0}^{N_0} a_n z^n - \sum_{n=0}^{N_0} a_n z_0^n| + \sum_{n=N_0+1}^{\infty} |a_n||z|^n + \sum_{n=N_0+1}^{\infty} |a_n||z_0|^n \\ &= |p(z) - p(z_0)| + \underbrace{2 \sum_{n=N_0+1}^{\infty} |a_n| r^n}_{<2\cdot\frac{\varepsilon}{4}} \end{aligned}$$

with $p(w) = \sum_{n=0}^{N_0} a_n w^n$. Since polynomials are continuous, there exists $\delta \in (0, r - |z_0|)$ with $|p(z) - p(z_0)| < \frac{\varepsilon}{2}$ if $|z - z_0| < \delta$. Therefore $|f(z) - f(z_0)| < \varepsilon$ if $|z - z_0| < \delta$. □

When applied to the exponential function, the above theorem implies that the exponential function is continuous for all $z \in \mathbb{C}$.

1.8 Corollary. (Exponential Function (Exponentialfunktion)) *The* exponential function

$$\exp : \mathbb{C} \to \mathbb{C}, \qquad z \mapsto \exp(z)$$

is continuous.

Proof. By Example II.5.5a) the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has radius of convergence $\varrho = \infty$. Theorem 1.7 implies the assertion. □

Many existence statements in analysis depend on the so-called *intermediate value theorem*. Bolzano was the first to realise the necessity of proving this apparently 'self-evident' statement. From the modern point of view, the following theorem is a variation of the completeness of $\mathbb{R}$. In the following, we again set $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ for $a, b \in \mathbb{R}$, $a < b$.

1.9 Theorem. (Intermediate Value Theorem (Zwischenwertsatz)) *Let $a, b \in \mathbb{R}$ with $a < b$. Further, let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) < 0$ and $f(b) > 0$ (or $f(a) > 0$ and $f(b) < 0$). Then there exists an $x_0 \in [a, b]$ with $f(x_0) = 0$.*

Though the above theorem is intuitively obvious, caution is required: for example, let $D = \{x \in \mathbb{Q} : 1 \leq x \leq 2\}$ and $f : D \to \mathbb{R}$ be given by $x \mapsto x^2 - 2$. Then $f(1) = -1 < 0$ and $f(2) = 2 > 0$, but there exists *no* $x_0 \in D$ with $f(x_0) = 0$.

Proof. Consider the set $M := \{x \in [a, b] : f(x) \leq 0\}$. Then $a \in M$ and hence $M \neq \emptyset$. Additionally M is bounded from above by b. The completeness axiom implies $x_0 := \sup M$ exists. Now we show that $f(x_0) = 0$.
We assume that $f(x_0) < 0$. By hypothesis, f is continuous; thus for $\varepsilon := \frac{-f(x_0)}{2} > 0$, there exists a $\delta > 0$ with $\delta < b - x_0$ and

$$f(x) - f(x_0) < \varepsilon, \text{ whenever } x \in (x_0 - \delta, x_0 + \delta) \cap [a, b].$$

Therefore, $f(x) < \frac{f(x_0)}{2} < 0$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$ and, therefore,

$$(x_0 - \delta, x_0 + \delta) \cap [a, b] \subset M.$$

Then $x_0 + \frac{\delta}{2} \in M$ in contradiction to the definition of x_0.
We now assume that $f(x_0) > 0$. Analogously to the above, the hypothesis f being continuous implies that to $\varepsilon := \frac{f(x_0)}{2} > 0$ there exists a $\delta > 0$ with $\delta < x_0 - a$ and

$$f(x_0) - f(x) < \varepsilon \qquad x \in (x_0 - \delta, x_0 + \delta) \cap [a, b].$$

Therefore $0 < \frac{f(x_0)}{2} < f(x)$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$ and, therefore,

$$(x_0 - \delta, x_0 + \delta) \cap [a, b] \cap M = \emptyset.$$

This implies that $x_0 - \delta/2$ is an upper bound of M in contradiction to the definition of x_0. Summarizing, we have that $f(x_0) = 0$.

□

1.10 Remarks.
a) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f takes every value between $f(a)$ and $f(b)$. In other words, let (without loss of generality) $f(a) < c < f(b)$. Then there exists an $x_0 \in [a, b]$ with $f(x_0) = c$. Proof as exercise.

b) Every polynomial of odd degree with real coefficients has at least one root.

c) For all $y > 0$ there exists exactly one $x \in \mathbb{R}$ with $\exp(x) = y$. We denote x by

$$x := \log y$$

and call it the *natural logarithm (natürlichen Logarithmus)* of y.

In order to see this property, observe that for $n \in \mathbb{N}$,

$$\exp(n) = 1 + n + \frac{n^2}{2!} + \cdots \geq 1 + n \to \infty.$$

The functional equation of the exponential function implies that $\exp(-n) = \frac{1}{\exp(n)} \to 0$. Thus, there exists an $N_0 \in \mathbb{N}$ with $\exp(-N_0) < y < \exp(N_0)$. Because $\exp : [-N_0, N_0] \to \mathbb{R}$ is continuous (see Corollary 1.8), by the intermediate value theorem, there exists an $x \in [-N_0, N_0]$ with $\exp(x) = y$.
In order to prove the uniqueness, we assume that there exists a z, $x < z$, with $\exp x = y = \exp z$. The functional equation of the exponential function then implies that $\exp(z) = \exp(h)\exp(x)$ for $h = z - x$. Since $h > 0$ and

$$\exp h = 1 + h + \ldots > 1$$

we get $\exp(x) < \exp(z)$. This is a contradiction.

The function $\log : (0, \infty) \to \mathbb{R}$ is called *logarithm function.*

As a further application of the intermediate value theorem, we now consider the image of an interval under a continuous function. Here the following subsets of $\mathbb{R}$ are called *intervals (Intervalle)*:

$$\begin{aligned}
(a, b) &:= \{x \in \mathbb{R} : a < x < b\} \\
[a, b) &:= \{x \in \mathbb{R} : a \leq x < b\} \\
(a, b] &:= \{x \in \mathbb{R} : a < x \leq b\} \\
[a, b] &:= \{x \in \mathbb{R} : a \leq x \leq b\} \\
(-\infty, b) &:= \{x \in \mathbb{R} : x < b\} \\
(-\infty, b] &:= \{x \in \mathbb{R} : x \leq b\} \\
(a, \infty) &:= \{x \in \mathbb{R} : x > a\} \\
[a, \infty) &:= \{x \in \mathbb{R} : x \geq a\} \\
(-\infty, \infty) &:= \mathbb{R}
\end{aligned}$$

1.11 Theorem. (Continuous Images of Intervals) *Let I be an interval and $f : I \to \mathbb{R}$ be a continuous function. Then $f(I)$ is an interval.*

Proof. First let $I = [a, b]$ be a closed and bounded interval.
We first show that $f(I)$ is bounded: Assume $f(I)$ is not bounded. Then there exists a sequence $(y_n)_n$ of elements of $f(I)$ such that $|y_n| > n$ for all $n \in \mathbb{N}$. On the other hand, there exists a sequence $(x_n)_n$ in I with $f(x_n) = y_n$ for all $n \in \mathbb{N}$. The sequence

$(x_n)_n$ is bounded since I is bounded by assumption. By Bolzano-Weierstraß, $(x_n)_n$ has therefore a convergent subsequence $(x_{n_k})_k$ with $x_{n_k} \overset{k\to\infty}{\longrightarrow} x_0 \in [a,b]$. By the continuity of f, we have $f(x_{n_k}) \overset{k\to\infty}{\longrightarrow} f(x_0)$. On the other hand, we have $|f(x_{n_k})| > n_k$, and this is a contradiction.

Next we show that $f(I)$ is closed: Since $f(I)$ is bounded, $\inf f(I)$ and $\sup f(I)$ exist (as finite numbers). We show that infimum and supremum are in fact minimum and maximum, respectively, and we carry out the proof for the maximum.
By the characterisation of the supremum we know that there exists a sequence $(x_n)_n$ in I such that $f(x_n) \overset{n\to\infty}{\longrightarrow} \sup f(I)$, and again by Bolzano-Weierstraß, this sequence has a convergent subsequence $(x_{n_k})_k$, and its limit p is contained in I, because we assumed I to be closed. By continuity, we have $f(x_{n_k}) \overset{k\to\infty}{\longrightarrow} f(p)$ and because the limit of a sequence is unique, this implies $f(p) = \sup f(I)$, whence $\sup f(I)$ is an element of $f(I)$. Analogously, we can show that there exists $q \in [a,b]$ with $f(q) = \inf f(I)$.

By the intermediate value theorem, we can infer that f takes any value between $f(q)$ and $f(p)$ on I, hence $f(I) = [f(p), f(q)]$ is an interval.

Next let I be a <u>bounded and open interval</u>, i.e. I is of the form $I = (a,b)$. Then we can express I as the following infinite union.

$$I = \bigcup_{\substack{n\in\mathbb{N} \\ 1/n<(b-a)/2}} [a+1/n, b-1/n]$$

By the argument above, each $f([a+1/n, b-1/n])$ is an interval, and hence

$$f(\bigcup_{\substack{n\in\mathbb{N} \\ 1/n<(b-a)/2}} [a+1/n, b-1/n]) \overset{\text{see ex. T 4.1}}{=} \bigcup_{\substack{n\in\mathbb{N} \\ 1/n<(b-a)/2}} f([a+1/n, b-1/n])$$

is an ascending union of intervals, which is again an interval.

If the interval is of the form $[a,b)$ or $(a,b]$ or unbounded in one or both directions, we can apply essentially the same argument and express the interval as an ascending union of closed and bounded intervals.
Below we give a list of possible representations for the remaining forms of I.

$$\begin{array}{lll} [a,b) = \bigcup [a, b-1/n] & [a,\infty) = \bigcup [a,n] & (-\infty, b] = \bigcup (-n, b] \\ (a,b] = \bigcup [a+1/n, b] & (a,\infty) = \bigcup [a+1/n, n] & (-\infty, b) = \bigcup [-n, b-1/n] \\ (-\infty,\infty) = \bigcup [-n,n] & & \end{array}$$

□

Our next task is to check the continuity of the inverse function of a given continuous function (so long as it exists). We introduce the following concepts:

1.12 Definition. A function $f : D \subset \mathbb{R} \to \mathbb{R}$ is called
(monotone) increasing, if $x, y \in D, x < y \Rightarrow f(x) \leq f(y)$
strictly (monotone) increasing, if $x, y \in D, x < y \Rightarrow f(x) < f(y)$
(monotone) decreasing, if $x, y \in D, x < y \Rightarrow f(x) \geq f(y)$
strictly (monotone) decreasing, if $x, y \in D, x < y \Rightarrow f(x) > f(y)$
monotone, if f is increasing or decreasing.
strictly monotone, if f is strictly increasing or strictly decreasing.

At this point we recall the definition of injectivity: A function $f : D \subset \mathbb{R} \to \mathbb{R}$ is called *injective (injektiv)*, if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. A strictly monotone function $f : D \subset \mathbb{R} \to \mathbb{R}$ is injective and it is possible to define the *inverse function (Umkehrfunktion)* $f^{-1} : f(D) \to D$ via the following:

$$f^{-1} : f(D) \to D, f^{-1}(y) = x :\Leftrightarrow y = f(x).$$

The graph of f^{-1} is simply the reflection of the graph of f around the line $x = y$, i.e.

$$\text{graph}(f^{-1}) = \{(y, f^{-1}(y)) : y \in f^{-1}(D)\} = \{(f(x), x) : x \in D\}.$$

We now consider the question whether or not the inverse function of a continuous function is also continuous.

1.13 Theorem. *Let I be an interval and $f : I \to \mathbb{R}$ be a continuous, strictly monotone function. Then the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is continuous.*

Proof. W.l.o.g. let f be strictly increasing. We divide the proof into three steps:

1) By Theorem 1.11, $f(I) =: J$ is an interval. We set $g := f^{-1} : J \to I$.

2) The function g is strictly increasing: If $s_1 < s_2$ in J then $g(s_1) < g(s_2)$. Otherwise $g(s_1) \geq g(s_2)$ and the monotonicity of f would give

$$s_1 = f(g(s_1)) \geq f(g(s_2)) = s_2 \qquad \text{– a contradiction.}$$

3) The inverse function g is continuous:

Consider first the case of a closed bounded interval $I := [a, b]$. Then by the proof of Theorem 1.11, $f(I) =: J$ is a closed bounded interval. Now assume that g is discontinuous at $s_0 \in J$. Then there exists an $\varepsilon_0 > 0$ and a sequence $(s_n)_n \subset J$ with

$$|s_n - s_0| < \frac{1}{n} \quad \text{and} \quad |g(s_n) - g(s_0)| \geq \varepsilon_0 \quad \text{for all } n \in \mathbb{N}.$$

Set $t_n := g(s_n) \in [a, b]$. By the theorem of Bolzano-Weierstraß the sequence $(t_n)_n$ contains a convergent subsequence $(t_{n_k})_{k\in\mathbb{N}}$ with limit $t_0 \in [a, b]$. Because f is continuous, it follows that $f(t_{n_k}) \overset{k\to\infty}{\longrightarrow} f(t_0)$.
On the other hand, $f(t_{n_k}) = s_{n_k} \overset{k\to\infty}{\longrightarrow} s_0$ and the uniqueness of the limit implies that $s_0 = f(t_0)$. Therefore,

$$g(s_{n_k}) = t_{n_k} \overset{k\to\infty}{\longrightarrow} t_0 = g(s_0)$$

in contradiction to the above property of the sequence $(g(s_n))_n$.
Next let $I = (a, b)$ be an open bounded interval and $s_0 \in J$ arbitrary. Then to $g(s_0) =: t_0 \in (a, b)$ one can find a closed bounded interval $[c, d] \subset (a, b)$ with $t_0 \in (c, d)$. Thus, by the $[a, b]$-case, the continuity of g in s_0, hence on J. As in the proof of Theorem 1.11 the remaining cases of intervals can be reduced to the above two cases. □

We conclude this chapter with some examples.

1.14 Examples. a) For $n \in \mathbb{N}$, the n-th root function

$$f : [0, \infty) \to [0, \infty), \quad x \mapsto \sqrt[n]{x}$$

is continuous and strictly increasing. To see this, consider the function

$$g : [0, \infty) \to [0, \infty), \quad t \mapsto t^n.$$

Then g is continuous and strictly monotone, because for $0 \le s < t$, we have

$$g(t) - g(s) = t^n - s^n = t^n(1 - (\frac{s}{t})^n) > 0.$$

The claim then follows from 1.13.

b) The exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is strictly increasing. We repeat the argument from Remark 1.10 c). Because we have $e^{x+h} = e^h e^x$ for all $x \in \mathbb{R}$ and $h > 0$, the strict monotonicity of the exponential function follows from the estimate

$$e^h = 1 + h + \frac{h^2}{2!} + \ldots > 1, \quad h > 0.$$

Furthermore, the exponential function $\exp : \mathbb{R} \to (0, \infty)$ is continuous by Corollary 1.8. The above theorem about inverse functions then states that the logarithm function $\log : (0, \infty) \to \mathbb{R}, x \mapsto \log x$,which was defined in 1.10 c) as the inverse function of the exponential function, is continuous as well.

c) For $x > 0$ and $\alpha \in \mathbb{R}$, we define the *general power* by

$$x^\alpha := \exp(\alpha \log x).$$

Then, the two functions

$$\begin{aligned} f_x : \mathbb{R} \to \mathbb{R}, \quad f_x(\alpha) &:= x^\alpha \qquad \text{for fixed } x > 0 \text{ and} \\ g_\alpha : (0,\infty) \to \mathbb{R}, \quad g_\alpha(x) &:= x^\alpha \qquad \text{for fixed } \alpha \in \mathbb{R} \end{aligned}$$

are continuous.
At this point, we note that the above definition extends the previous definition of powers with rational exponents from Remark II.1.14 to arbitrary exponents $\alpha \in \mathbb{R}$.
To see this, for given $x > 0$ and $\alpha = \frac{p}{q} \in \mathbb{Q}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, we deduce from the uniqueness of the root that

$$\exp(\frac{p}{q} \log x) = \Big(\exp(\frac{\log x}{q})\Big)^p = \Big(\sqrt[q]{\exp(\log x)}\Big)^p = (\sqrt[q]{x})^p.$$

d) If $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is continuous and strictly monotone, then f^{-1} is not continuous in general if D is not an interval. Consider for example the function $f : D = [0,1) \cup \{2\}$, given by

$$f(x) = \begin{cases} x, & \text{for } x \in [0,1) \\ 1, & \text{for } x = 2. \end{cases}$$

Then f is continuous and strictly monotone, but $f^{-1} : f(D) = [0,1] \to \mathbb{R}$, given by

$$f^{-1}(y) = \begin{cases} y, & \text{for } y \in [0,1) \\ 2, & \text{for } y = 1 \end{cases}$$

is not continuous at $y = 1$.

2 Basics of Topology

We begin this section with the concept of vector spaces, which play an important role in modern analysis. Throughout the section we let the scalar field be $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

We start by recalling the definition of a *vector space*, as known from linear algebra.

2.1 Definition. A *vector space (Vektorraum) over* $\mathbb{K}$, or a $\mathbb{K}$-VS is a triple $(V, +, \cdot)$ consisting of a set $V \neq \emptyset$, an addition $+ : V \times V \to V$, $(u, v) \mapsto u + v$, and a multiplication by scalars $\cdot : \mathbb{K} \times V \to V$, $(\lambda, v) \mapsto \lambda \cdot v$, are defined in accordance with the following rules:

(VR1) $(V, +)$ is an abelian group

(VR2) Distributivity:

$$\lambda(v + w) = \lambda v + \lambda w, \qquad (\lambda + \mu)v = \lambda v + \mu v, \quad \lambda, \mu \in \mathbb{K}, \;\; v, w \in V$$

(VR3) $\lambda \cdot (\mu v) = (\lambda\mu) \cdot v, \quad 1 \cdot v = v, \quad \lambda, \mu \in \mathbb{K}, \;\; v \in V$

The vector space is called *real* if $\mathbb{K} = \mathbb{R}$, and *complex* if $\mathbb{K} = \mathbb{C}$.

The elements of V are called *vectors (Vektoren)*, while the elements of $\mathbb{K}$ are called *scalars (Skalare)*. More information about the concept of vector spaces will be given in the Linear Algebra lectures.

2.2 Examples.

a) Let $n \in \mathbb{N}$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{K}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{K}^n$. Then $\mathbb{K}^n$ is a $\mathbb{K}$-VS equipped with

$$\begin{aligned} x + y &:= (x_1 + y_1, \cdots, x_n + y_n) \\ \lambda \cdot x &:= (\lambda x_1, \cdots, \lambda x_n), \quad \lambda \in \mathbb{K}. \end{aligned}$$

In particular, $\mathbb{R}^n$ and $\mathbb{C}^n$ are vector spaces.

b) Let X be a set. Then $V^X := \{f : X \longrightarrow V : f \text{ is a map}\}$ is a vector space with

$$\begin{aligned} (f + g)(x) : &= f(x) + g(x), \quad x \in X \\ (\lambda f)(x) : &= \lambda f(x), \quad x \in X, \; \lambda \in \mathbb{K} \end{aligned}$$

c) The set $c_0 := \{(x_n)_{n\geq 1} \subset \mathbb{K} : (x_n) \text{ is a null sequence}\}$ is a $\mathbb{K}$-vector space with coordinate-wise addition and scalar multiplication

$$\begin{aligned} (x_n)_n + (y_n)_n : &= (x_1 + y_1, x_2 + y_2, \cdots) = (x_n + y_n)_n\,, \\ \lambda(x_n)_n : &= (\lambda x_1, \lambda x_2, \cdots) = (\lambda x_n)_n\,. \end{aligned}$$

This follows from the calculation rules for convergent sequences.

We now want to equip the vector space $\mathbb{R}^n$ with a *Euclidean structure* and, therefore, introduce the idea of a distance between two elements $x, y \in \mathbb{R}^n$. We call

$$|x-y| := d(x,y) := \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$$

the *Euclidean distance (euklidischer Abstand)* between x and y. In particular, the Euclidean distance of x to the origin is $|x| = d(x,0) = \sqrt{\sum_{i=1}^{n} x_i^2}$. We sometimes write $\|x-y\|$ instead of $|x-y|$. We call

$$B_r(x) := \{y \in \mathbb{R}^n : d(y,x) = |y-x| < r\}, \quad x \in \mathbb{R}^n, \ \ r > 0,$$

the *open ball (offene Kugel) with center x and radius r with respect to d.*

In the following, we transfer the concept of convergence for sequences and series of real numbers, which we introduced earlier, to sequences and series in the Euclidean space $\mathbb{R}^n$. For this, it proves useful to introduce some basic topological concepts for subsets of $\mathbb{R}^n$. These concepts are mostly due to Felix Hausdorff.

2.3 Definition. a) A subset $U \subset \mathbb{R}^n$ is called a *neighborhood (Umgebung)* of $x \in \mathbb{R}^n$, if there exists an $\varepsilon > 0$ with $B_\varepsilon(x) \subset U$. The set $B_\varepsilon(x)$ is also called an ε-neighborhood of x.
b) A set $A \subset \mathbb{R}^n$ is called *open (offen)*, if for every $x \in A$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A$.

Examples. Let $a, b \in \mathbb{R}$ with $a < b$. Then:
a) The set $U = (a,b)$ is open. In fact, let $x \in (a,b)$; set $\varepsilon := \min(|a-x|, |b-x|)$, then $B_\varepsilon(x) \subset (a,b)$.

b) The intervals (a,∞) and $(-\infty,a)$ are both open.

c) The inverval $[a,b]$ is not open, since $B_\varepsilon(a) \not\subset [a,b]$ for every $\varepsilon > 0$.

d) $B_r(x)$ is an open subset of $\mathbb{R}^n$. (Exercise).

2.4 Definition. A set $A \subset \mathbb{R}^n$ is called *closed (abgeschlossen)*, if $\mathbb{R}^n \setminus A$ is open in $\mathbb{R}^n$. Here, $\mathbb{R}^n \setminus A := \{x \in \mathbb{R}^n : x \notin A\}$.

Examples. Let $a, b \in \mathbb{R}$ with $a < b$. Then:

a) (a,b) is not closed in $\mathbb{R}$,

b) $[a,b]$ is closed in $\mathbb{R}$,

c) $[0,1)$ is not open and not closed in $\mathbb{R}$,

d) Q given by $Q := \{(x_1, \cdots x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i, 1 \leq i \leq n\}$ where $a_i, b_i \in \mathbb{R}$ with $a_i \leq b_i$ is closed in $\mathbb{R}^n$.

In the following two theorems, we examine unions and intersections of open respectively closed sets.

2.5 Theorem. *The following statements hold:*

a) The empty set $\emptyset$ and also $\mathbb{R}^n$ are open in $\mathbb{R}^n$. (Thus the property open is not a negation of the property closed.)
b) Let $O_\alpha \subset \mathbb{R}^n$, $\alpha \in I$ be open sets. Then $\bigcup\limits_{\alpha\in I} O_\alpha$ is open in $\mathbb{R}^n$, i.e. a union of arbitrarily many open sets is open.
c) Let $U_1, U_2 \cdots U_N$ be open sets. Then $\bigcap\limits_{i=1}^{N} U_i$ is open in $\mathbb{R}^n$, i.e. a finite intersection of open sets is open.

Proof as exercise. The example of the open intervals $(-\frac{1}{n}, 1+\frac{1}{n}) = I_n$ with $[0,1] = \bigcap\limits_{n=1}^{\infty} I_n$ shows that in general, arbitrary intersections of open sets are not open.

2.6 Theorem. (analogous to Theorem 2.5)
a) The empty set $\emptyset$ and $\mathbb{R}^n$ are closed.
b) Intersections of arbitrarily many closed sets are closed.
c) Finite unions of closed sets are closed.

The proof follows from Theorem 2.5 and de Morgan's Rule. (Exercise).

Observe that the statement from Theorem 2.6 c) does not hold for arbitrary unions of closed sets. In fact, $B_{\frac{1}{n}}(0)^C$ is closed for all n, but $\bigcup\limits_{n=1}^{\infty} [B_{\frac{1}{n}}(0)^c] = \mathbb{R}^n \setminus \{0\}$ is not closed.

In the following, we continue to introduce basic topological concepts.

2.7 Definition. a) Let $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then x is called a *boundary point (Randpunkt)* of A, if every neighborhood U of x contains both a point from A and from $\mathbb{R}^n \backslash A$.

b) The set

$$\partial A := \{x \in \mathbb{R}^n : x \text{ is a boundary point of } A\}$$

is called the *boundary (Rand)* of A and

$$\mathring{A} := A \backslash \partial A$$

is called the *interior (Inneres)* of A. An element $a \in \mathring{A}$ is called an *interior point (innerer Punkt)* of A.

c) Additionally, $x \in \mathbb{R}^n$ is called an *accumulation point (Häufungspunkt)* of $A \subset \mathbb{R}^n$, if every neighborhood of x contains infinitely many elements of A.

d) We call

$$\overline{A} := \{x \in \mathbb{R}^n : x \in A \text{ or } x \text{ is an accumulation point of } A\}$$

the *closure (Abschluß)* of A.

e) Finally, $A \subset \mathbb{R}^n$ is called *bounded (beschränkt)*, if there exists an $x \in \mathbb{R}^n$ and an $r > 0$ with $A \subset B_r(x)$.

As an example, we consider the closed unit ball $A = \{x \in \mathbb{R}^n : |x| \leq 1\}$. Its interior is the *open* unit ball $\mathring{A} = \{x \in \mathbb{R}^n : |x| < 1\}$, and its boundary is the *unit sphere* $\partial A = \{x \in \mathbb{R}^n : |x| = 1\}$.

The following properties of open respectively closed sets often prove useful.

2.8 Remarks. (Interior, Boundary, Closure) Let $M \subset \mathbb{R}^n$. Then:
a)

$$\mathring{M} = \bigcup_{O \subseteq M,\, O \text{ open}}$$

is open. i.e. $\mathring{M}$ is the largest open set that is contained in M.
b)

$$\overline{M} = \mathring{M} \cup \partial M = \bigcap_{M \subseteq A,\, A \text{ closed}} A,$$

i.e. $\overline{M}$ is the smallest closed set in which M is contained.
c) $\partial M = \overline{M} \bigcap \overline{\mathbb{R}^n \backslash M}$ is closed.

2.9 Theorem. (Hausdorff's Separation Axiom) *Let $x, y \in \mathbb{R}^n$ with $x \neq y$. Then there exist neighborhoods U_x of x and U_y of y with $U_x \cap U_y = \emptyset$.*

The proof is not difficult: set $U_x := U_\varepsilon(x), U_y := U_\varepsilon(y)$ with $\varepsilon := \frac{|x-y|}{2}$. We assume that a $z \in \mathbb{R}^n$ exists with $z \in U_x \cap U_y$. However, we then have $2\varepsilon = |x - y| \leq \underbrace{|x - z|}_{<\varepsilon} + \underbrace{|z - y|}_{<\varepsilon} < 2\varepsilon$. Contradiction!

After the analysis of the convergence of real or complex sequences $(a_j)_j$ in Chapter 2, we now consider the convergence of sequences $(a_j)_j \subset \mathbb{R}^n$.

2.10 Definition. Let $(a_j)_{j\in\mathbb{N}} \subset \mathbb{R}^n$ be a sequence. Then $(a_j)_j$ is called *convergent to (konvergent gegen)* $a \in \mathbb{R}^n$, if for each neighborhood U of a there exists an $N_0 \in \mathbb{N}$ with $a_j \in U$ for all $j \geq N_0$. In this case, we write $\lim\limits_{j\to\infty} a_j = a$.

The following result says that a sequence in $\mathbb{R}^n$ is convergent if an only if each of its coordinate sequences converges.

2.11 Lemma. *A sequence $(a_j)_{j\in\mathbb{N}} \subset \mathbb{R}^n$ converges to $a = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$ if and only if*

$$\lim_{j\to\infty} a_{l,j} = a_l, \qquad l = 1, \cdots n,$$

i.e. if and only if the l-th coordinate of a_j converges to a_l for all $l = 1, \cdots n$.

Proof. $\Longrightarrow$: By assumption, there exists to each $\varepsilon > 0$ an $N_0 \in \mathbb{N}$ with $\|a_j - a\| = (\sum\limits_{l=1}^{n} |a_{l,j} - a_l|^2)^{\frac{1}{2}} < \varepsilon$ for all $j \geq N_0$. Therefore $|a_{j,l} - a_l| \leq \|a_j - a\| < \varepsilon$ for all $l = 1, \ldots, n,\ j \geq N_0$.
$\Longleftarrow$: For $\varepsilon > 0$ there exists an $N_l \in \mathbb{N}$ with $|a_{l,j} - a_l| < \frac{\varepsilon}{\sqrt{n}}$ for all $j \geq N_l$. Thus, for $j \geq N_0 := \max(N_1, \cdots, N_n)$

$$\|a_j - a\| = (\sum_{l=1}^{n} |a_{l,j} - a_l|^2)^{\frac{1}{2}} < (\frac{\varepsilon^2}{n} n)^{\frac{1}{2}} = \varepsilon.$$

$\square$

2.12 Definition. A sequence $(a_j)_j \subset \mathbb{R}^n$ is called a *Cauchy sequence (Cauchyfolge)*, if for all $\varepsilon > 0$ there exists an $N_0 \in \mathbb{N}$ with

$$\|a_n - a_m\| < \varepsilon, \quad n, m \geq N.$$

The following theorem about the convergence of Cauchy sequences in $\mathbb{R}^n$ again relies ultimately on the the completeness of the real numbers.

2.13 Theorem. *In $\mathbb{R}^n$ every Cauchy sequence is convergent.*

Proof. Let $(a_j)_j \subset \mathbb{R}^n$ with $a_j = (a_{1,j}, a_{2,j}, \cdots a_{n,j})$, $j \in \mathbb{N}$, be a Cauchy sequence in $\mathbb{R}^n$. Since

$$|a_{\nu,\ell} - a_{\nu,m}| \leq (\sum_{\nu=1}^{n} |a_{\nu,\ell} - a_{\nu,m}|^2)^{\frac{1}{2}}, \qquad \nu = 1, \cdots n\,,$$

every coordinate $(a_{\nu,j})_{j\geq 1}$ of $(a_j)_j$ is a Cauchy sequence in $\mathbb{R}$. Because $\mathbb{R}$ is complete, $(a_{\nu,j})_{j\geq 1}$ converges for each $\nu = 1, \cdots, n$. Lemma 2.11 now implies the assertion.

□

The following theorem describes closedness of a set in terms of convergent sequences.

2.14 Theorem. (Characterization of closed sets via sequences) *Let $A \subset \mathbb{R}^n$. Then A is closed if and only if for every sequence $(a_j)_j \subset A$ with $\lim_{j\to\infty} a_j = a \in \mathbb{R}^n$ it holds that $a \in A$.*

Proof. $\Longrightarrow$: Let $a_j \in A$ for all $j \in \mathbb{N}$ with $\lim_{j\to\infty} a_j = a \in \mathbb{R}^n$. We assume that $a \notin A$, i.e. that $a \in \mathbb{R}^n \backslash A$. Because $A^c := \mathbb{R}^n \backslash A$ is open, A^c is a neighborhood of a. By the definition of convergence (see 2.10) there exists an $N_0 \in \mathbb{N}$ with $a_j \in A^c$ for all $j \geq N_0$. Contradiction!
$\Longleftarrow$: We again assume that the assertion is false, i.e. that A^c is not open. Then there exists an $a \in \mathbb{R}^n \backslash A$ such that for all $\varepsilon > 0$ the neighborhood $U_\varepsilon(a)$ is not contained in $\mathbb{R}^n \backslash A$, i.e. $U_\varepsilon(a) \cap A \neq \emptyset$. For $j \in \mathbb{N}$ now choose $a_j \in U_{\frac{1}{j}}(a) \cap A$. Then $\lim_{j\to\infty} a_j = a \notin A$. Contradiction!

□

For a set $M \subset \mathbb{R}^n$ we define its diameter diam M as

$$\text{diam } M := \sup\{\|x - y\| : x, y \in M\}.$$

Then we have the following theorem.

2.15 Theorem. *Let $(A_j)_{j\geq 0}$ be a sequence of non-empty, closed subsets of $\mathbb{R}^n$ with*

$$A_0 \supset A_1 \supset A_2 \supset \cdots$$

and $\lim_{j\to\infty} \text{diam } (A_j) = 0$. Then there exists exactly one $x \in \mathbb{R}^n$ with $x \in \bigcap_{j=0}^{\infty} A_j$.

Proof. We begin with the existence of an element x with the desired properties. Here, choose for each $j \in \mathbb{N}$ an $x_j \in A_j$. Then for given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ with

$$||x_j - x_k|| \leq \text{ diam } (A_N) < \varepsilon, \qquad j, k \geq N.$$

Therefore, $(x_j)_j$ is a Cauchy sequence in $\mathbb{R}^n$ and Theorem 2.13 implies that $(x_j)_j$ converges to some $x \in \mathbb{R}^n$. Because $x_j \in A_k$ for $j \geq k$ and A_k is closed, it follows from Theorem 2.14 that $x \in A_k$ for all $k \in \mathbb{N}$. The uniqueness is clear.

□

Finally, we extend the definition of continuity of real functions of one varible to those of n real variables.

2.16 Definition. Let $M \subset \mathbb{R}^n$ and $f : M \to \mathbb{R}$ be a function. Then f is called *continuous (stetig) at* $x_0 \in M$, if for every sequence $(x_j)_j \subset M$ with $\lim\limits_{j\to\infty} x_j = x_0$, there holds that $\lim\limits_{j\to\infty} f(x_j) = f(x_0)$. If f is continuous for all $x_0 \in M$, then f is called *continuous.*

Analogously to Theorem 1.2 one shows: $f : M \to \mathbb{R}$ is continuous at $x_0 \in M$ if to each $\varepsilon > 0$ there exists a $\delta > 0$ (which depends upon ε and x_0) such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{for all } x \in M,\ \|x - x_0\| < \delta.$$

If $f : \mathbb{R}^n \to \mathbb{R}$ we can thus reformulate its continuity property at $x_0 \in \mathbb{R}^n$ in the form: $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}^n$ if and only if for every neighborhood V of $f(x_0)$ in $\mathbb{R}$ (in particular for $V_\varepsilon(f(x_0) \subset \mathbb{R})$ there exists a neighborhood U of $x_0 \in \mathbb{R}^n$ (in particular $U_\delta(x_0) \subset \mathbb{R}^n$) with $f(U) \subset V$.

For the following theorem, which is a fundamental characterization of continuous functions, we need the notion of a *pre-image.* Let $f : \mathbb{R}^n \to \mathbb{R}$ and $B \subset \mathbb{R}$. Then

$$f^{-1}(B) := \{x \in \mathbb{R}^n : f(x) \in B\}$$

is called *pre-image* of B w.r.t. f or *inverse image of* B under the function f.

2.17 Theorem. *For a function $f : \mathbb{R}^n \to \mathbb{R}$ the following are equivalent:*

i) f is continuous.

ii) $f^{-1}(O)$ is open in $\mathbb{R}^n$ for every open set O in $\mathbb{R}$, i.e. pre-images of open sets are open.

iii) $f^{-1}(A)$ is closed in $\mathbb{R}^n$ for every closed set A in $\mathbb{R}$, i.e. pre-images of closed sets are closed.

Proof. *i)* $\Rightarrow$ ii): Let $O \subset \mathbb{R}$ be open. If $f^{-1}(O) = \emptyset$, then the assertion follows from Theorem 2.5 a). Let $f^{-1}(O) \neq \emptyset$. Because f is continuous, there exists to each $x \in f^{-1}(O)$ an open neighborhood $U_x \subset \mathbb{R}^n$ of x with $f(U_x) \subset O$, i.e. $x \in U_x \subset f^{-1}(O)$ for all $x \in f^{-1}(O)$. Therefore

$$\bigcup_{x \in f^{-1}(O)} U_x = f^{-1}(O),$$

and then, by Theorem 2.5, $f^{-1}(O)$ is open as a union of open sets.
ii) $\Leftrightarrow$ iii): $A \subset \mathbb{R}$ is closed if and only if A^c is open in $\mathbb{R}$. Since $f^{-1}(A^c) = (f^{-1}(A))^c$ we have $f^{-1}(A)$ is closed if and only if $(f^{-1}(A))^c$ is open in $\mathbb{R}^n$.
ii) $\Rightarrow$ i): Let $x \in \mathbb{R}^n$ and V be an open neighborhood of $f(x)$ in $\mathbb{R}$. By the definition

of an open neighborhood, there exists an $\varepsilon > 0$ such that $V_\varepsilon(f(x)) \subset V$. Then by assumption $U := f^{-1}(V_\varepsilon(f(x)))$ is open in $\mathbb{R}^n$. Since $x \in U$ there exists some $\delta > 0$ such that $U_\delta(x) \subset U$, i.e., $f(U_\delta(x)) \subset V_\varepsilon(f(x))$ and f is continuous at $x \in \mathbb{R}^n$.

□

Before we now consider examples with the aim to exemplify the statement of the above theorem, we remark that Theorem 2.17 implies that a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous iff inverse images of open sets are open, or alternatively iff inverse images of closed sets are closed.

2.18 Examples.
a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $y \in \mathbb{R}$. Then $f^{-1}(y)$ is closed in $\mathbb{R}^n$. This is obvious, since $\{y\}$ is closed in $\mathbb{R}$.

b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then

$$\{x \in \mathbb{R}^n : f(x) \leq r\} \text{ is closed and } \{x \in \mathbb{R}^n : f(x) < r\} \text{ is open.}$$

This is clear, since $\{x \in \mathbb{R}^n : f(x) \leq r\} = f^{-1}((-\infty, r])$ and $(-\infty, r]$ is closed, resp. $\{x \in \mathbb{R}^n : f(x) < r\} = f^{-1}((-\infty, r))$ and $(-\infty, r)$ is open.

c) The closed n-dimensional unit cube

$$Q := \{x \in \mathbb{R}^n : 0 \leq x_j \leq 1, 1 \leq j \leq n\}$$

is closed in $\mathbb{R}^n$. In fact, the projection $p_j : \mathbb{R}^n \to \mathbb{R}, (x_1 \cdots x_n) \mapsto x_j$ on the j-th coordinate is continuous. Because

$$Q = \bigcap_{j=1}^{n} (\underbrace{\{x \in \mathbb{R}^n : p_j(x) \leq 1\}}_{\text{closed by (iii)}} \cap \underbrace{\{x \in \mathbb{R}^n : p_j(x) \geq 0\}}_{\text{closed by (iii)}})$$

and finite intersections of closed sets are closed (Satz 2.5), the assertion follows from Theorem 2.17.

d) Continuous images of open sets are, in general, *not* open:
Consider the interval $O = (-1, 1)$ and the continuous function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$. Then $f(O) = [0, 1)$ which is not open in $\mathbb{R}$.

Continuous images of closed sets are, in general, *not* closed:
Consider the set $A := \{(x, y) \in \mathbb{R}^2 : xy = 1\} \subset \mathbb{R}^2$ and the continuous function $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto xy$. Then $A = f^{-1}(\{1\})$ and by statment (iii) A is closed in $\mathbb{R}^2$. Now $p_1 : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$ is continuous, but $p_1(A) = \mathbb{R} \setminus \{0\}$ is not closed.

3 Compactness

The notion of compactness is of central importance in analysis. In particular, important existence statements of analysis depend on properties of continuous functions on compact sets. Exemplary, we mention the fact that a real-valued function on a compact set attains a minimum and a maximum value.

We define the concept of compactness of a subset of $\mathbb{R}^n$ by means of open covers and we show that this definition is equivalent to the later introduced 'compactness by sequences'. Furthermore, the Theorem of Heine-Borel states that a subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

The reason to introduce compactness via open covers is that this concept can be straightforwardly generalized to normed and metric spaces to be defined in Analysis II, while the characterisation of Heine-Borel only works in finite dimensions.

In this section, K is always a compact subset of $\mathbb{R}^n$. We start with the definitions of 'open cover' and compactness.

3.1 Definition.

a) Let I be an index set. Then $(O_i)_{i\in I}$ is called an *open cover (offene Überdeckung) of* K, if O_i are open sets for all $i \in I$ and

$$K \subset \bigcup_{i\in I} O_i\,.$$

b) The set $K \subset \mathbb{R}^n$ is called *compact (kompakt)*, if *every* open cover $(O_i)_{i\in I}$ of K contains a finite subcover, i.e. if there exist $i_1, \cdots, i_N$ with

$$K \subset \bigcup_{l=1}^{N} O_{i_l}\,.$$

3.2 Examples.

a) The set of real numbers $\mathbb{R}$ is not compact, because $\mathbb{R} \subseteq \bigcup\limits_{n\in\mathbb{N}} (-n, n)$.

b) The interval $(0, 1]$ is not compact in $\mathbb{R}$, because $(0, 1] \subseteq \bigcup\limits_{j\geq 1} (\frac{1}{j}, 2)$.

c) Let (a_j) be a convergent sequence in $\mathbb{R}^n$ with $\lim\limits_{j\to\infty} a_j = a$. Then

$$K := \{a_j, j \in \mathbb{N}\} \cup \{a\}$$

is compact. To see this, let $(O_i)_{i\in I}$ be an open cover of K. Then there exists $j \in I$ with $a \in O_j$. Since O_j is a neighborhood of a, there exists $N_0 \in \mathbb{N}$ with $x_n \in O_j$ for all $n \geq N_0$. Choose now $i_0, \cdots, i_{N_0}$ such that $x_n \in O_{i_n}, n = 1 \cdots N_0$. Then

$$K \subset (\bigcup_{n=0}^{N_0} O_{i_n}) \cup O_j.$$

d) The statement of c) does not hold in general, if we remove a from $\mathbb{K}$. To realize this, consider the sequence $(1/n)_n$ and let

$$\begin{aligned} M &= \{\frac{1}{j} : j \in \mathbb{N}\} \subset \mathbb{R}, \\ O_1 &= (\frac{1}{2}, 2), \quad \text{and} \\ O_j &= (\frac{1}{j+1}, \frac{1}{j-1}) \text{ for } j \geq 2. \end{aligned}$$

Then we have

$$M \subset \bigcup_{j\geq 1} O_j$$

and each O_j contains exactly one element of M. Therefore, the open cover $(O_j)_{j\in\mathbb{N}}$ does not contain a finite subcover.

3.3 Theorem. *Let $K \subset \mathbb{R}^n$ be a compact set. Then K is closed and bounded.*

Proof. First we show that K is bounded: Let $x \in \mathbb{R}^n$ be arbitrary, then fixed. Then $\mathbb{R}^n = \bigcup_{k=1}^{\infty} B_k(x)$ and since K is assumed to be compact, there exists $N \in \mathbb{N}$ with

$$K \subset \bigcup_{j=1}^{N} B_{k_j}(x).$$

For $R := \max\{k_1, \cdots, k_N\}$, we have $K \subset B_R(x)$, therefore, K is bounded.

Next we show that K is closed or, equivalently, that $\mathbb{R}^n \backslash K$ is open: To this end consider $x \in \mathbb{R}^n \backslash K$ and set $U_n := \{y \in \mathbb{R}^n : ||y - x|| > \frac{1}{n}\}$. Then U_n is open and

$$K \subset \mathbb{R}^n \backslash \{x\} = \bigcup_{n=1}^{\infty} U_n.$$

Since K is compact, there exists $N \in \mathbb{N}$ with $K \subset \bigcup_{j=1}^{N} U_{n_j}$. For $R := \max\{n_1, \cdots, n_N\}$, we have $B_{\frac{1}{R}}(x) \subset \mathbb{R}^n \backslash K$, i.e. $\mathbb{R}^n \backslash K$ is open, and therefore K is closed.

□

3.4 Lemma. *Let $A \subset K \subset \mathbb{R}^n$, where A is closed and K is compact. Then A is compact.*

Proof. Let $(O_i)_{i\in I}$ be an open cover of A. By assumption $\mathbb{R}^n\backslash A$ is open and

$$K \subset \mathbb{R}^n = \bigcup_{i\in I} O_i \cup \mathbb{R}^n\backslash A.$$

Since K is compact, there exists a finite subcover of K, i.e., there exist $i_1, \cdots, i_N \in I$ with

$$K \subset (O_{i_1} \cup \cdots O_{i_N}) \cup \mathbb{R}^n\backslash A.$$

Therefore $A \subset O_{i_1} \cup \cdots O_{i_N}$.

□

3.5 Theorem. (Heine-Borel)
A set $K \subset \mathbb{R}^n$ is compact if and only if K is closed and bounded.

Proof. $\underline{\Longrightarrow}$: This is Theorem 3.3.

$\underline{\Longleftarrow}$: Conversely, let K be closed and bounded. Then K is contained in a cuboid of the form:

$$Q = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : a_l \leq x_l \leq b_l\,,\ l = 1, \ldots, n\}$$

with $a_l, b_l \in \mathbb{R}, a_l \leq b_l$. If we can show that Q is compact, the assertion follows from Lemma 3.4. This, however, is exactly the statement of the following lemma.

3.6 Lemma. *Let $Q \subset \mathbb{R}^n$ be as above. Then Q is compact.*

Proof. Let $(O_i)_{i\in I}$ be an open cover of Q. We assume that there does not exist an open subcover of Q. Now we construct a sequence of closed sub-cuboids

$$Q_0 \supset Q_1 \supset Q_2 \supset \ldots$$

with the properties

i) each Q_m has no finite subcover

ii) diam $(Q_m) = 2^{-m}$ diam (Q)

by the following procedure: Set $Q_0 = Q$ and assume Q_m is already constructed. Then $Q_m = I_1 \times I_2 \times \cdots \times I_n$ where $I_l \subset \mathbb{R}$ are closed intervals. Now halve I_l, $I_l = I_l^1 \cup I_l^2$ and set

$$Q_m^{s_1,\ldots,s_n} := I_1^{s_1} \times I_2^{s_2} \times \cdots \times I_n^{s_n}, \quad s_i = 1, 2.$$

Then

$$Q_m = \bigcup_{s_1,\ldots,s_n} Q_m^{s_1,\ldots,s_n}.$$

Since Q_m does not have a finite subcover and is represented by a finite union of sub-cuboids, there must exist at least one sub-cuboid $Q_m^{s_1\cdots s_n}$ which has not a finite subcover. We denote this by Q_{m+1}. Then

$$\operatorname{diam}(Q_{m+1}) = \frac{1}{2}\operatorname{diam}(Q_m) = 2^{-m-1}\operatorname{diam}(Q)$$

and therefore Q_{m+1} has properties i) and ii). By Theorem 2.15 there exists exactly one a with $a \in \bigcap\limits_{m\geq 1} Q_m$. Additionally, since $(O_i)_{i\in I}$ is a cover of Q, a is an element of O_{i_0} for some i_0. Since O_{i_0} is open there exists some $\varepsilon > 0$ such that $B_\varepsilon(a) \subset O_{i_0}$. Choose now m so big that $\operatorname{diam} Q_m < \frac{\varepsilon}{2}$. Since $a \in Q_m$, we have

$$Q_m \subset B_\varepsilon(a) \subset O_{i_0}$$

in contradiction to property i).

□

The notion of compactness, in particular the Theorem of Heine-Borel, has many important consequences in analysis. First of all, we consider basic properties of continuous images of compact sets.

3.7 Theorem. *Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous map. If $K \subset D$ is compact, then $f(K) \subset \mathbb{R}$ is also compact. In other words: Continuous images of compact sets are compact.*

Proof. Let $(O_i)_{i\in I}$ be an open cover of $f(K)$. For any point $x \in K$ we have $f(x) \in O_{i_0}$ for some $i_0 \in I$. Since O_{i_0} is open, there exists an open interval $B_\varepsilon^{\mathbb{R}}(f(x)) \subset O_{i_0}$, where $B_\varepsilon^{\mathbb{R}}(f(x)) := \{s \in \mathbb{R} : |s - f(x)| < \varepsilon\}$, for some $\varepsilon = \varepsilon(f(x)) > 0$. By the continuity of f there exists some $\delta = \delta(\varepsilon, x) > 0$ such that $f(B_\delta^{\mathbb{R}^n}(x) \cap D) \subset B_\varepsilon^{\mathbb{R}}(f(x))$; here $B_\delta^{\mathbb{R}^n}(x) := \{y \in \mathbb{R}^n : \|y - x\| < \delta\}$. Clearly, $K \subset \bigcup_{x\in K} B_\delta^{\mathbb{R}^n}(x)$. Since K is compact, there are finitely many x_j such that $K \subset \bigcup_{j=1}^N B_{\delta_j}^{\mathbb{R}^n}(x_j)$ and $f(B_{\delta_j}^{\mathbb{R}^n}(x_j) \cap D) \subset B_{\varepsilon_j}^{\mathbb{R}}(f(x_j)) \subset O_{i_j}$. Hence $f(K) \subset \bigcup_{j=1}^N O_{i_j}$.

□

The following corollary is a direct consequence of Theorem 3.7 and Theorem 3.3.

3.8 Corollary. *Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be continuous and $K \subset D$ a compact set. Then $f(K)$ is bounded, i.e. there exists $M > 0$ with $|f(x)| \leq M$ for all $x \in K$.*

In fact, $f(K)$ is compact by the above Theorem 3.7 and Theorem 3.3 implies that $f(K)$ is bounded.

3.9 Theorem. (Minimum and Maximum). *Let $f : K \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function and K compact. Then the function f has a maximum and a minimum, i.e. there exist $x_0, x_1 \in K$ with*

$$f(x_0) = \min_{x\in K} f(x) \quad \textit{and} \quad f(x_1) = \max_{x\in K} f(x).$$

The proof is as follows: By Theorem 3.7 $f(K)$ is compact and, therefore, by Theorem 3.3 closed and bounded. Thus

$$m := \inf f(K) > -\infty \quad \text{and} \quad M := \sup f(K) < \infty.$$

Then there exist sequences $(y_j)_j, (z_j)_j \subset f(K)$ with $y_j \to m$ and $z_j \to M$. Since $f(K)$ is closed, it follows from Theorem 2.14 that m and M are in $f(K)$. Therefore, there exist $x_0, x_1 \in K$ with $f(x_0) = m$ and $f(x_1) = M$.

□

The above theorem implies that a closed set and a compact set whose intersection is empty always have a strictly positive distance.
Here the distance between two sets is defined as follows:

Let $M_1, M_2 \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then

$$d(x, M_1) := \inf\{||x - y|| : y \in M_1\}$$

is called the *distance (Abstand) of* x *from* M_1 and

$$d(M_1, M_2) := \inf\{||x - y|| : x \in M_1, y \in M_2\}$$

is the *distance between the two sets* M_1 and M_2.

3.10 Corollary.
Let $A \subset \mathbb{R}^n$ *be closed and* $K \subset \mathbb{R}^n$ *a compact set with* $A \cap K = \emptyset$. *Then* $d(A, K) > 0$.

Proof. The function $d(\cdot, A) : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto d(x, A)$ is continuous (Exercise) and K is compact by assumption. By Theorem 3.9 there exists an $x_0 \in K$ with $d(x_0, A) = d(K, A)$. If we had $d(x_0, A) = 0$, there would exist a sequence $(a_j)_j \subset A$ with $a_j \to x_0$. A being closed implies that $x_0 \in A$, i.e., $x_0 \in A \cap K$ in contradiction to $A \cap K = \emptyset$.

□

3.11 Theorem. (Sequential compactness) *For a set* $K \subset \mathbb{R}^n$ *the following statements are equivalent:*

i) K is compact. (cover compactness)

ii) Every sequence in K has a subsequence that converges to an element $a \in K$. (sequential compactness)

Proof. **(i)** $\Longrightarrow$ **(ii)** : We assume that the assertion is false. Then there exists a sequence $(a_n)_{n\in\mathbb{N}} \in K$ that does not have any convergent subsequence with limit in K. Therefore, for every $x \in K$ there exists a neighborhood U_x of x that contains only finitely many terms of the sequence. Since clearly $K \subset \bigcup_{x \in K} U_x$ and K is compact, there exists a finite subcover of K. Then K contains only finitely many terms of the sequence. Contradiction!

(ii) $\Longrightarrow$ **(i)** : By assumption K is bounded, because otherwise, there would exist a sequence $(a_j)_j \subset K$ with $|a_j| \geq j$ for all $j \in \mathbb{N}$, which would then, however, contain no convergent subsequences.
By the Theorem of Heine-Borel we now only have to show that K is closed. Here, let $(a_j)_j \subset K$ be a sequence with $\lim_{j \to \infty} a_j = a \in \mathbb{R}^n$. By assumption, there exists a subsequence $(a_{j_l})_{l \in \mathbb{N}}$ with $\lim_{l \to \infty} a_{j_l} = a' \in K$. From the uniqueness of the limit, it follows that $a = a'$ and therefore $a \in K$. Theorem 2.14 implies that K is closed. By the above it is also bounded, therefore, by the Theorem of Heine-Borel K is compact. $\square$

We now consider the concept of *uniform continuity* of a function defined on a set $M \subset \mathbb{R}^n$. The continuity of the function $f : M \subset \mathbb{R}^n \to \mathbb{R}$ at a point $x_0 \in M$ means the following:

$$(\forall \, \varepsilon > 0) \; (\exists \, \delta = \delta(\varepsilon, x_0) > 0) \; (\forall \, x \in M, ||x - x_0|| < \delta) \quad |f(x) - f(x_0)| < \varepsilon.$$

Here, δ depends on ε and ! x_0 . If we can choose δ independent of x_0, then f is called uniformly continuous on M.

3.12 Definition. Let $f : M \subset \mathbb{R}^n \to \mathbb{R}$ be a function. Then f is called *uniformly continuous (gleichmässig stetig)*, if to each $\varepsilon > 0$ there exists a (universal) $\delta(\varepsilon) > 0$ with

$$x, y \in M, \;\; ||x - y|| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon.$$

or in short notation

$$(\forall \, \varepsilon > 0) \; (\exists \, \delta > 0) \; (\forall \, x, y \in M, ||x - y|| < \delta) \quad |f(x) - f(y)| < \varepsilon.$$

We easily verify that $f : (0,1) \to \mathbb{R}$, $x \mapsto 1/x$, is continuous, but not uniformly continuous. However, $f : [0, \infty) \to [0, \infty)$, $x \mapsto \sqrt{x}$, is uniformly continuous.

The following theorem says that a continuous function on a compact set is uniformly continuous.

3.13 Theorem.
Let $f : K \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function and K a compact set. Then f is uniformly continuous, i.e., continuous functions on compact sets are uniformly continuous.

Proof. Let $\varepsilon > 0$. The continuity of f says that for all $y \in K$, there exists a radius $r(y) > 0$ with
$$|f(y) - f(z)| < \frac{\varepsilon}{2} \quad \text{if} \quad z \in B_{r_y}(y) \cap K.$$
Since $K \subseteq \bigcup_{y \in K} B_{\frac{r_y}{2}}(y)$ and K is compact, there exist finitely many $y_1, \cdots, y_n$ with

$$K \subseteq \bigcup_{j=1}^{N} B_{\frac{r_{y_i}}{2}}(y_j).$$

Let $\delta := \frac{1}{2}\min(r(y_1), \cdots, r(y_N))$ and $x, x' \in K$ with $||x - x'|| \leq \delta$. Then there exists a $j \in \{1 \cdots N\}$ with $x \in B_{\frac{r_{y_j}}{2}}(y_j)$ and $x' \in B_{r_{y_j}}(y_j)$ and

$$|f(x) - f(x')| \leq \underbrace{|f(x) - f(y_j)|}_{<\frac{\varepsilon}{2}} + \underbrace{|f(y_j) - f(x')|}_{<\frac{\varepsilon}{2}} < \varepsilon.$$

□

The extension of a given continuous function $f : M \subset \mathbb{R}^n \to \mathbb{C}$ to a continuous function on $\overline{M}$ is closely related to the concept of uniform continuity. More precisely, let $x_0 \in \mathbb{R}^n \backslash M$ be an accumulation point of M. We want to examine the question under which circumstances there exists a continuous extension of f to $M \cup \{x_0\}$.

At first, we introduce the concept of limit of a function (as opposed to sequence).

3.14 Definition. A function $f : M \subset \mathbb{R}^n \to \mathbb{C}$ has a limit a in the accumulation point x_0 of M, if for *each* sequence $(x_j)_j \subset M \backslash \{x_0\}$ with $x_j \to x_0$, we have

$$\lim_{j \to \infty} f(x_j) = a.$$

In this case, we also say that $f(x)$ converges to a for $x_j \to x_0$, and we write

$$\lim_{x \to x_0} f(x) = a \quad \text{or} \quad f(x) \to a \quad \text{for} \quad x \to x_0.$$

If $x_0 \in M$ and f is continuous at x_0, then the value of the function at x_0 is equal to the limit, i.e. we have $\lim_{x \to x_0} f(x) = f(x_0)$. Furthermore, we call the function

$$F : M \cup \{x_0\} \to \mathbb{R},$$
$$x \mapsto \begin{cases} f(x) & x \in M \\ y_0 & x = x_0 \end{cases}$$

a *continuous extension* if $\lim_{x \to x_0} f(x) = y_0$ exists.

For the special case $M \subset \mathbb{R}$, we furthermore define the *limit from the left* (linksseitiger Grenzwert) of f in x_0 to be y_0, in symbols

$$\lim_{x \to x_0 - 0} f(x) := y_0 \qquad \text{(or in short} \quad \lim_{x \to x_0 -} f(x) := y_0),$$

if for all sequences $(x_j)_j \subset M \cap (-\infty, x_0)$ with $x_j \to x_0$, we have $\lim_{j\to\infty} f(x_j) = y_0$.

Analogously, we call

$$\lim_{x\to x_0+0} f(x) := \lim_{x\to x_0+} f(x) := y_0,$$

the limit from the right (rechtsseitiger Grenzwert) of f in x_0, if for all sequences $(x_j)_j \subset M \cap (x_0, \infty)$ with $x_j \to x_0$, we have $\lim_{j\to\infty} f(x_j) = y_0$.

If $M \subset \mathbb{R}$ is not bounded from above and we are given a function $f : M \to \mathbb{C}$, we call $a \in \mathbb{C}$ *the limit of f in ∞*, if for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$|f(x) - a| < \varepsilon \quad \text{for all } x \in M \text{ with } x > N_0.$$

Analogously, one defines the limit in $-\infty$.

3.15 Examples.
a) Let $M = \mathbb{R}\backslash\{1\}$ and $f : M \to \mathbb{R}$ be defined by $f(x) = \frac{x^n-1}{x-1}$. Then

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^n - 1}{x - 1} = n,$$

because $\frac{x^n-1}{x-1} = 1 + x + x^2 + \cdots x^{n-1}$.

b)

$$\lim_{z\to 0} \frac{e^z - 1}{z} = 1,$$

because

$$\frac{e^z - 1}{z} = \frac{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \ldots$$

Therefore $|\frac{e^z-1}{z} - 1| \le |\frac{z}{2}|(1 + |z| + |z^2| + \ldots) = \frac{|z|}{2(1-|z|)} \to 0$ for $z \to 0$ if $|z| < 1$. (geometric sum)

c) The limit

$$\lim_{x\to 0} \frac{x}{|x|}$$

does not exist: Define the function

$$f : \mathbb{R}\backslash\{0\} \to \mathbb{R}, \quad x \mapsto \begin{cases} 1 & : \ x > 0 \\ -1 & : \ x < 0 \end{cases}$$

Then the limit from the left $\lim_{x\to 0-} = -1$ does not coincide with the limit from the right $\lim_{x\to 0+} = 1$.

The following theorem characterizes when a function can be continuously extended in terms of uniform continuity.

3.16 Theorem. *Let $M \subset \mathbb{R}$ be a bounded set and $f : M \to \mathbb{R}$ be a function. Then the following are equivalent:*

i) There exists a unique continuous extension $F : \overline{M} \subset \mathbb{R} \to \mathbb{R}$ of f on $\overline{M}$, i.e. $F(x) = f(x)$ for $x \in M$.

ii) f is uniformly continuous.

Proof. (**i**) $\Rightarrow$ (**ii**) : Because $\overline{M}$ is bounded and closed, it is compact by the Heine-Borel Theorem, and the claim follows from Theorem 3.13.
(**ii**) $\Rightarrow$ (**i**) : See the Exercises.

$\square$

To conclude this section, we consider a function g (see below) constructed via the so-called 'saw tooth function' f, which is defined by

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \left| x - [x] - \frac{1}{2} \right| .$$

Clearly f is continuous on $\mathbb{R}$. Now define

$$g : (0, 1] \to \mathbb{R}, \quad g(x) = f(\frac{1}{x}),$$

which as a composition of two continuous functions is continuous on $(0, 1]$. But g is not uniformly continuous since

$$g(\frac{1}{n}) - g(\frac{1}{n + 1/2}) = f(n) - f(n + 1/2) = 1/2, \quad n \in \mathbb{N}.$$

Therefore, by the above theorem, we cannot extend g continuously to the closed interval $[0, 1]$; in particular the limit $\lim_{x \to 0+} g(x)$ does not exist.

4 The exponential function and related functions

Central for this section is the *exponential function*, one of the most important functions of all mathematics. With its help, we will firstly introduce the trigonometric functions *sine* and *cosine*, and secondly we will examine the already known power and logarithm functions more closely.

Many of the following definitions and arguments can be traced back directly to LEONHARD EULER (1707-1783), one of the all-time greatest mathematicians. Born in Basel in 1707, he enrolled at the university of Basel at the age of 13 where he was a student of Johann Bernoulli. In 1727, he went to the Academy of St. Petersburg where he was appointed a professorship in 1733. During this time, the academies were the center of scientific research, and Euler spent his whole life at the academies of St. Petersburg and Berlin (1741-1766).

Euler was most influential in mathematics through his textbooks. His " *Introductio in analysin infinitorum*" of 1748 paved the way for analysis as a branch of mathematics on a par with geometry and algebra. Our contemporary mathematical notation is due to Euler in great parts.

Before we — following Euler — define sine and cosine as power series, we recall the *exponential series*, already known from Chapter II,

$$e^z = \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \dots, \quad z \in \mathbb{C},$$

with infinite radius of convergence. The power series of sine and cosine, which we will examine thoroughly in the following, have a close relationship to the exponential series. Here it is essential to work with complex numbers. Only then the connection between the exponential and the trigonometric functions becomes fully apparent. Retrospectively, from considering the trigonometric functions, we will also gain new insights concerning the exponential function; e.g. that the exponential function is periodic with a complex period.

4.1 Definition. The *sine series* and *cosine series* are defined as

$$\begin{aligned} \sin z &:= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots, \quad z \in \mathbb{C}, \\ \cos z &:= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots, \quad z \in \mathbb{C}. \end{aligned}$$

These series have the following elementary properties:

4.2 Theorem. *a) The sine and cosine series have an infinite radius of convergence.*

b) Euler's formula

$$e^{iz} = \cos z + i \sin z, \quad z \in \mathbb{C}.$$

holds.

c) The functions $z \mapsto \sin z$ *and* $z \mapsto \cos z$ *are continuous on* $\mathbb{C}$.

The claim about the radius of convergence follows from the Cauchy-Hadamard formula II.5.2. Euler's formula is a direct consequence of the presentation

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{(iz)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(iz)^{2n+1}}{(2n+1)!} = \cos z + i \sin z, \quad z \in \mathbb{C}.$$

The continuity of $z \mapsto \sin z$ and $z \mapsto \cos z$ follows from Theorem 1.7.

Further properties of sine and cosine can be deduced directly from the definition likewise.

4.3 Corollary. *a) The cosine function* $\cos : \mathbb{C} \to \mathbb{C}, z \mapsto \cos z$ *is an* even *function, and the sine function* $\sin : \mathbb{C} \to \mathbb{C}, z \mapsto \sin z$ *is an* odd *function, , i.e. we have*

$$\cos z = \cos(-z) \quad \textit{and} \quad \sin z = -\sin(-z), \quad z \in \mathbb{C}.$$

b) For all $z \in \mathbb{C}$ *we have*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \textit{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}.$$

c) For $x \in \mathbb{R}$ *we have* $\cos x = Re\,(e^{ix})$ *and* $\sin x = Im\,(e^{ix})$.

d) For all $x \in \mathbb{R}$ *we have* $|e^{ix}| = 1$.

The functional equation of the exponential function implies the addition theorems for the sine and cosine functions which express how we can rewrite these functions applied to sums of angles.

4.4 Theorem. (Angle sum and difference identities). *The following equations hold for all* $z, w \in \mathbb{C}$.

$$\begin{aligned}
\cos(z \pm w) &= \cos z \cos w \mp \sin z \sin w, \\
\sin(z \pm w) &= \sin z \cos w \pm \cos z \sin w, \\
\sin z - \sin w &= 2 \cos(\frac{z+w}{2}) \sin(\frac{z-w}{2}), \\
\cos z - \cos w &= -2 \sin(\frac{z+w}{2}) \sin(\frac{z-w}{2}).
\end{aligned}$$

Proof. For all $z, w \in \mathbb{C}$, we have

$$\begin{aligned}\cos z \cos w - \sin z \sin w &= \frac{1}{4}[(e^{iz}+e^{-iz})(e^{iw}+e^{-iw}) + (e^{iz}-e^{-iz})(e^{iw}-e^{-iw})] \\ &= \frac{1}{4}[e^{i(z+w)} + e^{-i(z+w)} + e^{i(z+w)} + e^{-i(z+w)}] \\ &= \frac{1}{2}[e^{i(z+w)} + e^{-i(z+w)}] = \cos(z+w).\end{aligned}$$

by Corollary 4.3 b). The proof of the remaining identities is similar and left to the reader.

□

From the first of the above identities we infer (take $z = w$)

$$\cos^2 z + \sin^2 z \overset{4.4}{=} \cos(z-z) = \cos 0 = 1, \quad z \in \mathbb{C}.$$

We write down this important relation explicitly in the following corollary.

4.5 Corollary. *For all $z \in \mathbb{C}$ we have*

$$\cos^2 z + \sin^2 z = 1.$$

In the following we examine the exponential function specifically for real arguments. The proof of the following properties is left to the reader as an exercise.

4.6 Theorem. *The following statements hold:*

a) $e^x < 1$ if $x < 0$ and $e^x > 1$ if $x > 0$.

b) The function $\exp : \mathbb{R} \to \mathbb{R}_+$ is strictly monotone increasing.

c) For each (fixed) $\alpha \in \mathbb{R}$ we have

$$\lim_{x\to\infty} \frac{e^x}{x^\alpha} = \infty\,;$$

in other words, the exponential function grows faster for $x \to \infty$ than every power x^α.

d) For every $\alpha \in \mathbb{R}$ we have

$$\lim_{x\to\infty} x^\alpha e^{-x} = \lim_{x\to\infty} \frac{x^\alpha}{e^x} = 0\,;$$

in other words, e^{-x} decreases faster than every power x^α.

Since the exponential function $\exp : \mathbb{R} \to (0, \infty)$ is continuous, surjective, and strictly monotone increasing, by III.1, there exists an inverse function

$$\log : (0, \infty) \to \mathbb{R}$$

of the exponential function. As in Chapter II, this function will be called *logarithm function*. In particular, we have

$$\log 1 = 0 \quad \text{and} \quad \log e = 1.$$

Furthermore, the logarithm function has the properties

$$\begin{aligned} \log(xy) &= \log x + \log y, \quad x, y \in (0, \infty) \\ \log(\frac{x}{y}) &= \log x - \log y, \quad x, y \in (0, \infty). \end{aligned}$$

This follows directly from the functional equation of the exponential function, because if we let $a := \log x$ and $b := \log y$, we have $x = e^a$ and $y = e^b$ and it follows that $xy = e^a \cdot e^b = e^{a+b}$; hence $\log(xy) = \log x + \log y$.

The exponential function also allows to define *general powers* a^z for $a > 0$ and $z \in \mathbb{C}$ in accordance with the previous definition of powers, compare Example 1.14 c).
If we define

$$a^z := e^{z \log a}, \quad z \in \mathbb{C}, a > 0,$$

then we have the following calculation rules for $z, w \in \mathbb{C}$ and $a, b > 0$:

$$a^z a^w = a^{z+w}, \quad a^w b^w = (ab)^w, \qquad z, w \in \mathbb{C},$$

$$\log(a^x) = x \log a, \quad (a^x)^y = a^{xy}, \qquad x, y \in \mathbb{R}.$$

To prove the first rule observe $a^z a^w = e^{z \log a} e^{w \log a} = e^{(z+w) \log a} = a^{(z+w)}$. The others follow analogously.

We also verify that for each $\alpha > 0$ there holds

$$\lim_{x \to \infty} \frac{\log x}{x^\alpha} = 0 \quad \text{and} \quad \lim_{x \to 0+} x^\alpha \log x = 0$$

In other words, the logarithm function grows slower than any (positive) power x^α for $x \to \infty$, and its singularity at the origin is controlled by any (tiny) positive x-power.

Let us now discuss the sine and cosine functions for real arguments; in particular, we are interested in their roots.

4.7 Lemma. *For $x \in (0, 2]$ we have:*

$$x - \frac{x^3}{6} < \sin x < x \quad \textit{and} \quad 1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

In particular, $\sin x > 0$ *for* $x \in (0, 2]$.

Proof. For $x \in (0, 2]$ we have

$$\sin x = x - \frac{x^3}{3!} + \underbrace{\frac{x^5}{5!} - \frac{x^7}{7!}}_{>0} + \underbrace{\frac{x^9}{9!} - \frac{x^{11}}{11!}}_{>0} + \ldots > x - \frac{x^3}{3!},$$

because

$$\frac{x^n}{n!} - \frac{x^{n+2}}{(n+2)!} = \frac{x^n[(n+1)(n+2) - x^2]}{(n+2)!} > 0.$$

On the other hand,

$$\sin x = x - \underbrace{\left(\frac{x^3}{3!} - \frac{x^5}{5!}\right)}_{>0} - \underbrace{\left(\frac{x^7}{7!} - \frac{x^9}{9!}\right)}_{>0} + \ldots < x,$$

and this implies the proposition for the sine function. The estimate for cos is analogous.

□

We also note that the cosine is a strictly decreasing function on the interval $[0, 2]$: For, if $x > y$, we have

$$\cos x - \cos y \overset{4.4}{=} -2 \underbrace{\sin(\frac{x+y}{2})}_{>0} \underbrace{\sin(\frac{x-y}{2})}_{>0} < 0, \quad x, y \in [0, 2].$$

We can now show that the cosine function has exactly one root in the interval $[0, 2]$.

4.8. Theorem and definition of the number π. *The cosine function has exactly one root x_0 in the interval $[0, 2]$. We define*

$$\pi := 2x_0.$$

Proof. We have $\cos(0) = 1$ and the above Lemma 4.7 implies that

$$\cos(2) < 1 - \tfrac{2^2}{2} + \tfrac{2^4}{24} = -\tfrac{1}{3} < 0.$$

Because cos is continuous, the intermediate value theorem implies that cos has at least one root x_0 in $[0, 2]$. The uniqueness follows from the strict monotonicity of cos in $[0, 2]$.

□

The term π became popular through the textbook of Euler that was mentioned above, and is possibly derived from the Greek word $\pi\varepsilon\varrho\iota\varphi\varepsilon\varrho\varepsilon\iota\alpha$ for circumference. If we try to compute π numerically, we obtain

$$\pi = 3.14159\,26535\,89793\,23846\ldots$$

With the following mnemonic these digits of π can be reproduced, if one assigns to each single word the number of its letters:

"Sir, I send a rhyme excelling in sacred truth and rigid spelling numerical sprites elucidate for me the lexicon's dull weight"

4.9 Remark. A real number is called *algebraic* if it is a root of a non-trivial polynomial with integer coefficients. For example, every rational number p/q is algebraic as a root of the polynomial $x \mapsto qx - p$. Real numbers which are not algebraic are called transcendental. In particular, they are irrational.

Already in 1761, H. J. Lambert proved that π is irrational. The transcendence proof of π was given 1882 by F. Lindemann. This theorem also decided the more than 2000 years old and still famous problem of squaring the circle: it is *impossible* to give a ruler-and-compass construction of a square that has the same area as a given circle.

The above definition of the number π implies in particular that

$$\cos(\frac{\pi}{2}) = 0 \quad \text{and} \quad \sin(\frac{\pi}{2}) = 1.$$

This identity holds because $\cos^2(\frac{\pi}{2}) + \sin^2(\frac{\pi}{2}) = 1$ implies firstly $\sin\frac{\pi}{2} = \pm 1$, and the positivity of the sine in $(0, 2]$ then yields $\sin\frac{\pi}{2} = 1$.

If we combine these formulas with Euler's formula of Theorem 4.2 b),we obtain $e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$. More generally we have the following table of values of $\cos x, \sin x$ and e^{ix} :

x	0	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$
$\cos x$	1	0	-1	0
$\sin x$	0	1	0	-1
e^{ix}	1	i	-1	$-i$

If we combine the above function values with the functional equation of the exponential function, we can deduce the important *periodicity* of the exponential function.

4.10 Theorem. *For all $z \in \mathbb{C}$ and $n \in \mathbb{Z}$ we have*

$$e^{z+i\frac{n}{2}\pi} = e^z i^n, \qquad \textit{and in particular} \qquad e^{z+2in\pi} = e^z.$$

This means that the exponential function has the purely imaginary period $2\pi i$.

This result transferred to the trigonometric functions gives the following corollary.

4.11 Corollary. *a) For $z \in \mathbb{C}$ we have*

i) $\cos(z + \frac{\pi}{2}) = -\sin z, \quad \cos(z + \pi) = -\cos z, \quad \cos(z + 2\pi) = \cos z,$

ii) $\sin(z+\frac{\pi}{2}) = \cos z, \quad \sin(z+\pi) = -\sin z, \quad \sin(z+2\pi) = \sin z.$

In particular, the functions sin *and* cos *are periodic functions with real period* 2π.

b) We have

$$\begin{aligned} \cos z = 0 \quad &\Leftrightarrow \quad z = \frac{\pi}{2} + n\pi \textit{ for an } n \in \mathbb{Z}, \\ \sin z = 0 \quad &\Leftrightarrow \quad z = n\pi \textit{ for an } n \in \mathbb{Z}, \\ e^z = 1 \quad &\Leftrightarrow \quad z = 2ni\pi \textit{ for an } n \in \mathbb{Z}. \end{aligned}$$

We conclude our discussion of the trigonometric functions for the time being by introducing the tangent and cotangent functions. We define the *tangent* and the *cotangent* functions by

$$\begin{aligned} \tan : \mathbb{C}\backslash\{\pi/2 + n\pi : n \in \mathbb{Z}\} \to \mathbb{C}, \qquad z \quad &\mapsto \frac{\sin z}{\cos z}, \\ \cot : \mathbb{C} \setminus \{n\pi : n \in \mathbb{Z}\} \to \mathbb{C}, \qquad z \quad &\mapsto \frac{\cos z}{\sin z}. \end{aligned}$$

To conclude this section, we consider the inverse functions of the trigonometric and the hyperbolic functions. We begin with the following properties of sin, cos and tan.

4.12 Lemma. *a) The function* $\cos : [0, \pi] \to [-1, 1]$ *is continuous, surjective and strictly decreasing.*

b) The function $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ *is continuous, surjective and strictly increasing.*

c) The function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ *is continuous, surjective and strictly increasing.*

Proof. (a) Since $\cos 0 = 1$, $\cos\pi = -1$, and the cosine is continuous by Theorem 2.4, we have surjectivity on account of the intermediate value theorem. Furthermore, since the cosine is in particular strictly decreasing on $[0, \pi/2]$ and $\cos x = -\cos(\pi - x)$ the cosine is also strictly decreasing on $[\pi/2, \pi]$, i.e., injectivity.

(b) Since $\sin x = \cos(\pi/2 - x)$ the assertions follow from (a).

(c) Since sine is strictly increasing and cosine strictly decreasing on $[0, \pi/2)$ and $\tan(-x) = -\tan x$ the tangent is strictly increasing and continuous on $(-\pi/2, \pi/2)$. Also $\lim_{x\to\pi/2-} \tan x = \infty$ and, therefore, surjectivity follows.

$\square$

The above lemma therefore implies that the inverse functions

$$\begin{aligned} \arccos : &\quad [-1, 1] \to [0, \pi] \\ \arcsin : &\quad [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}] \\ \arctan : &\quad \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2}) \end{aligned}$$

of $\sin, \cos$, and $\tan$, resp., exist on the respective intervals. These are called *inverse trigonometric functions* or *cyclometric functions* and they are all continuous by Theorem 1.13.

Our current state of knowledge now allows to treat the polar form of complex numbers. We have the following theorem.

4.13 Theorem. (Polar form of complex numbers). *Every $z \in \mathbb{C} \setminus \{0\}$ has a representation of the form*

$$z = re^{i\varphi},$$

where $r = |z|$ and $\varphi \in \mathbb{R}$ is determined up to addition of an integer multiple of 2π.

In the above representation, r is called the *absolute value* (or *modulus*) and φ the *argument* (or *angle*) of the complex number z.

Proof. For $z \in \mathbb{C} \setminus \{0\}$ there exist $x, y \in \mathbb{R}$ with $\frac{z}{|z|} = x + iy$. Then we have $x^2 + y^2 = 1$ and therefore $x, y \in [-1, 1]$. Therefore, $\alpha := \arccos x$ is well defined. Now $x = \cos\alpha$ implies $\sin\alpha = \pm\sqrt{1 - x^2} = \pm y$. We set

$$\varphi := \begin{cases} \alpha & : \quad \sin\alpha = y \\ -\alpha & : \quad \sin\alpha = -y \end{cases} = \begin{cases} \arccos x, & y \geq 0, \\ -\arccos x, & y < 0, \end{cases}$$

In either case we have that φ is well defined and $\varphi \in [0, \pi]$ provided $y \geq 0$: By Lemma 4.7, we have $\sin\varphi \geq 0$ for all $\varphi \in [0, 2]$ and because $\sin\varphi = \sin(\pi - \varphi)$ (cf. 4.11 b)), we deduce $\sin\varphi \geq 0$ for all $\varphi \in [0, \pi]$. Furthermore, because $\sin^2\varphi = 1 - \cos^2\varphi = y^2$, it follows that $\sin\varphi = y$. Therefore we obtain

$$e^{i\varphi} = \cos\varphi + i\sin\varphi = x + iy = \frac{z}{|z|},$$

and thus $z = re^{i\varphi}$ for $r = |z|$. The case $y < 0$ is treated analogously.

□

4.14 Remarks. a) The polar form gives us a nice geometric way to visualize the product of complex numbers in the complex plane. For $z = |z|e^{i\varphi}$ and $w = |w|e^{i\psi}$, we have

$$z \cdot w = |zw|e^{i(\varphi+\psi)}.$$

b) Furthermore, for each $z \in \mathbb{C} \setminus \{0\}$ and each $n \in \mathbb{N}$, there exist exactly n different numbers $z_1, \ldots, z_n \in \mathbb{C}$ with $(z_k)^n = z$ for all $k = 1, \ldots, n$. These numbers are called *n-th roots* of z. In particular, for $z = 1$ there exist exactly n different *roots of unity* $\xi_1, \xi_2, \ldots, \xi_n$, i.e., complex numbers ξ_k with $\xi_k^n = 1$ for all $k = 1, \ldots, n$. The n-th roots of a complex number $z = re^{i\varphi}$ are given explicitly by

$$z_k := \sqrt[n]{r}\,\xi_k \quad \text{with} \quad \xi_k = e^{i(\frac{\varphi+2\pi k}{n})} \quad \text{for all} \quad k = 1, \ldots, n.$$

In many concrete problems, the exponential function appears in the form $(e^z + e^{-z})/2$ or $(e^z - e^{-z})/2$. Based on this, we define the hyperbolic functions as follows:

$$\begin{aligned}
\cosh z &:= \frac{1}{2}(e^z + e^{-z}) \quad \text{hyperbolic cosine,}\\
\sinh z &:= \frac{1}{2}(e^z - e^{-z}) \quad \text{hyperbolic sine,}\\
\tanh z &:= \frac{\sinh z}{\cosh z} \quad \text{hyperbolic tangent,}\\
\coth z &:= \frac{\cosh z}{\sinh z} \quad \text{hyperbolic cotangent.}
\end{aligned}$$

The relations

$$\cosh z = \cos iz, \quad \sinh z = -i \sin iz, \quad z \in \mathbb{C}$$

and

$$\cosh^2 z - \sinh^2 z = 1, \quad z \in \mathbb{C}$$

are easily verified as well as their power series representation

$$\cosh z = \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!}, \qquad \text{and} \qquad \sinh z = \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(2j+1)!}, \quad z \in \mathbb{C}.$$

Chapter IV

Differential Calculus in one Variable

1 Differentiable Functions

The differential and integral calculus, which dates back to Leibniz and Newton, builds the core of all basic lectures on analysis. In this section, we restrict our attention to the differential calculus of functions in one *real variable*, however we do admit that the functions may have *complex values*.

We begin with the problem to approximate a given function $f : D \subset \mathbb{R} \to \mathbb{K}$ at the point $x_0 \in D$ by an affine function. If we have $\mathbb{K} = \mathbb{R}$, we can interpret this geometrically as the problem to find the tangent line of the graph of f at the point $(x_0, f(x_0))$.

The basic idea to solve the above problem is to approximate the tangent lines by the lines through the points $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ for small h. The slope of these lines is given by $\frac{f(x_0+h)-f(x_0)}{h}$. This motivates the following definition.

1.1 Definition. Let $D \subset \mathbb{R}$ and assume that $x_0 \in D$ is an accumulation point of D. We call a function $f : D \to \mathbb{K}$ *differentiable (differenzierbar) at* $x_0 \in D$, if the limit

$$\lim_{\substack{x \to x_0 \\ x \in D\setminus\{x_0\}}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\substack{h \to 0,\, h \neq 0 \\ x_0 + h \in D}} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. This limit is called *derivative (Ableitung) of* f *at* x_0 and is denoted by $f'(x_0)$ or $\frac{df}{dx}(x_0)$. If f is differentiable at every $x \in D$, we say that f is *differentiable on* D and we call the function $f' : D \to \mathbb{K}$, $x \mapsto f'(x)$ the *derivative* of f.

1.2 Examples. a) The function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^n$ is differentiable for each $n \in \mathbb{N}$ and one has $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}$. Just observe that

$$\frac{x^n - x_0^n}{x - x_0} = x_0^{n-1} + xx_0^{n-2} + x^2x_0^{n-3} + \ldots + x^{n-1} \overset{x\to x_0}{\longrightarrow} x_0^{n-1} + x_0^{n-1} + \ldots + x_0^{n-1} = nx_0^{n-1}.$$

b) The function $f : \mathbb{R} \to \mathbb{C}, f(x) = e^{\alpha x}$ is differentiable for all $\alpha \in \mathbb{C}$ and we have $f'(x) = \alpha e^{\alpha x}$, because we have

$$\frac{e^{\alpha(x_0+h)} - e^{\alpha x_0}}{h} = e^{\alpha x_0}\left(\frac{e^{\alpha h} - 1}{h}\right) \overset{h\to 0}{\longrightarrow} \alpha e^{\alpha x_0},$$

in analogy to Example 3.15 b).

In the following theorem, we give an equivalent reformulation of the concept of differentiability. For this, we require that $x_0 \in D$ is an accumulation point of D.

1.3 Theorem. *Let $f : D \subset \mathbb{R} \to \mathbb{K}$ be a map and $x_0 \in D$ an accumulation point. The following statements are equivalent.*

i) The function f is differentiable at x_0.

ii) There exists a function $\varphi : D \to \mathbb{K}$ which is continuous at x_0, such that

$$f(x) = f(x_0) + (x - x_0)\varphi(x), \quad x \in D.$$

In this case, we have $f'(x_0) = \varphi(x_0)$.

iii) There exists a linear mapping $L : \mathbb{R} \to \mathbb{K}$ such that

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - Lh}{h} = 0.$$

In this case, we have $f'(x_0)h = Lh$ for all $h \in \mathbb{R}$.

Proof. *i)* $\Longrightarrow$ *ii)*: By assumption, the function $x \mapsto \frac{f(x)-f(x_0)}{x-x_0}$ for $x \in D \setminus \{x_0\}$ has an extension φ which is continuous at x_0. In x_0 we then have $\varphi(x_0) = f'(x_0)$.

ii) $\Longrightarrow$ *iii)*: The linear mapping $Lh := \varphi(x_0)h = f'(x_0)h$ has the properties which are required in statement iii).

iii) $\Longrightarrow$ *i)*: Let L be a linear mapping for which statement iii) holds. If we have $Lh = \alpha h$, $h \in \mathbb{R}$, for some $\alpha \in \mathbb{C}$, it follows that

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} - \alpha = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - \alpha h}{h} = 0;$$

This means that f is differentiable in x_0 and we have $f'(x_0) = \alpha$.

□

The statement iii) of the above theorem says that for a differentiable function f the increment $f(x_0+h)-f(x_0)$ is approximated so well by Lh, that the difference $f(x_0+h)-f(x_0)-Lh$ tends to 0 faster for $h \to 0$ than h itself. This formulation aims at approximating functions locally by linear functions and will be further extended later by the theorem of Taylor (Taylor formula). Also, this formulation is the starting point for the generalization of the notion of differentiability to functions of several variables.

In particular, Theorem 1.3 immediately implies that a function, differentiable at x_0, is continuous at this point.

1.4 Corollary. *A function $f : D \to \mathbb{K}$, which is differentiable in $x_0 \in D \subset \mathbb{R}$, is also continuous at x_0.*

We note that the converse of Corollary 1.4 does *not* hold in general. For this, consider for example the absolute value function $f(x) = |x|$ in the point 0. We further remark that there exist continuous functions on $\mathbb{R}$ which are differentiable in *no* point of their domain of definition.

1.5 Theorem. (Calculation rules for differentiable functions) *Let $f, g : D \subset \mathbb{R} \to \mathbb{K}$ be functions differentiable in $x_0 \in D$. Then the following statements hold:*

a) The function $\alpha f + \beta g : D \subset \mathbb{R} \to \mathbb{K}$ is differentiable in x_0 for all $\alpha, \beta \in \mathbb{K}$ and

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0).$$

Thus, differentiation is a linear mapping*; here differentiation is interpreted as an operation acting from some set of functions into another set of functions.*

b) (Product rule). *The product $f \cdot g$ is differentiable at x_0 and we have*

$$(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

c) (Quotient rule). *If $g(x_0) \neq 0$, then there exists a $\delta > 0$ such that $g(x) \neq 0$ for all $x \in D \cap (x_0 - \delta, x_0 + \delta)$ and $\dfrac{f}{g} : D \cap (x_0 - \delta, x_0 + \delta) \to \mathbb{K}$ is differentiable in x_0 and*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}.$$

Proof. The statement a) follows directly from the calculation rules for limits.

To prove b), let $h \neq 0$ and $x_0 + h \in D$. Then we have

$$\begin{aligned}
&\frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0)}{h}\\
&\quad=\frac{f(x_0+h)-f(x_0)}{h}g(x_0+h)+\frac{g(x_0+h)-g(x_0)}{h}f(x_0)\\
&\quad\overset{h\to 0}{\longrightarrow} f'(x_0)g(x_0)+g'(x_0)f(x_0).
\end{aligned}$$

To prove c) we note that $\dfrac{f}{g}=f\cdot\dfrac{1}{g}$, thus by (b) it is sufficient to discuss $1/g$.

$$\frac{1}{h}\Big(\frac{1}{g(x_0+h)}-\frac{1}{g(x_0)}\Big)=\frac{1}{g(x_0+h)g(x_0)}\cdot\frac{g(x_0)-g(x_0+h)}{h}\overset{h\to 0}{\longrightarrow}-\frac{g'(x_0)}{g^2(x_0)}.$$

$\square$

1.6 Examples. a) A polynomial p of the form $p(x)=5x^3+7x^2+3x$ is differentiable with derivative $p'(x)=15x^2+14x+3$. This follows from Example 1.2 a) and Theorem 1.5 a).

b) The sine as well as the cosine functions are differentiable for all $x\in\mathbb{R}$ and we have

$$\sin'(x)=\cos x,\qquad \cos'(x)=-\sin x,\quad x\in\mathbb{R},$$

because $\sin x=\frac{1}{2i}(e^{ix}-e^{-ix})$ and Example 1.2 b) and Theorem 1.5 a) imply

$$(\sin x)'=\frac{1}{2i}(ie^{ix}+ie^{-ix})=\cos x.$$

c) The quotient rule implies that the derivative of the tangent function is given by

$$(\tan x)'=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}\overset{\text{III.4.3}}{=}\frac{1}{\cos^2 x}=1+\tan^2 x,\quad x\in\mathbb{R}\setminus\{\frac{\pi}{2}+k\pi:k\in\mathbb{Z}\}.$$

d) For $n\in\mathbb{N}$ let $f:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be given by $x\mapsto x^{-n}$. Then we have $f'(x)=-nx^{-n-1}$, because if we define $h(x)=x^n$, then we have $f=\frac{1}{h}$, and by the quotient rule we can deduce $f'(x)=\frac{-nx^{n-1}}{x^{2n}}=-nx^{-n-1}$ for all $x\in\mathbb{R}\setminus\{0\}$.

1.7 Theorem. (Chain rule) *Let $f:D_f\subset\mathbb{R}\to\mathbb{K}$ and $g:D_g\subset\mathbb{R}\to\mathbb{R}$ be two functions with $g(D_g)\subset D_f$. If g is differentiable in $x_0\in D_g$ and f is differentiable in $y_0:=g(x_0)\in D_f$, then $f\circ g:D_g\subset\mathbb{R}\to\mathbb{K}$ is differentiable in x_0 and we have*

$$(f\circ g)'(x_0)=g'(x_0)\cdot f'(y_0)\Big|_{y_0=g(x_0)}=f'(g(x_0))\cdot g'(x_0).$$

Proof. By Theorem 1.3, there exist functions φ_g and φ_f which are continuous at x_0 and $y_0 := g(x_0)$, resp., such that

$$\begin{aligned} f(y) - f(y_0) &= (y - y_0)\varphi_f(y), \quad \varphi_f(y_0) = f'(y_0),\ y \in D_f \\ g(x) - g(x_0) &= (x - x_0)\varphi_g(x), \quad \varphi_g(x_0) = g'(x_0),\ x \in D_g. \end{aligned}$$

Therefore, we have

$$(f \circ g)(x) - (f \circ g)(x_0) = (g(x) - g(x_0))\varphi_f(g(x)) = (x - x_0)\underbrace{\varphi_g(x)\varphi_f(g(x))}_{=:\varphi(x)}$$

with a function $\varphi := \varphi_g \cdot (\varphi_f \circ g)$, which is continuous at x_0. Now, Theorem 1.3 implies that $f \circ g$ is differentiable at x_0 and, by the preceding formula,

$$(f \circ g)'(x_0) = \varphi(x_0) = \varphi_g(x_0)\varphi_f(g(x_0)) = f'(g(x_0)) \cdot g'(x_0).$$

□

To conclude this section, we examine the derivative of the inverse of a given differentiable function.

1.8 Theorem. (Derivative of the inverse function). *Let $J \subset \mathbb{R}$ be an interval and let g be the inverse function of a continuous and strictly monotone function $f : J \to \mathbb{R}$. If f is differentiable at $x_0 \in J$ and $f'(x_0) \neq 0$, then $g : f(J) \to \mathbb{R}$ is differentiable at $y_0 := f(x_0)$ and we have*

$$g'(y_0) = g'(f(x_0)) = \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}.$$

Proof. By assumption and Theorem 1.3, there exists a function φ which is continuous at x_0 such that $f(x) - f(x_0) = (x - x_0)\varphi(x)$ for all $x \in J$. Because we have $\varphi(x_0) = f'(x_0) \neq 0$, there exists a $\delta > 0$ such that $\varphi(x) \neq 0$ for all $x \in J_\delta := J \cap [x_0 - \delta, x_0 + \delta]$. If we let $x = g(y)$ for $y \in f(J_\delta)$, we have

$$y - y_0 = f(g(y)) - f(g(y_0)) = (g(y) - g(y_0))\varphi(g(y)), \quad y \in f(J_\delta).$$

Therefore, $g(y) - g(y_0) = (y - y_0)\frac{1}{\varphi(g(y))}$ holds, and $\varphi \circ g$ is continuous at x_0 by Section III.1. Theorem 1.3 now implies that g is differentiable at y_0 and that we have

$$g'(y_0) = \frac{1}{\varphi(g(y_0))} = \frac{1}{\varphi(x_0)} = \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}.$$

□

1.9 Example. The function $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is differentiable by 1.6 c) and we have $\tan'(x) = 1 + \tan^2 x$ for all $x \in (-\pi/2, \pi/2)$. Therefore, $\arctan : \mathbb{R} \to \mathbb{R}$ is also differentiable and we have

$$\arctan'(y) = \frac{1}{1 + \tan^2(\arctan y)} = \frac{1}{1 + y^2}.$$

2 The Mean Value Theorem and Applications

In Section III.3, we saw that a continuous real valued function f on a compact set has a global maximum and a global minimum. We shall now see that if the function is furthermore differentiable, the derivative gives an additional information on the location of the extrema. More precisely, we have the following (necessary) criterion for extremal values; as an application of the mean value theorem, we will later also give a sufficient criterion.

2.1 Definition. If $f : D \subset \mathbb{R} \to \mathbb{R}$ is a function, we call $x_0 \in D$ a *local maximum (minimum)* (*lokales Maximum (Minimum)*) of f, if there exists a $\delta > 0$ such that

$$f(x) \leq f(x_0) \quad (f(x) \geq f(x_0)) \quad \text{ for all } x \in D \cap (x_0 - \delta, x_0 + \delta).$$

Local minima and maxima are also called *local extrema* (*lokale Extrema*) of a given function f. In the following, we will give criteria which allow to examine a given function for local extrema. Firstly, we begin with a necessary criterion.

2.2 Theorem. *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : (a, b) \to \mathbb{R}$ be a function which has a local extremum at $x_0 \in (a, b)$. If f is differentiable in x_0, then $f'(x_0) = 0$.*

Proof. Let x_0 be a local minimum of f. Then there exists a $\delta > 0$ with

$$f(x) - f(x_0) \geq 0, \quad \text{ for all } x \in (x_0 - \delta, x_0 + \delta).$$

Therefore we have, when we let x tend to x_0 from the left hand side,

$$f'(x_0) = \lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0} \leq 0\,.$$

For the right sided limit we obtain

$$f'(x_0) = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0} \geq 0\,.$$

This implies $f'(x_0) = 0$. The proof for a local maximum is analogous.

□

At this point we remark that the converse of the above theorem does not hold in general, and that a function f that is defined on a closed interval $[a, b]$ can attain an extremum at a or b even if $f'(a) \neq 0$ and $f'(b) \neq 0$.

The following theorem is an easy consequence of the above theorem.

2.3 Corollary (Rolle's Theorem (Satz von Rolle)). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). If $f(a) = f(b)$, then there exists $\xi \in (a, b)$ with $f'(\xi) = 0$.*

Proof. If f is a constant function, then we have $f' = 0$ and, therefore, the proposition holds. Let us now assume that f is not constant. By Theorem III.3.9, f attains its maximum $\max f$ and minimum $\min f$ on the compact interval $[a, b]$. Then $\max f \neq f(a) = f(b)$ or $\min f \neq f(a) = f(b)$. Thus, there is a $\xi \in (a, b)$ which is an extremum of f. By Theorem 2.2 above, we thus have $f'(\xi) = 0$.

□

The following *mean value theorem* is the central theorem of this section. It has far reaching consequences for the analysis in one real variable.

2.4 Theorem (Mean value theorem (Mittelwertsatz)). *If $f : [a, b] \to \mathbb{R}$ is a continuous, real-valued function which is differentiable on (a, b), then there exists a $\xi \in (a, b)$ with*

$$f(b) - f(a) = f'(\xi)(b - a).$$

Proof. We define a function $F : [a, b] \to \mathbb{R}$ by

$$F(x) := f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then F is continuous on $[a, b]$ and differentiable on (a, b). We have $F(a) = f(a) = F(b)$. Therefore, by Rolle's Theorem 2.3, there exists a $\xi \in (a, b)$ with

$$F'(\xi) = 0 = f'(\xi) - \frac{f(b) - f(a)}{b - a}.$$

□

At this point, we remark that the mean value theorem does *not* hold for complex-valued differentiable functions $f : [a, b] \to \mathbb{C}$. A counterexample is given by the function $f : [0, 2\pi] \to \mathbb{C}$, defined by $f(x) = e^{ix}$. We have $f(0) = 1 = f(2\pi)$, but $f'(x) = ie^{ix} \neq 0$ for all $x \in [0, 2\pi]$.

The mean value theorem has many important consequences. Some of these are assembled in the following corollary.

2.5 Corollary. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Then the following propositions hold.*

(a) *f is constant $\Leftrightarrow$ $f'(x) = 0$ for all $x \in (a, b)$.*

(b)

$$
\begin{aligned}
&f'(x) \geq 0 \text{ for all } x \in (a, b) &&\Leftrightarrow && f \text{ is increasing on } [a, b].\\
&f'(x) \leq 0 \text{ for all } x \in (a, b) &&\Leftrightarrow && f \text{ is decreasing on } [a, b].\\
&f'(x) > 0 \text{ for all } x \in (a, b) &&\Rightarrow && f \text{ is strictly increasing on } [a, b].\\
&f'(x) < 0 \text{ for all } x \in (a, b) &&\Rightarrow && f \text{ is strictly decreasing on } [a, b].
\end{aligned}
$$

(c) *If* $f'(x_0) = 0$ *for an* $x_0 \in (a, b)$ *then, for a sufficiently small* $\delta > 0$, x_0 *is a*

i) local minimum, if $f' \leq 0$ *in* $(x_0 - \delta, x_0)$ *and* $f' \geq 0$ *on* $(x_0, x_0 + \delta)$*;*

ii) local maximum, if $f' \geq 0$ *in* $(x_0 - \delta, x_0)$ *and* $f' \leq 0$ *on* $(x_0, x_0 + \delta)$*.*

(d) *If* $|f'(x)| \leq L$ *for all* $x \in [a, b]$*, we have*

$$|f(x) - f(y)| \leq L|x - y|, \qquad \textit{for all } x, y \in [a, b],$$

i.e., f *is Lipschitz continuous with Lipschitz constant* L*.*

(e) *The function* f' *has the intermediate value property even though it is not continuous in general. More precisely, let* $f'(a) \neq f'(b)$ *and* $\min\{f'(a), f'(b)\} < \alpha < \max\{f'(a), f'(b)\}$*. Then, there exists* $\xi \in (a, b)$ *with* $f'(\xi) = \alpha$*.*

Proof. a) If f is constant, it is clear that $f'(x) = 0$ for all $x \in (a, b)$. Conversely, let $x \in (a, b]$. By the mean value theorem and the assumption, there exists $\xi \in (a, x)$ with $f(x) - f(a) = f'(\xi)(x - a) = 0$. Therefore, $f(x) = f(a)$.

b) The definition of differentiability immediately implies that $f'(x) \geq 0$ for all $x \in (a, b)$, given that f is increasing. Conversely, let $a \leq x < y \leq b$. Again, by the mean value theorem, there exists a $\xi \in (x, y)$ with

$$f(y) - f(x) = \underbrace{f'(\xi)}_{\geq 0}\underbrace{(y - x)}_{>0} \geq 0,$$

if $f' \geq 0$.

The propositions c), d) and e) are left as exercises.

□

A further corollary of the mean value theorem is the following characterization of the exponential function on $\mathbb{R}$.

2.6 Corollary. *The exponential function* exp *is the only differentiable function* $f : \mathbb{R} \to \mathbb{C}$ *with* $f' = f$ *and* $f(0) = 1$*.*

As proof, consider the function $g(x) := f(x)e^{-x}$ for $x \in \mathbb{R}$. We have $g'(x) = [f'(x) - f(x)]e^{-x} = 0$ for all $x \in \mathbb{R}$ and, therefore, g is a constant with the value $g(0) = 1$.

2.7 Theorem (Generalized mean value theorem[1]). *Let* $f, g : [a, b] \to \mathbb{R}$ *be continuous functions which are differentiable in* (a, b) *and assume that* $g'(x) \neq 0$ *for all* $x \in (a, b)$*. Then we have* $g(a) \neq g(b)$ *and there exists a* $\xi \in (a, b)$ *with*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

[1]also known as *Cauchy mean value theorem*

Proof. Firstly, we have $g(a) \neq g(b)$, because otherwise, by Rolle's Theorem 2.3, there would exist a $x \in (a,b)$ with $g'(x) = 0$, contradicting the assumption.
To prove the theorem, we define $F : [a,b] \to \mathbb{R}$ by

$$F(x) := f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$

Then $F(a) = f(a) = F(b)$ and by Rolle's theorem, there exists a $\xi \in (a,b)$ with

$$0 = F'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(\xi).$$

□

Making use of the generalized mean value theorem, we also prove the rules of l'Hospital. They allow to compute limits of the form $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ where $f(x)$ as well as $g(x)$ tend to ∞ for $x \to x_0$.

2.8 Corollary (L'Hospital's Rules (l'Hospitalsche Regeln)). *Let $-\infty < a < b < \infty$ and let $f, g : (a,b) \to \mathbb{R}$ be two differentiable functions with $g'(x) \neq 0$ for all $x \in (a,b)$. If*
a) $\lim_{x\to a+} f(x) = 0 = \lim_{x\to a+} g(x)$ or
b) $\lim_{x\to a+} f(x) = \infty = \lim_{x\to a+} g(x)$,
and $\lim_{x\to a+} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a+} \frac{f(x)}{g(x)}$ exists as well, and we have

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = \lim_{x\to a+} \frac{f'(x)}{g'(x)}.$$

The corresponding result does also hold for $x \to b-$, $x \to \infty$ or $x \to -\infty$.

Proof. To prove proposition a), we view f and g as continuous in a by setting $f(a) = g(a) = 0$. By the generalized mean value theorem, for each $x \in (a,b)$ there exists a $\xi \in (a,x)$ with

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

If $x \to a$, it follows that $\xi \to a$, which in turn entails the proposition.
For the case b) let $q := \lim_{x\to a} \frac{f'(x)}{g'(x)}$. Then for each $\varepsilon > 0$ there exists a $c \in (a,b)$ with

$$\Big|\frac{f'(x)}{g'(x)} - q\Big| \leq \varepsilon, \quad \text{for all } x \in (a,c).$$

Then, by the generalized mean value theorem,

$$\Big|\frac{f(x) - f(y)}{g(x) - g(y)} - q\Big| \leq \varepsilon\,, \quad x, y \in (a,c),\ x \neq y.$$

Now fix $y \in (a,c)$. Because $\lim_{x\to a} g(x) = \infty$ by assumption, there exists a $c' \in (a,c)$ with

$$|\frac{g(y)}{g(x)}| \leq \varepsilon \text{ and } |\frac{f(y)}{g(x)}| \leq \varepsilon \quad \text{for all } x \in (a,c').$$

Thus, we have

$$\begin{aligned} |\frac{f(x)}{g(x)} - q| &= |(1-\frac{g(y)}{g(x)})(\frac{f(x)-f(y)}{g(x)-g(y)} - q) + \frac{f(y)}{g(x)} - q\frac{g(y)}{g(x)}| \\ &\leq \varepsilon(2 + |q| + \varepsilon) \end{aligned}$$

for all $x \in (a,c')$, i.e. we have $\lim_{x\to a} \frac{f(x)}{g(x)} = q = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.
The remaining cases are proved analogously.

□

L'Hospital's rules are often very convenient to calculate limits.

2.9 Examples. The following propositions hold.

a) $\lim\limits_{x\to 0} \frac{\log(1+x)}{x} \overset{2.8}{=} \lim\limits_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1.$

b) $\lim\limits_{x\to\infty} \frac{\log x}{x^\alpha} \overset{2.8}{=} \lim\limits_{x\to\infty} \frac{1}{x}\frac{1}{\alpha x^{\alpha-1}} = \lim\limits_{x\to\infty} \frac{1}{\alpha x^\alpha} = 0, \quad \alpha > 0.$

c) $\lim\limits_{x\to 0}(\frac{1}{\sin x} - \frac{1}{x}) = \lim\limits_{x\to 0} \frac{x-\sin x}{x\sin x} \overset{2.8}{=} \lim\limits_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} \overset{2.8}{=} \lim\limits_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = 0.$

We now consider derivatives of higher order. More precisely, let $f : D \subset \mathbb{R} \to \mathbb{K}$ be a differentiable function. If f' is also differentiable, then f is called *two times differentiable* and we call $f'' := (f')'$ the *second derivative* of f. More generally, one defines the *n-th derivative* $f^{(n)}$ recursively as the derivative of $f^{(n-1)}$. For $f^{(n)}$, we also write $\frac{d^n f}{dx^n}$ or $D^n f$.

2.10 Definition. A function $f : D \subset \mathbb{R} \to \mathbb{K}$ is called *n times continuously differentiable* if f is n times differentiable and the n-th derivative is still continuous.
Notation: $f \in C^n(D, \mathbb{K})$.

The second derivative of a function can also be interpreted geometrically. For this, we introduce the notion of a convex function.

2.11 Definition. If $J \subset \mathbb{R}$ is an interval, $f : J \to \mathbb{R}$ a function, we call f *convex* (*konvex*), if for all $x_1, x_2 \in J$ and all $\lambda \in (0,1)$ we have

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)f(x_1) + \lambda f(x_2).$$

The following theorem describes the relation between convex functions f and properties of f'.

2.12 Theorem. *Let $f : J \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then f is convex if and only if f' is monotone increasing.*

Proof. $\Longrightarrow$: Let $x, x_1, x_2 \in J$ with $x_1 < x < x_2$. We choose $\lambda \in (0,1)$ such that $x = (1-\lambda)x_1 + \lambda x_2$. Because f is convex by assumption, we have $f(x) \leq (1-\lambda)f(x_1) + \lambda f(x_2)$. Therefore,

$$\begin{aligned} f(x) - f(x_1) &\leq \lambda\,[f(x_2) - f(x_1)] \\ f(x_2) - f(x) &\geq (1-\lambda)\,[f(x_2) - f(x_1)] \end{aligned}$$

and because of $x - x_1 = \lambda(x_2 - x_1) > 0$ and $x_2 - x = (1-\lambda)(x_2 - x_1) > 0$, it follows for all x, $x_1 < x < x_2$, that

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}, \quad x_1 < x < x_2.$$

Therefore, we have

$$f'(x_1) = \lim_{x \to x_1+} \frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \lim_{x \to x_2-} \frac{f(x_2) - f(x)}{x_2 - x} = f'(x_2),$$

thus f is increasing.
$\Longleftarrow$: The proof is similar to the preceding one and is left to the reader as an exercise.
□

2.13 Corollary. *If $f : (a,b) \to \mathbb{R}$ is a two times differentiable function, we have*

$$f \textit{ is convex} \iff f'' \geq 0 \textit{ in } (a,b).$$

2.14 Example. The function $-\log$ is convex on $\mathbb{R}_+$, because we have $(\log x)'' = -\frac{1}{x^2} \leq 0$ for all $x > 0$. Functions f with the property "$-f$ is convex" are called *concave* (*konkav*). In particular, log is a concave function on $\mathbb{R}_+$.

Convex and concave functions are important notions in analysis and have interesting applications. We consider in particular Young's and Hölder's inequalities. For $p \in (1,\infty)$, we call $q \in (1,\infty)$ with

$$\frac{1}{p} + \frac{1}{q} = 1$$

the *Hölder conjugate* of p (zu p konjugierter Index) .

2.15 Theorem (Young's inequality). *For $1 < p, q < \infty$ with $1/p + 1/q = 1$, we have*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q, \quad a, b \geq 0.$$

Proof. Let $a > 0$ and $b > 0$, otherwise the statement is trivial. Since $\log$ is a concave function, it follows from the definition of convexity with $\lambda = 1/p$ and $(1-\lambda) = 1/q$, that

$$\log(\frac{a^p}{p} + \frac{b^p}{q}) \geq \frac{1}{p}\log a^p + \frac{1}{q}\log b^q = \log a + \log b = \log(ab).$$

Because the exponential function is increasing, the proposition follows by applying the exponential function on both sides of the above inequality.

□

For a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{K}^n$ and p with $1 < p < \infty$, we define

$$||x||_p := \big(\sum_{j=1}^n |x_j|^p\big)^{\frac{1}{p}}.$$

2.16 Corollary (Hölder's Inequality). *For $1 < p, q < \infty$ with $1/p + 1/q = 1$ and $x, y \in \mathbb{K}^n$, we have*

$$\sum_{j=1}^n |x_j y_j| \leq ||x||_p ||y||_q.$$

We observe that the special case $p = q = 2$ is precisely the Cauchy-Schwarz inequality known from linear algebra.

Proof. W.l.o.g. let $x, y \neq 0$. Young's equality above implies

$$\frac{|x_j|}{||x||_p}\frac{|y_j|}{||y||_q} \leq \frac{1}{p}\frac{|x_j|^p}{||x||_p^p} + \frac{1}{q}\frac{|y_j|^q}{||y||_q^q}.$$

Summing up yields

$$\sum_{j=1}^n \frac{|x_j y_j|}{||x||_p ||y||_q} \leq \frac{1}{p} + \frac{1}{q} = 1,$$

which is equivalent to the assertion.

□

3 Taylor's theorem

The differential calculus, as presented previously, approximates a function, which is differentiable at a, by an affine function, i.e., we have the representation

$$f(x) = f(a) + f'(a)(x-a) + R(x)$$

of f as a sum of an affine function and an error term $R(x)$ for which

$$\lim_{x \to a} R(x)(x-a) = 0$$

holds. Now, we want to use polynomials instead of affine functions to get even more accurate approximations. More precisely, for a given n times differentiable function f, we seek a polynomial p of degree at most n such that

$$p(a) = f(a),\ p'(a) = f'(a),\ \ldots,\ p^{(n)}(a) = f^{(n)}(a). \tag{3.1}$$

Considering such a polynomial $p(x) = \sum_{j=0}^{n} a_j (x-a)^j$ we get for its coefficients $a_0, \ldots, a_n$, since $p^{(k)}(a) = k! a_k$,

$$a_k = \frac{f^k(a)}{k!}, \quad k = 0, \ldots, n.$$

That means that there exists exactly one polynomial of degree at most n for which (3.1) holds, namely

$$(T_n f)(x,a) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This motivates the following definition.

3.1 Definition. Let $I \subset \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$ an n times differentiable function and $a \in I$. Then we call $T_n f$ the *n-th Taylor polynomial (n-tes Taylorpolynom)* of f near a.

The question of how good f is approximated of course depends on the *remainder* term

$$(R_n f)(x,a) := f(x) - (T_n f)(x,a).$$

Taylor's theorem provides a conclusive answer to this question.

3.2 Theorem (Taylor's Theorem). *Let $I \subset \mathbb{R}$ be an interval, $a, x \in I$ with $a \neq x$. Let $k \in \mathbb{N}$ and $f : I \to \mathbb{R}$ be an $(n+1)$ times continuously differentiable function. Then there exists a $\xi \in (\min\{a,x\}, \max\{a,x\})$ such that*

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j + \frac{f^{(n+1)}(\xi)}{kn!}\left(\frac{x-\xi}{x-a}\right)^{n-k+1}(x-a)^{n+1}.$$

Proof. In the following, we will show that the remainder term of the approximation is given by

$$(R_n f)(x,a) = \frac{f^{(n+1)}(\xi)}{kn!}\left(\frac{x-\xi}{x-a}\right)^{n-k+1}(x-a)^{n+1}$$

To this end we define functions $g : J \to \mathbb{R}$ and $h : J \to \mathbb{R}$ by

$$g(t) := \sum_{j=0}^{n} \frac{f^{(j)}(t)}{j!}(x-t)^j, \qquad h(t) := (x-t)^k,$$

where J denotes the interval $J := (\min\{a,x\}, \max\{a,x\})$. Then we have

$$g'(t) = \sum_{j=0}^{n} \left(\frac{f^{(j+1)}(t)}{j!}(x-t)^j - \frac{f^{(j)}(t)}{j!}j(x-t)^{j-1}\right) = f^{(n+1)}(t)\frac{(x-t)^n}{n!}$$

and $h'(t) = -k(x-t)^{k-1}$ for all $t \in J$. By the generalized mean value theorem, there exists a $\xi \in J$ with

$$\frac{g(x)-g(a)}{h(x)-h(a)} = \frac{g'(\xi)}{h'(\xi)}.$$

Further, we have $g(x) - g(a) = R_n f(x,a)$ and $h(x) - h(a) = -(x-a)^k$ and, therefore,

$$R_n f(x,a) = \frac{f^{(n+1)}(\xi)}{kn!}\left(\frac{x-\xi}{x-a}\right)^{n-k+1}(x-a)^{n+1}.$$

□

If we let $k = n+1$ or $k = 1$ in the above theorem, we obtain the Lagrange form and the Cauchy form of the remainder term, respectively.

3.3 Corollary. *With the assumptions of the theorem, we have*

$$R_n f(x,a) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \quad \textit{(Lagrange form of the remainder term)}$$

and

$$R_n f(x,a) = \frac{f^{(n+1)}(\xi)}{n!}\left(\frac{x-\xi}{x-a}\right)^{n}(x-a)^{n+1} \quad \textit{(Cauchy form of the remainder term).}$$

In the following, we consider an arbitrarily often differentiable function f on an interval $J \subset \mathbb{R}$. For $a \in J$ we call

$$(Tf)(x,a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \lim_{n\to\infty}(T_n f)(x,a)$$

the *Taylor series (Taylorreihe)* of f in a.

It is now natural to ask the following questions:

a) Does the Taylor series converge, and if yes, to which value?

b) Does the Taylor series converge to f at least in a neighbourhood of a?

A first answer to question b) is given by the following Theorem:

3.4 Theorem. *Let $f : J \to \mathbb{R}$ be an arbitrarily often differentiable function and $x, a \in J$. Then we have*

$$(Tf)(x,a) = f(x) \quad \Longleftrightarrow \quad \lim_{n\to\infty} R_n f(x,a) = 0.$$

Of course, this theorem follows directly from Taylor's theorem (3.2) and the definition of convergence of a series. At first sight, the statement of this theorem seems quite banal; however there exist functions f for which $\lim\limits_{n\to\infty} R_n f(x,a)$ exists, but is not equal to 0. In this case, the Taylor series converges at the point x, but *not* to $f(x)$! In the following example, we explicitly state such a function.

3.5 Example. Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) := \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then f is arbitrarily often differentiable on $\mathbb{R}$ and we have $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}_0$ (compare with the exercises). Therefore, we have

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 \text{ for all } x \in \mathbb{R}, \text{ but } f(x) \neq 0 \text{ for } x \neq 0.$$

A sufficient criterion for the convergence of the Taylor series to f is given by the following corollary.

3.6 Corollary. *Let $f : J \to \mathbb{R}$ be an arbitrarily often differentiable function and $x, a \in J$. Assume there exists an $M > 0$ with*

$$\sup_{n\in\mathbb{N}_0} \max_{\xi\in[a,x]} |f^{(n)}(\xi)| \leq M \quad \textit{or} \quad \sup_{n\in\mathbb{N}_0} \max_{\xi\in[x,a]} |f^{(n)}(\xi)| \leq M.$$

Then we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

The proof is easy. Since we have

$$|(R_nf)(x,a)| = |\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}| \leq \frac{M|x-a|^{n+1}}{(n+1)!} \overset{n\to\infty}{\longrightarrow} 0,$$

the claim follows from 3.4.

We further exemplify the theorem using some examples.

3.7 Examples. a) We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R},$$

because the exponential function is arbitrarily often differentiable on $\mathbb{R}$, and we have $f^{(n)}(x) = e^x$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Therefore we have

$$\frac{f^{(0)}(0)}{0!} = 1, \quad \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}, \quad (T_nf)(x,0) = \sum_{j=0}^{n} \frac{x^j}{j!}$$

and furthermore

$$\max_{\xi\in[0,x]} |f^{(n)}(\xi)| = e^x, \quad \max_{\xi\in[x,0]} |f^{(n)}(\xi)| = 1, \quad x \in \mathbb{R}$$

for all $n \in \mathbb{N}$. Corollary 3.6 therefore implies the proposition.

b) For $x \in (-1,1]$, we have:

$$\log(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{x^n}{n}.$$

For the proof consider $f(x) := \log(1+x)$ for all $x > -1$. Then, f is arbitrarily often differentiable and we have

$$f^{(n)}(x) = \frac{(n-1)!(-1)^{n+1}}{(1+x)^n}, \quad \frac{f^{(0)}(0)}{0!} = 0, \quad \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}}{n}, \quad n \in \mathbb{N},$$

and therefore

$$T_nf(x,0) = \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j} x^j.$$

The Lagrange form of the remainder term for $x \in [0,1]$ is

$$R_nf(x,0) = \frac{f^{(n+1)}(\xi)x^{n+1}}{(n+1)!} = \frac{(-1)^n x^{n+1}}{(1+\xi)^{n+1}(n+1)} \quad \text{for some } \ \xi \in (0,1).$$

Therefore, we have $|R_nf(x,0)| \leq \frac{1}{n+1}|\frac{x}{1+\xi}|^{n+1} \leq \frac{1}{n+1}$, and thus

$$R_nf(x,0) \overset{n\to\infty}{\longrightarrow} 0, \quad x \in [0,1].$$

If $-1 < x < 0$, we use the Cauchy form of the remainder term to deduce that

$$R_nf(x,0) = \frac{f^{(n+1)}(\xi)}{n!}\left(\frac{x-\xi}{x}\right)^n x^{n+1} = \frac{n!(-1)^n}{(1+\xi)^{n+1}n!}x^{n+1}\left(\frac{x-\xi}{x}\right)^n,$$

and therefore

$$|R_nf(x,0)| = \frac{|x-\xi|^n}{|1+\xi|\,|1+\xi|^n}|x|.$$

If $\xi \in (x,0)$ we have $\xi - x = \xi + 1 - (x+1)$, thus $|\frac{x-\xi}{1+\xi}| = \frac{\xi-x}{1+\xi} = 1 - \frac{1+x}{1+\xi} < 1$, hence

$$|R_nf(x,0)| \overset{n\to\infty}{\longrightarrow} 0.$$

In particular, for $x = 1$ we have

$$\ln(x+1) = \ln 2 = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots,$$

which gives us an explicit value for the alternating harmonic series.

Furthermore, Taylor's theorem gives a sufficient criterion to determine local extrema.

3.8 Theorem (Sufficient criteron for local extrema). *Let $n \in \mathbb{N}$ be odd, $J \subset \mathbb{R}$ be an interval. Assume that $f : J \to \mathbb{R}$ is an $(n+1)$-times continuously differentiable function with*

$$f'(a) = \ldots = f^{(n)}(a) = 0, \quad \textit{and } f^{(n+1)}(a) \neq 0, \quad a \in J.$$

Then the following statements hold:

a) If $f^{(n+1)}(a) > 0$, then f has a local minimum at a.

b) If $f^{(n+1)}(a) < 0$, then f has a local maximum at a.

Proof. Let $f^{(n+1)}(a) > 0$. Since $f^{(n+1)}$ is continuous on J by assumption, there exists a neighbourhood $U_\delta(a) \subset J$ of a with $f^{(n+1)}(x) > 0$ for all $x \in U_\delta(a)$. Taylor's theorem with the Lagrange form of the remainder term implies that there exists a $\xi \in U_\delta(a)$ with

$$f(x) = f(a) + \frac{\overbrace{f^{(n+1)}(\xi)}^{>0}}{(n+1)!}(x-a)^{n+1} > f(a) \quad \text{for all } \; x \in U_a,$$

which is just the proposition. For the case $f^{(n+1)}(a) < 0$, the proof is analogous.

$\square$

We conclude this section by showing how to find approximations for the roots of differentiable functions. To this end we at first consider an affine approximation of f given by $F(x) = f(x_0) + f'(x_0)(x - x_0)$. Geometrically, this is the tangent to f at the point x_0. If $f'(x_0) \neq 0$, we let

$$x_1 := x_0 - \frac{f(x_0)}{f'(x_0)},$$

thus x_1 is the root of the tangent. If $x_1 \in D_f$, we proceed by the same pattern and set $x_2 := x_1 - \frac{f(x_1)}{f'(x_1)}$. More general, we define the $(n+1)$-th iteration as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots .$$

This technique to approximate a root of a given function is called *Newton's method*.

3.9 Theorem (Convergence of Newton's method). *Let $f : [a, b] \to \mathbb{R}$ be a twice continuously differentiable function and assume that*

a) f has a root ξ in $[a, b]$,

b) $f'(x) \neq 0$ for all $x \in [a, b]$,

c) f is convex or concave on $[a, b]$,

d) We have $x_0 - \frac{f(x_0)}{f'(x_0)} \in [a, b]$ for $x_0 = a$ and $x_0 = b$.

Then Newton's method converges for each $x_0 \in [a, b]$ monotonously to ξ and we have the estimate

$$|x_k - \xi| \leq \frac{M}{2m} |x_k - x_{k-1}|^2, \quad k \in \mathbb{N},$$

where $m := \min\{|f'(\tau)| : \tau \in [a, b]\}$ and $M := \max\{|f''(\tau)| : \tau \in [a, b]\}$.

The above estimate means that Newton's method has a *quadratic rate of convergence*.

Proof. We note that by b) f is strictly monotone, hence ξ is the only root of f in the interval $[a, b]$. We distinguish the four cases

$$\begin{array}{ll} f' > 0,\ f'' \geq 0 & \quad f' < 0,\ f'' \geq 0 \\ f' > 0,\ f'' \leq 0 & \quad f' < 0,\ f'' \leq 0 \end{array}$$

and prove only the first one in detail. The proof of the other cases is analogous.

We define an auxiliary function $\varphi : [a, b] \to \mathbb{R}$ by

$$\varphi(x) := x - \frac{f(x)}{f'(x)}\,.$$

Then

$$\varphi'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2} = \begin{cases} \leq 0, & x \in [a,\xi] \\ \geq 0, & x \in [\xi,b] \end{cases}$$

where the inequalities follow from the facts that f is increasing, $f(\xi) = 0$, and $f'' \geq 0$. Furthermore, $\varphi(\xi) = \xi$ is a minimum of φ in $[a,b]$. By hypothesis d), it therefore follows that $\varphi(x) \in [\xi,b]$ for all $x \in [a,b]$ and we have $\varphi(x) \leq x$ for all $x \in [\xi,b]$ (by definition of φ since we are considering the first case). We now set

$$x_{k+1} := \varphi(x_k) = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Then we have $x_1 \in [\xi,b]$ and $x_k \in [\xi,b]$ implies $\xi \leq x_{k+1} \leq x_k$. Hence, $(x_k)_{k\in\mathbb{N}}$ is a bounded decreasing sequence with a limit ω. Since in particular φ is continuous, we have by the preceding formula (when $k \to \infty$)

$$\omega = \omega - \frac{f(\omega)}{f'(\omega)} \quad \Rightarrow \quad f(\omega) = 0 \quad \Rightarrow \quad \omega = \xi\,.$$

To prove the error estimate, we use the mean value theorem and obtain

$$\left|\frac{f(x_k) - f(\xi)}{x_k - \xi}\right| \geq m,$$

what in turn implies $|x_k - \xi| \leq \frac{|f(x_k)|}{m}$. Using Taylor's theorem at the point $a = x_{k-1}$ with the Lagrange form of the remainder term, we can estimate $|f(x_k)|$ by observing

$$f(x_k) = \underbrace{f(x_{k-1}) + f'(x_{k-1})(x_k - x_{k-1})}_{=0 \text{ by Construction}} + \frac{1}{2}\, f''(\tilde{x})(x_k - x_{k-1})^2$$

for some $\tilde{x} \in (x_{k-1}, x_k)$. Therefore we have $|f(x_k)| \leq \frac{M}{2}(x_k - x_{k-1})^2$ and thus

$$|x_k - \xi| \leq \frac{M}{2m}|x_k - x_{k-1}|^2.$$

$\square$

4 Convergence of Sequences of Functions

In analysis, methods to approximate functions f by sequences $(f_n)_{n\in\mathbb{N}}$ of functions with certain, often "better" properties than f, are of central importance. Our construction of the integral in the following chapter, for example, uses such an approximation method.

We start this section by considering a sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n : D \to \mathbb{K}$ with a common domain $D \subset \mathbb{R}$. We call the sequence $(f_n)_{n\in\mathbb{N}}$ *pointwise convergent* on D , if for each (fixed) $x \in D$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges in $\mathbb{K}$. By

$$f(x) := \lim_{n\to\infty} f_n(x)$$

we can define a *limit function* $f : D \to \mathbb{K}$. It is natural to ask the following questions:

a) Are central properties of the functions f_n, such as continuity and differentiability, transferred to f?

b) If so, is it possible to compute the derivative f' of the limit function from the derivatives of the functions f_n?

If the functions f_n are continuous at $x_0 \in D$ then the limit function f is continuous at $x_0 \in D$, *if and only if* we have $\lim_{x\to x_0} f(x) = f(x_0)$, i.e. if and only if

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x).$$

The question about the continuity of limit functions therefore leads us naturally to the problem of *interchanging limits*. In the following, we show that such limits *cannot* be interchanged in general.

4.1 Examples.
a) Let $D = [0,1]$, and $f_n(x) = x^n$ for all $x \in [0,1]$ and all $n \in \mathbb{N}$. Then the functions f_n are continuous on D for all $n \in \mathbb{N}$. However the limit function f, given by

$$f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1, \end{cases}$$

is not continuous at $x = 1$.

b) Let again $D = [0,1]$ and $g_n(x) = \frac{\sin nx}{\sqrt{n}}$ for all $n \in \mathbb{N}$. The limit function is $g \equiv 0$ with derivative $g' \equiv 0$. On the other hand, we have $g_n'(x) = \sqrt{n}\cos nx$ for all $n \in \mathbb{N}$ and the sequence $g_n'(x)$ diverges at each point $x \in D$.

4.2 Definition. Let $D \subset \mathbb{R}$ be an arbitrary set and $f_n : D \to \mathbb{K}$ for all $n \in \mathbb{N}$. The sequence $(f_n)_{n\in\mathbb{N}}$ is called *uniformly convergent on D to $f : D \to \mathbb{K}$*, if for each $\varepsilon > 0$, there exists a $N_0 \in \mathbb{N}$ with

$$|f(x) - f_n(x)| < \varepsilon \quad \text{for all } x \in D, \quad n \geq N_0,$$

or reformulated,

$$(\forall\, \varepsilon > 0)\,(\exists\, N_0 \in \mathbb{N})\,(\forall\, n \geq N_0)\,(\forall\, x \in D) \quad |f(x) - f_n(x)| < \varepsilon$$

4.3 Remarks. a) Of course, a sequence (f_n) of functions that converges uniformly to f, also converges pointwise to f. The converse, however, is wrong in general.

b) If we let

$$\|f\|_\infty := \sup_{x\in D} |f(x)|,$$

then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f if and only if we have

$$\|f_n - f\|_\infty \overset{n\to\infty}{\longrightarrow} 0$$

c) For the function sequences of the above examples a) and b) we have $\|f_n - f\|_\infty = 1$ and $\|g_n - g\|_\infty = 1/\sqrt{n}$ for all $n \in \mathbb{N}$, respectively. Clearly (g_n) converges uniformly on $[0,1]$ to $g \equiv 0$.
However, the sequence $(f_n)_{n\in\mathbb{N}}$ from example a) does not converge uniformly on $[0,1]$, because otherwise, for given $\varepsilon = \frac{1}{4}$, there would exist a global N_0 such that $x^n < \frac{1}{4}$ for all $x \in [0,1)$ and all $n \geq N_0$, but $((1 + \frac{1}{n})^{-1})^n \geq 1/3$ for all $n \in \mathbb{N}$. Contradiction!

d) The difference between pointwise and uniform convergence can be described as follows: In the case of *pointwise convergence*, if we consider an $x \in D$, then for each $\varepsilon > 0$, there exists a number $N = N(\varepsilon, x)$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$. Here, the number $N(\varepsilon, x)$ may depend on x. For *uniform convergence*, there is for each $\varepsilon > 0$ a universal number $N = N(\varepsilon)$ such that for all $n > N(\varepsilon)$ and *all* $x \in D$ we have $|f_n(x) - f(x)| < \varepsilon$.

e) For $x > 0$ and $n \in \mathbb{N}$ consider $f_n(x) = \frac{1}{nx}$.This sequence of functions converges pointwise to 0. It does not converge uniformly on its domain $(0, \infty)$ (consider $x_n = 1/n > 0$), however it *does* converge uniformly on $[a, \infty)$ for each $a > 0$.

The following theorem gives – in analogy to the treatment for series – an inner criterion for the uniform convergence of a function sequence which does not presuppose the knowledge of the limit function.

4.4 Theorem (Cauchy criterion for uniform convergence). *A sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n : D \to \mathbb{K}$ converges uniformly if and only if for all $\varepsilon > 0$, there exists an $N_0 \in \mathbb{N}$ with*

$$\|f_n - f_m\|_\infty < \varepsilon \quad \textit{for all } n, m \geq N_0.$$

Proof. $\Longrightarrow$: Assume that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to the limit function f. Then there exists, to each $\varepsilon > 0$ an N_0 with $\|f_n - f\|_\infty < \frac{\varepsilon}{2}$ for all $n > N_0$. This implies

$$\|f_n - f_m\|_\infty \le \|f_n - f\| + \|f - f_m\|_\infty < \varepsilon \quad \text{for all } n, m \ge N_0.$$

$\Longleftarrow$: The assumption implies that $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$ for each $x \in D$. Since $\mathbb{K}$ is complete, there is a unique pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$.
To show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, let $\varepsilon > 0$. We have to prove the existence of an N_0 such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \ge N_0$. By assumption there is an N_0 such that $|f_n(x) - f_m(x)| < \varepsilon/2$ for all $x \in D$ and all $n, m \ge N_0$. Hence, if we let $m \to \infty$, we get – as required – for all $n > N_0$

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \le \varepsilon/2 < \varepsilon, \quad x \in D.$$

□

In the following, we consider in detail the initially posed question, under which conditions certain properties of the functions f_n, such as continuity, boundedness and differentiability, are passed on to the limit function f. We begin with the property of boundedness.

4.5 Lemma. *Let $f_n : D \to \mathbb{K}$ be bounded functions for all $n \in \mathbb{N}$. If the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on D to a function f, then f is bounded on D as well.*

Proof. For $\varepsilon = 1$, there exists an $N_1 \in \mathbb{N}$ such that $|f(x) - f_{N_1}(x)| < 1$ for all $x \in D$. By assumption, there exists furthermore a constant M_{N_1} with $|f_{N_1}(x)| \le M_{N_1}$ for all $x \in D$. Hence,

$$|f(x)| \le \underbrace{|f(x) - f_{N_1}(x)|}_{<1} + \underbrace{|f_{N_1}(x)|}_{\le M_{N_1}} \le 1 + M_{N_1} \quad \text{for all } x \in D.$$

□

The following result says that the property of continuity of an approximating sequence of functions $(f_n)_{n\in\mathbb{N}}$ is inherited to the limit function f, provided the convergence is uniform.

4.6 Theorem. [*Uniform limit of continuous functions is continuous*] *Assume that $D \subset \mathbb{R}$ and $f_n : D \to \mathbb{K}$ are continuous functions for all $n \in \mathbb{N}$. If $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f : D \to \mathbb{K}$, then f is continuous. In other words, uniform limits of continuous functions are continuous.*

Proof. Let $x_0 \in D$ and $\varepsilon > 0$. Since $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, there exists an $N_0 \in \mathbb{N}$ with $|f_{N_0}(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in D$. Furthermore, because f_{N_0} is continuous by assumption, there exists a $\delta > 0$ with

$$|f_{N_0}(x) - f_{N_0}(x_0)| < \frac{\varepsilon}{3} \quad \text{for all } x \in U_\delta(x_0) \cap D.$$

Therefore,

$$|f(x) - f(x_0)| \le \underbrace{|f(x) - f_{N_0}(x)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_{N_0}(x) - f_{N_0}(x_0)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_{N_0}(x_0) - f(x_0)|}_{<\frac{\varepsilon}{3}} \le \varepsilon$$

for all $x \in U_\delta(x_0) \cap D$.

□

The above Example 4.1 b) shows that there can't be a result analogous to Theorem 4.6 for differentiable functions, i.e. uniform limits of differentiable functions are not necessarily differentiable.

Rather, in this situation, we have to require uniform convergence of the sequence $(f_n')_{n\in\mathbb{N}}$. This is made precise in the following theorem.

4.7 Theorem. *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f_n : [a,b] \to \mathbb{K}$ be continuously differentiable functions for all $n \in \mathbb{N}$, having the following properties:*

a) The sequence $(f_n(c))_{n\in\mathbb{N}} \subset \mathbb{K}$ converges for some $c \in [a,b]$.

b) There is a function $f^* : [a,b] \to \mathbb{K}$, such that the sequence $(f_n')_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$ to f^*.

Then the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly. The limit function f is differentiable, and $f' = f^*$.

Proof. We divide the proof in three steps:
Step 1: First of all, we show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly. Indeed,

$$|f_n(x) - f_m(x)| \le |f_n(x) - f_m(x) - [f_n(c) - f_m(c)]| + |f_n(c) - f_m(c)|$$

for all $n, m \in \mathbb{N}$ and all $x \in [a,b]$. The mean value theorem applied to the first term on the right hand side yields

$$|f_n(x) - f_m(x)| \le |f_n'(\xi) - f_m'(\xi)||x - c| + |f_n(c) - f_m(c)| \quad \text{for some } \xi \in (a,b).$$

For $\varepsilon > 0$, there exists by assumptions a) and b) an N_0 with

$$\|f_n' - f_m'\|_\infty \le \frac{\varepsilon}{2(b-a)}, \quad \text{for all } n, m \ge N_0$$

and $|f_n(c) - f_m(c)| \le \frac{\varepsilon}{2}$ for all $n, m \ge N_0$. Therefore,

$$|f_n(x) - f_m(x)| \le \frac{\varepsilon}{2(b-a)}(b-a) + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all } x \in [a,b].$$

Now the claim follows from the Cauchy criterion 4.4.

Step 2: We define
$$f := \lim_{n\to\infty} f_n.$$
Then f_n converges uniformly to f by step 1. Moreover, by Theorem 4.6, the limit function f is continuous on $[a,b]$ and the same is true for $f^* := \lim_{n\to\infty} f_n'$.

Step 3: We show that the limit function f is differentiable and that we have $f' = f^*$. For this, we consider the functions $g_n : [0,1] \to \mathbb{K}$, given by
$$g_n(t) = f_n(x_0 + t(x - x_0)) - tf_n'(x_0)(x - x_0).$$
for $x, x_0 \in [a,b]$. By the mean value theorem, we have $g_n(1) - g_n(0) = g_n'(\xi)$ for some $\xi \in (0,1)$. Therefore, we have
$$g_n(1)-g_n(0) = f_n(x)-f_n(x_0)-f_n'(x_0)(x-x_0) = g_n'(\xi) = [f_n'(x_0+\xi(x-x_0))-f_n'(x_0)](x-x_0),$$
and thus, for $n \to \infty$,
$$f(x) - f(x_0) - f^*(x_0)(x - x_0) = [f^*(x_0 + \xi(x - x_0)) - f^*(x_0)](x - x_0) =: \varphi(x)(x - x_0).$$
Now f^* is continuous by step 2, and we have
$$\lim_{x\to x_0} \varphi(x) = \lim_{x\to x_0} f^*(x_0 + \xi(x - x_0)) - f^*(x_0) = 0.$$
Therefore, f is differentiable in x_0 by Theorem 1.3, and we have $f'(x_0) = f^*(x_0)$.

□

4.8 Examples.
a) Let $D = \mathbb{R}$ and define f_n as
$$f_n(x) = \begin{cases} -1, & x < -\frac{\pi}{n} \\ \sin\frac{nx}{2}, & -\frac{\pi}{n} \le x \le \frac{\pi}{n} \\ 1, & \frac{\pi}{n} < x. \end{cases}$$

Then the functions f_n are continuous for all $n \in \mathbb{N}$, but the limit function f, given by
$$f(x) = \begin{cases} +1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$
is discontinuous at $x = 0$. Therefore, the sequence $(f_n)_{n\in\mathbb{N}}$ does *not* converge uniformly to f.

b) For $n \in \mathbb{N}$, consider the functions $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = \frac{1}{n}\sin(n^2 x)$. Then the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f \equiv 0$, because we have $\sin(n^2 x) \le 1$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. Therefore, $f' \equiv 0$ as well. On the other hand, the sequence $(f_n')(x) = (n\cos(n^2 x))_{n\in\mathbb{N}}$ is divergent for all $x \in \mathbb{R}$. This means that the assumption b) of Theorem 4.7 is indispensable.

To conclude this section, we consider criteria for the uniform convergence of series of functions.

4.9 Theorem (Weierstraß M-test (Weierstraßsches Konvergenzkriterium)). *Let $f_n : D \to \mathbb{K}$ for $n \in \mathbb{N}$ be a sequence of functions with $\sum_{n=0}^{\infty} \|f_n\|_\infty < \infty$. Then the series of functions $\sum_{n=0}^{\infty} f_n$ converges uniformly, i.e., the sequence of partial sums converges uniformly.*

The proof is left as an exercise.

The above criterion has important applications for power series. In particular, we have the following corollary.

4.10 Corollary. *A power series $\sum_{n=0}^{\infty} a_n x^n$ with radius $\rho > 0$ of convergence converges absolutely and uniformly on $\overline{U_r(0)} := \{z \in \mathbb{C} : |z| \leq r\}$ for each $r \in (0, \rho)$.*

Indeed, we know from earlier results about power series that $\sum_{n=0}^{\infty} |a_n| r^n$ converges. Considering the function $f_n : \overline{U_r(0)} \to \mathbb{C}$ given by $f_n(x) := a_n x^n$, we know that $\|f_n\|_\infty \leq |a_n| r^n$ and the claim follows from the Weierstraß M-test 4.9.

An immediate consequence is that power series define continuous functions in the interior of their disc of convergence.

4.11 Corollary. *A power series with radius $\rho > 0$ of convergence defines a continuous function on $U_\rho(0)$.*

4.12 Examples. a) The series

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$$

converges absolutely and uniformly on $\mathbb{R}$, because we have $|\frac{\cos(n \cdot x)}{n^2}| \leq \frac{1}{n^2}$ for all $x \in \mathbb{R}$.

b) The *Riemann Zeta function* ζ, given by

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

converges absolutely and uniformly on the set $\{z \in \mathbb{C} : \Re z \geq \alpha\}$ when $\alpha > 1$, since $|\frac{1}{n^z}| = |\frac{1}{n^{\Re z}}| \leq \frac{1}{n^\alpha}$.

Finally we consider the question whether a function that is given by a power series is differentiable. We begin with a lemma.

4.13 Lemma. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius $\varrho > 0$ of convergence. Then the formal derivative*

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

has radius ϱ of convergence.

The proof is left as an exercise.

4.14 Theorem. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius $\rho > 0$ of convergence. Then $f : (-\varrho, \varrho) \to \mathbb{K}$ is differentiable and we have*

$$\Big(\sum_{n=0}^{\infty} a_n x^n\Big)' = f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (a_n x^n)', \quad x \in (-\varrho, \varrho),$$

i.e. power series can be differentiated termwise.

The proof is a consequence of Corollary 4.10 and Theorem 4.7.

For $|x| < 1$ consider the following example:

$$\sum_{n=1}^{\infty} n x^n = x \sum_{n=1}^{\infty} n x^{n-1} = x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = x \frac{d}{dx} \frac{1}{(1-x)} = \frac{x}{(1-x)^2}.$$

If we iterate the statement of the above theorem, we obtain the following strengthening of Theorem 4.14.

4.15 Corollary. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius $\rho > 0$ of convergence. Then $f : (-\varrho, \varrho) \to \mathbb{K}$ is arbitrarily often differentiable and we have*

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \textit{for all } n \in \mathbb{N}_0.$$

We conclude the section with Abel's theorem, which we cite without proof.

4.16 Theorem (Abel's theorem (Abelscher Grenzwertsatz)). *Let $\sum_{n=0}^{\infty} a_n$ be a convergent series. Then the power series*

$$f(x) := \sum_{n=0}^{\infty} a_n x^n$$

converges uniformly for $x \in [0,1]$ and therefore defines a continuous function $f : [0,1] \to \mathbb{K}$.

4.17 Example. We calculate the power series expansion of the arctan-function, given by

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad x \in (-1,1).$$

The series $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ converges by the Leibniz criterion. Therefore, by Abel's theorem we have

$$\frac{\pi}{4} \overset{\text{III.4}}{=} \arctan(1) \overset{\text{Abel}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Chapter V

Integration in one Variable

The calculation of areas, volumes, and lengths of curves belongs to the oldest mathematical problems. Today these questions — having remained as relevant as ever — form the central motivations for modern integration theory.

To ARCHIMEDES (287-212 B.C.), it was evident, that a figure that is bounded by curved lines has a well defined area. To determine this area, it was approximated 'from inside' and 'from outside' by easier objects with known area.

The systematic investigation of the integral concept began only much later with the discovery of the connection between differentiation and integration by G.W. LEIBNIZ (1646-1716) and I. NEWTON (1642-1727) in the 17th century. A. L. CAUCHY, in his famous textbook *Calcul infinitésimal*, was the first to point out the necessity of a definition of the integral and, building on this, of a development of the theory of integration. B. RIEMANN (1826-1866) extended this concept to a bigger class of functions. A different, but very general concept of integral was finally introduced by H. L. LEBESGUE in the year 1902. We will examine Lebesgue's integral intensively in the lecture Analysis IV.

In this chapter, we restrict our attention at first on the integral for so-called jump continuous functions, a less general class of functions than that of Riemann integrable functions. The advantage of this approach is that we can first define the integral directly for so-called *step functions* and later extend this definition — via an approximation process — to more general functions.

We begin this chapter in Section 1 by defining step functions and jump continuous functions. The approximation theorem of jump continuous functions by step functions is the main result of this section and is the basis of our approach to the integral. Section 2 is devoted to the fundamental theorem of calculus and to the comparison of of our concept of integral with the so-called Riemann integral. Afterwards, in Section 3, we consider classical integration techniques such as partial integration and substitution, before we consider improper integrals in the last section.

1 Step functions and jump continuous functions

In this section, a and b are always real numbers with $a < b$, and I denotes the compact interval $I := [a, b]$. We begin with the notion of a partition of the interval I.

1.1 Definition. a) We call $Z := (x_0, \ldots, x_n)$ a *partition of* I, if

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

holds.

b) A partition $\overline{Z} := (y_0, \ldots, y_k)$ is called *refinement of* Z , if $\{x_0, \ldots, x_n\} \subset \{y_0, \ldots, y_k\}$. If this is the case, we also write $Z \subset \overline{Z}$.

c) A function $f : I \to \mathbb{K}$ is called *step function (Treppenfunktion)*, if there exists a partition $Z = (x_0, \ldots, x_n)$ of I, such that f is constant on all intervals (x_{j-1}, x_j), where $j = 1, \ldots, n$.

d) A function $f : I \to \mathbb{K}$ is called *jump continuous (sprungstetig)* on I, if

i) f has one sided limits from the left and from the right at each $c \in (a, b)$, i.e., the limits

$$\lim_{x\to c-} f(x) \quad \text{and} \quad \lim_{x\to c+} f(x)$$

exist, and,

ii) f has a limit from the right at a and a limit from the left at b.

1.2 Remarks. a) The set of step functions

$$\mathcal{T}([a, b], \mathbb{K}) := \{\varphi : [a, b] \to \mathbb{K} \,:\, \varphi \text{ is a step function on } [a, b]\}$$

as well as the set of jump continuous functions

$$\mathcal{S}([a, b], \mathbb{K}) := \{\varphi : [a, b] \to \mathbb{K} \,:\, \varphi \text{ is jump continuous on } [a, b]\}$$

are vector spaces over $\mathbb{K}$. Further, $\mathcal{T}([a, b], \mathbb{K})$ is a linear subspace of $\mathcal{S}([a, b], \mathbb{K})$.

b) Every continuous function on $[a, b]$ is jump continuous.

c) Every monotone function on $[a, b]$ is jump continuous.

d) If we define

$$\begin{aligned}
C([a, b], \mathbb{K}) &:= \{f : [a, b] \to \mathbb{K} \,:\, f \text{ is continuous on } [a, b]\},\\
C^1([a, b], \mathbb{K}) &:= \{f : [a, b] \to \mathbb{K} \,:\, f \text{ is continuously differentiable on } [a, b]\},\\
B([a, b], \mathbb{K}) &:= \{f : [a, b] \to \mathbb{K} \,:\, f \text{ is bounded on } [a, b]\},
\end{aligned}$$

then $C([a, b], \mathbb{K})$, $C^1([a, b], \mathbb{K})$ and $B([a, b], \mathbb{K})$ are also vector spaces over $\mathbb{K}$ and we have

$$C^1([a, b], \mathbb{K}) \subset C([a, b], \mathbb{K}) \subset \mathcal{S}([a, b], \mathbb{K}) \subset B([a, b], \mathbb{K}),$$

where the inclusions are linear subspace inclusions.

We now define the integral for step functions.

1.3 Definition. Let $f : [a, b] \to \mathbb{K}$ be a step function and $Z = (x_0, \dots, x_n)$ a partition of I. Furthermore assume that $f(x) = c_j$ for all $x \in (x_{j-1}, x_j)$ and all $j = 1, \dots, n$. Then

$$\int_Z f := \sum_{j=1}^{n} c_j(x_j - x_{j-1})$$

is called the *integral of f (with respect to Z).*

First of all, we have to show that the integral $\int_Z f$ of a function f depends only on f and not on the chosen partition Z.

1.4 Lemma. *Let Z and Z' be partitions of I and assume that f is a step function with respect to Z as well as Z'. Then we have*

$$\int_Z f = \int_{Z'} f.$$

Proof. We prove the claim first for pairs of partitions partitions where one is a refinement of the other. In particular, consider the partitions $Z = (x_0, \dots, x_n)$ and $Z' = (x_0, \dots, x_k, y, x_{k+1}, \dots, x_n)$ of I. Then we have

$$\begin{aligned} \int_Z f &= \sum_{j=1}^{n} c_j(x_j - x_{j-1}) \\ &= \sum_{j=1}^{k} c_j(x_j - x_{j-1}) + \underbrace{c_{k+1}(x_{k+1} - x_k)}_{=c_{k+1}(x_{k+1}-y)+c_{k+1}(y-x_k)} + \sum_{j=k+2}^{n} c_j(x_j - x_{j-1}) = \int_{Z'} f. \end{aligned}$$

If Z' is an arbitrary refinement of Z, then the claim follows by iteration of the above argument. If Z and Z' are arbitrary partitions of I, then $Z \cup Z'$ is a refinement of Z as well as Z'. Therefore,

$$\int_Z f = \int_{Z \cup Z'} f = \int_{Z'} f.$$

□

The above Lemma implies that we can now define the integral of a step function as

$$\int_I f := \int_a^b f(x)dx := \int_I f\, dx := \int f := \int_Z f.$$

The following properties of the inegral are immediately evident.

1.5 Lemma. *Assume that $\varphi, \psi \in \mathcal{T}([a,b],\mathbb{K})$ and $\alpha, \beta \in \mathbb{K}$. Then the following propositions hold:*

a) $\int_I (\alpha\varphi + \beta\psi) = \alpha \int_I \varphi + \beta \int_I \psi$ *(Linearity of the integral).*

b) $|\int_I \varphi| = |\int_a^b \varphi(x)dx| \leq (b-a)\|\varphi\|_\infty$.

c) If φ and ψ are real valued with $\varphi \leq \psi$, we have $\int_I \varphi \leq \int_I \psi$ (Monotonicity of the integral).

In the following our aim is to extend the integral — which we have only defined for step functions up to now — to jump continuous functions in such a way that the above properties of the integral are preserved. To this end, the following approximation theorem for jump continuous functions is crucial.

1.6 Theorem. (Approximation theorem for jump continuous functions). *A function $f : [a,b] \to \mathbb{K}$ is jump continuous on $[a,b]$ if and only if there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{T}([a,b],\mathbb{K})$ of step functions on $[a,b]$ such that $(\varphi_n)_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$ to f, i.e. if $\|f-\varphi_n\|_\infty \overset{n\to\infty}{\longrightarrow} 0$ holds.*

Proof. $\Longrightarrow$: Let $f \in \mathcal{S}([a,b],\mathbb{K})$ be a jump continuous function and $n \in \mathbb{N}$. Then for all $x \in I = [a,b]$ there exist real numbers α_x and β_x with $\alpha_x < x < \beta_x$ and

$$|f(s) - f(t)| < \frac{1}{n}, \quad s,t \in (\alpha_x, x) \cap I \ \text{ or } \ s,t \in (x, \beta_x) \cap I.$$

Now the set $\{(\alpha_x, \beta_x) : x \in I\}$ is an open cover of the compact interval $[a,b]$. Therefore there exists a finite subcover of I, i.e. there exist $x_0 < x_1 < \ldots < x_m$ with $I \subset \bigcup_{j=0}^m (\alpha_{x_j}, \beta_{x_j})$. If we set $y_0 := a$, $y_{j+1} := x_j$ for $j = 0, \ldots, m$, as well as $y_{m+2} := b$, we obtain a partition $Z_0 = (y_0, \ldots, y_{m+2})$ of I. Now we choose a refinement $Z_1 = (z_0, \ldots, z_k)$ of Z_0 with

$$|f(s) - f(t)| < \frac{1}{n}, \quad s,t \in (z_{j-1}, z_j), \quad j = 1, \ldots, k$$

(from the construction of Z_0 it follows that this property can be obtained by inserting at most one additional point between each pair (y_{j-1}, y_j) of points of Z_0) and define the function φ_n approximating the function f by

$$\varphi_n(x) := \begin{cases} f(x), & x \in \{z_0, \ldots, z_k\} \\ f(\frac{z_{j-1}+z_j}{2}), & x \in (z_{j-1}, z_j),\ j = 1, \ldots, k. \end{cases}$$

Then φ_n is a step function on $[a,b]$ for each $n \in \mathbb{N}$ and by construction we have $|f(x) - \varphi_n(x)| < \frac{1}{n}$ for all $x \in I$, i.e. we have $\|f - \varphi_n\|_\infty < \frac{1}{n}$ for all $n \in \mathbb{N}$.

$\Longleftarrow$: By assumption, $\varphi_n \in \mathcal{T}([a,b],\mathbb{K})$ and we have $\|\varphi_n - f\|_\infty < \frac{1}{n}$ for all $n \in \mathbb{N}$. We have to show that f is jump continuous. For $\varepsilon > 0$ we choose $n \in \mathbb{N}$ in such a way that we have $|f(x) - \varphi_n(x)| < \frac{\varepsilon}{2}$ for all $x \in I$. Further, since φ_n is a step function, there exists for all $x \in (a,b]$ an $a' \in [a,x)$ with $\varphi_n(s) = \varphi_n(t)$ for all $s,t \in (a',x)$. Therefore we have

$$|f(s) - f(t)| \leq |f(s) - \varphi_n(s)| + |\varphi_n(t) - f(t)| < \varepsilon, \quad \text{for all } s,t \in (a',x).$$

Now assume that $(s_j)_{j\in\mathbb{N}} \subset I$ is such that $s_j \to x-$. Then there exists an $N \in \mathbb{N}$ such that $s_j \in (a',x)$ for all $j \geq N$ and thus

$$|f(s_j) - f(s_k)| < \varepsilon, \quad \text{for all } j,k \geq N.$$

Thus, $(f(s_j))_{j\in\mathbb{N}}$ is a Cauchy sequence with $\lim_{j\to\infty} f(s_j) = r$. If $(t_k)_{k\in\mathbb{N}}$ is another sequence as above, we have $\lim_{n\to\infty} f(t_n) = r'$. But since $|f(s_j) - f(t_k)| < \varepsilon$ for all $j,k > N$, we have $r = r'$, and thus the limit from the left, $\lim_{y\to x-} f(y)$, exists.
The proof for the limit from the right is analogous.

$\square$

1.7 Corollary. *A function $f : [a,b] \to \mathbb{K}$ is jump continuous if and only if it can be written as*

$$f = \sum_{n=1}^{\infty} \varphi_n \quad \textit{with} \quad \varphi_n \in \mathcal{T}([a,b],\mathbb{K}) \quad \textit{such that} \quad \sum_{n=1}^{\infty} \|\varphi_n\|_\infty < \infty \textit{ holds.}$$

Proof. $\Longrightarrow$: By the above Theorem 1.6 we can choose a function $\psi_n \in \mathcal{T}([a,b],\mathbb{K})$ for each $n \in \mathbb{N}$ in such a way that $\|f - \psi_n\|_\infty \leq \frac{1}{2^n}$. If we further set $\varphi_1 := \psi_1$ and $\varphi_k := \psi_k - \psi_{k-1}$ for $k \geq 2$, we have

$$|f(x) - \sum_{j=1}^{n} \varphi_j(x)| = |f(x) - \psi_n(x)| \leq \|f - \psi_n\|_\infty \leq \frac{1}{2^n},$$

and therefore $\sum_{j=1}^{\infty} \varphi_j(x) = f(x)$ for all $x \in [a,b]$. Further we have

$$\|\varphi_j\|_\infty \leq \underbrace{\|\psi_j - f\|_\infty}_{\leq \frac{1}{2^j}} + \underbrace{\|f - \psi_{j-1}\|_\infty}_{\leq \frac{1}{2^{j-1}}} = \frac{3}{2^j},$$

and thus $\sum_{n=1}^{\infty} \|\varphi_n\|_\infty < \infty$.
$\Longleftarrow$: For $n \in \mathbb{N}$ we define $\psi_n := \sum\limits_{j=1}^{n} \varphi_j$. Then $\psi_n \in \mathcal{T}([a,b],\mathbb{K})$ for all $n \in \mathbb{N}$ and we have

$$\|f - \psi_n\|_\infty = \|f - \sum_{j=1}^{n} \varphi_j\|_\infty = \|\sum_{j=n+1}^{\infty} \varphi_j\|_\infty \leq \sum_{j=n+1}^{\infty} \|\varphi_j\|_\infty \overset{n\to\infty}{\longrightarrow} 0.$$

Thus, the claim follows from Theorem 1.6

□

1.8 Corollary. *A jump continuous function $f \in \mathcal{S}([a,b],\mathbb{K})$ has at most countably many points of discontinuity. This holds in particular for monotone functions.*

Proof. By the above Theorem 1.6, we can express f as a limit of a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of step functions. By unfolding the ε-δ-definition of continuity, and applying a $\frac{\varepsilon}{3}$-argument, it is easy to see that f is continuous at a given x whenever all φ_n are continuous at x. Thus, the points of discontinuity of f are contained in the union of the sets of points of discontinuity of all φ_n. This is a countable union of finite sets, hence at most countable.

□

2 The integral and its properties

In this section, let again $a, b \in \mathbb{R}$ with $a < b$ and $I = [a, b]$. We consider the following situation: Let $f \in \mathcal{S}([a, b], \mathbb{K})$ be a jump continuous function which is approximated uniformly by a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{T}([a, b], \mathbb{K})$ of step functions, as described in Theorem 1.6; i.e. we have $\|f - \varphi_n\|_\infty \overset{n\to\infty}{\longrightarrow} 0$. If we set $I_n := \int_a^b \varphi_n$, we have

$$|I_n - I_m| \overset{1.5b)}{\leq} (b - a)\|\varphi_n - \varphi_m\|_\infty \leq (b - a)(\|\varphi_n - f\|_\infty + \|f - \varphi_m\|_\infty) \overset{n\to\infty}{\longrightarrow} 0,$$

i.e., $(I_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and thus convergent. Let further $(\psi_n)_{n\in\mathbb{N}} \subset \mathcal{T}([a, b], \mathbb{K})$ be another sequence of step functions with $\|\psi_n - f\|_\infty \overset{n\to\infty}{\longrightarrow} 0$. If we consider the sequence $\varphi_1, \psi_1, \varphi_2, \psi_2, \ldots =: (g_n)_{n\in\mathbb{N}}$ of step functions, we have $\|f - g_n\|_\infty \overset{n\to\infty}{\longrightarrow} 0$. Thus, the sequence $(\int_a^b g_n)_{n\in\mathbb{N}}$ converges and the subsequences $(\int_a^b \varphi_n)_{n\in\mathbb{N}}$ and $(\int_a^b \psi_n)_{n\in\mathbb{N}}$ have the same limit. These considerations show that we have the following result.

2.1 Theorem and Definition. *Let $f \in \mathcal{S}([a, b], \mathbb{K})$ and $\varphi_n \in \mathcal{T}([a, b], \mathbb{K})$ for all $n \in \mathbb{N}$ with $\|f - \varphi_n\|_\infty \overset{n\to\infty}{\longrightarrow} 0$. Then the limit*

$$\lim_{n\to\infty} \int_a^b \varphi_n(x)\,dx =: \int_a^b f(x)\,dx.$$

exists and is independent of the choice of φ_n. This limit is called the integral of f on $[a, b]$.

In the following, we also use the notations $\int f$, $\int_I f$ or $\int_I f(x)\,dx$ for the integral of a jump continuous function f. Since continuous functions and monotone functions are jump continuous, the following corollary is immediately evident:

2.2 Corollary. *The integral $\int_a^b f(x)dx$ exists for every continuous and every monotone (real-valued) function f on $[a, b]$.*

On the other hand, we remark that not every function on $[a, b]$ is integrable. A counterexample is the Dirichlet function already known from Chapter III. More precisely, the integral of the function f, given by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0, 1] \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \cap [0, 1] \end{cases}$$

does *not* exist.

2.3 Theorem. *Let $\alpha, \beta \in \mathbb{K}$ and $f, g \in \mathcal{S}([a, b], \mathbb{K})$. Then we have* *a)* $\int_a^b(\alpha f+\beta g)=\alpha\int_a^b f+\beta\int_a^b g$ *(Linearity of the integral).*

b) $|f|\in\mathcal{S}([a,b],\mathbb{R})$ *and*

$$|\int_a^b f|\le\int_a^b|f|\le(b-a)\|f\|_\infty.$$

c) If we have $f\le g$*, i.e.* $f(x)\le g(x)$ *for all* $x\in[a,b]$*, we also have*

$$\int_a^b f\le\int_a^b g,\quad \textit{(Monotonicity of the integral).}$$

Proof. Assume that $\varphi_n,\psi_n\in\mathcal{T}([a,b],\mathbb{K})$ are step functions for all $n\in\mathbb{N}$, and $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ converge uniformly to f and g, respectively.

Then $(\alpha\varphi_n+\beta\psi_n)$ converges uniformly to $\alpha f+\beta g$ and we have

$$\int_a^b(\alpha f+\beta g)=\lim_{n\to\infty}(\int_a^b(\alpha\varphi_n+\beta\psi_n))=$$
$$\alpha(\lim_{n\to\infty}\int_a^b\varphi_n)+\beta(\lim_{n\to\infty}\int_a^b\psi_n)=\alpha\int_a^b f+\beta\int_a^b g,$$

which is claim a).
b) Since the sequence $(|\varphi_n|)_{n\in\mathbb{N}}$ converges uniformly to $|f|$, and since $|f|\in\mathcal{S}([a,b],\mathbb{R})$ (compare Theorem 1.6), it follows that $\int|f|\overset{\text{Thm.2.1}}{=}\lim_{n\to\infty}\int|\varphi_n|$. Thus,

$$|\int f|=|\lim_{n\to\infty}\int\varphi_n|=\lim_{n\to\infty}|\int\varphi_n|\le\underbrace{\lim_{n\to\infty}\int|\varphi_n|}_{\int|f|}\le\lim_{n\to\infty}\|\varphi_n\|_\infty|b-a|=\|f\|_\infty(b-a).$$

c) Assume that φ_n and ψ_n are real valued step functions on $[a,b]$. Then $\Phi_n:=\varphi_n-\|f-\varphi_n\|_\infty$ and $\Psi_n:=\psi_n+\|g-\psi_n\|_\infty$ are also step functions on $[a,b]$ with $\Phi_n\le f\le g\le\Psi_n$ and $(\Phi_n)_{n\in\mathbb{N}}$ and $(\Psi_n)_{n\in\mathbb{N}}$ converge uniformly to f and g, respectively. Thus, we have

$$\int_a^b f=\lim_{n\to\infty}\int_a^b\Phi_n\le\lim_{n\to\infty}\int_a^b\Psi_n=\int_a^b g.$$

□

We now consider a jump continuous function $f\in\mathcal{S}([a,b],\mathbb{K})$, real numbers $c,d\in[a,b]$ and define

$$\int_c^d f:=\int_c^d f(x)dx:=\begin{cases}\int_{[c,d]}f, & c<d\\ 0, & c=d\\ -\int_{[d,c]}f, & d<c.\end{cases}$$

In particular, we have

$$\int_c^d f = -\int_d^c f.$$

2.4 Lemma. (Additivity of the integral). *Let $f \in \mathcal{S}([a,b],\mathbb{K})$ and $c \in [a,b]$. Then we have*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. Let $a \le c \le b$. Then the claim is obviously true for all step functions $f \in \mathcal{T}([a,b],\mathbb{K})$. Therefore, we consider a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{T}([a,b],\mathbb{K})$ which converges uniformly on $[a,b]$ to f. Then $\varphi_n|_J \in \mathcal{T}(J,\mathbb{K})$ and $(\varphi_n|_J)_{n\in\mathbb{N}}$ converges uniformly to $f|_J$ for each compact subinterval J of $[a,b]$. Since $\int_a^b \varphi_n = \int_a^c \varphi_n + \int_c^b \varphi_n$, it follows that $\int_a^b f = \int_a^c f + \int_c^b f$.

□

2.5 Lemma. *Let $f \in \mathcal{S}([a,b],\mathbb{R})$ be a jump continuous function with $f(x) \ge 0$ for all $x \in [a,b]$. If f is continuous at $c \in [a,b]$ and $f(c) > 0$, it follows that $\int_a^b f > 0$.*

Proof. First, let $a < c < b$. Since f in continuous at c by assumption, there exists a $\delta > 0$ with $[c-\delta, c+\delta] \subset [a,b]$ and

$$f(x) \ge \frac{1}{2} f(c), \qquad \text{for all } x \in [c-\delta, c+\delta].$$

Since $f \ge 0$, the monotonicity of the integral (Theorem 2.3) implies that $\int_a^{c-\delta} f \ge 0$ and $\int_{c+\delta}^b f \ge 0$. Therefore we have

$$\int_a^b f \overset{2.4}{=} \int_a^{c-\delta} f + \int_{c-\delta}^{c+\delta} f + \int_{c+\delta}^b f \ge \int_{c-\delta}^{c+\delta} f \ge \frac{1}{2} f(c) \int_{c-\delta}^{c+\delta} 1 = \delta f(c) > 0.$$

The proof for the cases $c = a$ and $c = b$ is similar.

□

2.6 Theorem (Mean value theorem for integrals (Mittelwertsatz für das Integral)). *Let $f \in C([a,b],\mathbb{R}), \varphi \in \mathcal{S}([a,b],\mathbb{R})$ with f being real valued and $\varphi \ge 0$. Then there exists a point $\xi \in [a,b]$ with*

$$\int_a^b f(x)\varphi(x)\,dx = f(\xi) \int_a^b \varphi(x)\,dx.$$

Proof. Because f is continuous on a compact interval, there exist $m, M \in [a,b]$ such that $f(m) = \min_{x\in[a,b]} f(x)$ and $f(M) = \max_{x\in[a,b]} f(x)$. Since $\varphi \geq 0$, we have

$$f(m)\varphi(x) \leq f(x)\varphi(x) \leq f(M)\varphi(x),$$

and by the monotonicity of the integral

$$f(m)\int_a^b \varphi(x)\,dx \leq \int_a^b f(x)\varphi(x)\,dx \leq f(M)\int_a^b \varphi(x)\,dx.$$

Now the function $g(t) := f(t)\int_a^b \varphi(x)\,dx$ is continuous, whence by the intermediate value theorem there exists a ξ between m and M such that $f(\xi)\int_a^b \varphi(x)dx = \int_a^b f(x)\varphi(x)dx$. This is precisely the claim.

□

If we consider the above theorem for the particular case $\varphi \equiv 1$, we obtain the following corollary.

2.7 Corollary. *To each $f \in C([a,b],\mathbb{R})$ there exists a point $\xi \in [a,b]$ such that*

$$\int_a^b f(x)\,dx = f(\xi)(b-a).$$

Now, for $f \in \mathcal{S}([a,b],\mathbb{K})$ we consider the mapping

$$F : [a,b] \to \mathbb{K}, \quad F(x) := \int_a^x f(s)\,ds.$$

Then the additivity of the integral implies

$$F(x) - F(y) = \int_a^x f(s)\,ds - \int_a^y f(s)\,ds = \int_y^x f(s)\,ds, \quad \text{for all } x, y \in [a,b].$$

Now Theorem 2.3 b) immediately implies the estimate

$$|F(x) - F(y)| \leq \|f\|_\infty |x-y|, \quad x, y \in [a,b].$$

2.8 Theorem. (Differentiability of the integral by the upper bound). *Assume that $f \in \mathcal{S}([a,b],\mathbb{K})$ is continuous at $c \in [a,b]$ and let $F : [a,b] \to \mathbb{K}$ be defined by*

$$F(x) := \int_a^x f(s)\,ds.$$

Then F is differentiable in c and we have $F'(c) = f(c)$.

Proof. Let $h \neq 0$ such that $c+h \in [a,b]$. Then we have

$$\frac{F(c+h)-F(c)}{h} = \frac{1}{h}\left(\int_a^{c+h} f(s)\,ds - \int_a^c f(s)\,ds\right) \overset{\text{Add.}}{=} \frac{1}{h}\int_c^{c+h} f(s)\,ds.$$

Since $\int_c^{c+h} f(c)\,ds = f(c)h$, we have

$$\frac{F(c+h)-F(c)-f(c)h}{h} = \frac{1}{h}\int_c^{c+h} (f(s)-f(c))\,ds,$$

and thus

$$\left|\frac{F(c+h)-F(c)-f(c)h}{h}\right| \leq \frac{1}{|h|}\int_c^{c+h} |f(s)-f(c)|\,ds \leq \sup_{s\in[c,c+h]} |f(s)-f(c)| \xrightarrow{h\to 0} 0,$$

since f is continuous in c. Therefore, F is differentiable in c and we have $F'(c) = f(c)$. $\square$

We summarize our previous considerations in the following *fundamental theorem of calculus.*

2.9 Theorem. (Fundamental Theorem of calculus (Hauptsatz der Differential- und Integralrechnung)). *Let $f : [a,b] \to \mathbb{K}$ be a continuous function and for $c \in [a,b]$ let*

$$F(x) := \int_c^x f(s)\,ds, \quad x \in [a,b].$$

Then we have

a) F is differentiable for all $x \in [a,b]$ and we have $F'(x) = f(x)$ for all $x \in [a,b]$.

b) If $\phi : [a,b] \to \mathbb{K}$ is a differentiable function with $\phi'(x) = f(x)$ for all $x \in [a,b]$, we have

$$\phi(x) = \phi(y) + \int_y^x f(s)\,ds, \quad x,y \in [a,b].$$

Proof. Claim a) follows directly from Theorem 2.8. To show claim b), let F and ϕ as in the assumption. Then we have $(F-\phi)' = 0$, thus $F = \phi + \alpha$ for a constant $\alpha \in \mathbb{C}$. Therefore,

$$\int_y^x f(s)\,ds = F(x) - F(y) = \phi(x) + \alpha - \phi(y) - \alpha = \phi(x) - \phi(y).$$

$\square$

2.10 Definition.
Let $f \in \mathcal{S}([a,b],\mathbb{K})$. A differentiable function $F:[a,b]\to\mathbb{K}$ with $F'(x)=f(x)$ for all $x\in[a,b]$ is called *antiderivative (Stammfunktion) of f* .

The above fundamental theorem of calculus implies the following corollary.

2.11 Corollary. *Every continuous function $f:[a,b]\to\mathbb{K}$ has an antiderivative F and we have:*

$$\int_y^x f(s)\,ds = F(x)-F(y) =: F\Big|_y^x, \quad x,y\in[a,b].$$

Thus, the above corollary guarantees the existence of antiderivatives for continuous functions. However, we remark that in the most cases, it is not possible to give an explicit definition of antiderivatives.

2.12 Examples. a) In the following table, we collect examples of functions f for which antiderivatives F can be given explicitly.

$f(x)$	$F(x)$
x^a	$\frac{x^{a+1}}{a+1}$, $a\neq -1$
$\frac{1}{x}$	$\log\lvert x\rvert$
e^x	e^x
$\cos x$	$\sin x$
$\frac{1}{\cos^2 x}$	$\tan x$
$\frac{1}{1+x^2}$	$\arctan x$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$

b) If $f:(a,b)\to\mathbb{R}$ is differentiable and $f(x)\neq 0$ for all $x\in(a,b)$, we have

$$\int \frac{f'}{f} = \log|f|.$$

In analogy to the previous section we now consider a sequence of jump continuous functions $(f_n)_{n\in\mathbb{N}}$, which converge uniformly to a function f on $[a,b]$, and ask whether f is in turn integrable (i.e. jump continuous). The answer is given by the following theorem.

2.13 Theorem. *Let $(f_n)_{n\in\mathbb{N}}\subset\mathcal{S}([a,b],\mathbb{K})$ be a sequence of jump continuous functions which converge uniformly to f on $[a,b]$. Then $f\in\mathcal{S}([a,b],\mathbb{K})$ and we have*

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. For given $\varepsilon > 0$ we choose $n \in \mathbb{N}$ so big, that $\|f - f_n\|_\infty \le \varepsilon/2$, and for given f_n, we choose a step function φ with $\|f_n - \varphi\|_\infty \le \varepsilon/2$. Then we have $\|f - \varphi\| \le \varepsilon$ and thus $f \in \mathcal{S}([a,b],\mathbb{K})$. Furthermore, we have

$$\Big|\int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx\Big| \le \|f - f_n\|(b-a) \le \varepsilon(b-a),$$

and this is the proposition.

□

2.14 Remark. The above Theorem 2.13 allows to give an easy and elegant proof of Theorem IV.4.7. First of all, the limit function $f^* = \lim_{n\to\infty} f_n'$ of the derivatives is continuous on $[a,b]$ by Theorem IV.4.6. For fixed $a \in I$ and arbitrary $x \in I$ we have

$$f_n(x) = f_n(a) + \int_a^x f_n'(t)\,dt,$$

and thus, by Theorem 2.13, we have

$$f(x) = f(a) + \int_a^x f^*(t)\,dt$$

for $n \to \infty$. By the fundamental theorem of calculus, f is differentiable and we have $f'(x) = f^*(x) = \lim_{n\to\infty} f_n'(x)$.

In the following, we consider the approximation of the integral by so-called *Riemann sums*

2.15 Definition. Assume that $f : [a,b] \to \mathbb{K}$ is a function, $Z := (x_0, \dots, x_n)$ is a partition of the interval $[a,b]$ and $\xi_j \in [x_{j-1}, x_j]$ for $j \in \{1,\dots,n\}$. Then

$$\sum_{j=1}^n f(\xi_j)(x_j - x_{j-1})$$

is called the *Riemann sum (Riemann Summe)* of f with respect to Z. The norm (Feinheit) of the partition Z is defined as $\|Z\| := \max_{1\le j\le n}(x_j - x_{j-1})$.

We have the following theorem.

2.16 Theorem. *Let $f \in \mathcal{S}([a,b],\mathbb{K})$ be a jump continuous function. Then to each $\varepsilon > 0$ there exists some $\delta > 0$, such that for every partition Z of $[a,b]$ with norm $\|Z\| < \delta$ and every choice of points $\xi_j \in [x_{j-1}, x_j]$ we have*

$$\left|\sum_{j=1}^n f(\xi_j)(x_j - x_{j-1}) - \int_a^b f(x)\,dx\right| < \varepsilon.$$

Proof. First, we show the claim for step functions φ via induction on the number m of discontinuities of φ. Then we deduce the claim for general jump continuous functions using the approximation theorem (1.6).
a) Let $\varphi \in \mathcal{T}([a,b],\mathbb{K})$ be a step function and $\varepsilon > 0$. If we have $\varphi = c$ for all $x \in [a,b]$ and a $c \in \mathbb{K}$, the claim follows immediately. If φ has exactly one discontinuous point, the claim follows easily by setting $\delta := \frac{\varepsilon}{4\|\varphi\|}$.
For the induction step assume that the proposition holds for step functions with m discontinuities and consider a step function φ with $m+1$ discontinuous points. We then decompose φ into $\varphi = \varphi' + \varphi''$, where φ' is a step function with m discontinuities and φ'' is a step function with exactly one discontinuities. For a given $\varepsilon > 0$ we choose a $\delta'(\varepsilon/2)$ for φ' and a $\delta''(\varepsilon/2)$ for φ'' in such a way that the proposition holds for φ' and φ''; if we then set $\delta = \min(\delta', \delta'')$ the proposition also holds for φ.
b) For $f \in \mathcal{S}([a,b],\mathbb{K})$ choose $\varphi \in \mathcal{T}([a,b],\mathbb{K})$ with $\|f-\varphi\|_\infty < \frac{\varepsilon}{3(b-a)}$ and $\delta := \delta(\frac{\varepsilon}{3})$. By a), we have $|\sum_{j=1}^n \varphi(\xi_j)(x_j - x_{j-1}) - \int_a^b \varphi dx| < \frac{\varepsilon}{3}$; therefore,

$$\begin{aligned}
\Big|\sum_{j=1}^n f(\xi_j)(x_j - x_{j-1}) - \int_a^b f\,dx\Big| &\leq \Big|\sum_{j=1}^n f(\xi_j)(x_j - x_{j-1}) - \sum_{j=1}^n \varphi(\xi_j)(x_j - x_{j-1})\Big| \\
&\quad + \underbrace{\Big|\sum_{j=1}^n \varphi(\xi_j)(x_j - x_{j-1}) - \int_a^b \varphi\,dx\Big|}_{<\frac{\varepsilon}{3}} \\
&\quad + \underbrace{\Big|\int_a^b \varphi\,dx - \int_a^b f\,dx\Big|}_{<\frac{\varepsilon}{3}} \\
&< \sum_{j=1}^n \|f-\varphi\|_\infty (x_j - x_{j-1}) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon
\end{aligned}$$

□

2.17 Corollary. *Let $Z_1, Z_2 \ldots,$ be a sequence of partitions of the interval $[a,b]$ with $\|Z_n\| \to 0$, $n \to \infty$. Let $f \in \mathcal{S}([a,b],\mathbb{K})$ and S_n be the corresponding sequence of Riemann sums. Then we have*

$$\lim_{n\to\infty} S_n = \int_a^b f.$$

2.18 Remarks. a) A function $f : [a,b] \to \mathbb{C}$ is called *Riemann integrable* if there exists a $c \in \mathbb{C}$ with the following property: To each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\Big|c - \sum_{j=1}^n f(\xi_j)(x_j - x_{j-1})\Big| < \varepsilon$$

for every partition $Z := (x_0, \ldots, x_n)$ with norm $\|Z\| < \delta$ and every choice of $\xi_j \in [x_{j-1}, x_j]$.
b) The above Theorem 2.16 says that every jump continuous function $f \in \mathcal{S}([a,b],\mathbb{K})$ is Riemann integrable and the Riemann integral coincides with our integral for these functions.
c) There exist Riemann-integrable functions which are not jump continuous.

The above Corollary 2.17 allows in many cases to transfer statements about sums to integrals. As an example consider the Hölder inequality for integrals. To this end define

$$\|f\|_p := \Big(\int_a^b |f(x)|^p \, dx\Big)^{1/p}$$

for $f \in \mathcal{S}([a,b],\mathbb{K})$ and $1 < p < \infty$. Then the following inequality holds.

2.19 Corollary. *For $f, g \in \mathcal{S}([a,b],\mathbb{K})$ and $1 < p, q < \infty$ we have*

$$\int_a^b |f(x)g(x)| \, dx \le \|f\|_p \|g\|_q \,, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

For $p = q = 2$, this is the *Cauchy-Schwarz inequality* for integrals.

3 Integration techniques

The fundamental theorem of calculus from the previous section allows to transfrom the product rule and the substitution rule from differential calculus into very useful integration techniques. We start this relatively short section with the substitution rule. In the entire section, let $I \subset \mathbb{R}$ be a compact interval and $a, b \in \mathbb{R}$ with $a < b$.

3.1 Theorem (Substitution rule). (*Substitutionsregel*) *Let $f \in C(I, \mathbb{K})$ and $\varphi \in C^1([a,b], \mathbb{R})$ with $\varphi([a,b]) \subset I$. Then we have*

$$\int_a^b f(\varphi(x))\varphi'(x)\,dx = \int_{\varphi(a)}^{\varphi(b)} f(y)\,dy.$$

Proof. By the fundamental theorem, f has an antiderivative $F \in C^1(I, \mathbb{K})$. The chain rule implies that $F \circ \varphi \in C^1([a,b], \mathbb{K})$ and that

$$(F \circ \varphi)'(x) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x), \quad x \in [a,b]\,.$$

Therefore,

$$\int\limits_a^b f(\varphi(x))\varphi'(x)\,dx = (F \circ \varphi)\big|_a^b = F(\varphi(b)) - F(\varphi(a)) = F\big|_{\varphi(a)}^{\varphi(b)} = \int_{\varphi(a)}^{\varphi(b)} f(y)\,dy.$$

□

3.2 Examples. a) For $\alpha > 0$ and $\beta \in \mathbb{R}$ we have

$$\int_a^b \cos(\alpha x + \beta)\,dx = \frac{1}{\alpha}\int_{\alpha a+\beta}^{\alpha b+\beta} \cos u\,du = \frac{1}{\alpha}\sin\big|_{\alpha a+\beta}^{\alpha b+\beta} = \frac{1}{\alpha}(\sin(\alpha b + \beta) - \sin(\alpha a + \beta)).$$

b) We have

$$\int_0^1 x^{n-1}\sin(x^n)\,dx = \frac{1}{n}\int_0^1 \sin u\,du = -\frac{\cos u}{n}\Big|_0^1 = \frac{1}{n}(1 - \cos 1).$$

3.3 Theorem (Integration by parts). (*Partielle Integration*) *For functions $f, g \in C^1([a,b], \mathbb{K})$ we have*

$$\int_a^b fg'\,dx = (fg)\big|_a^b - \int_a^b f'g\,dx\,.$$

The proof is easy. By the product rule, we have $(fg)' = f'g + fg'$; and thus

$$\int_a^b (fg)'dx = \int_a^b f'gdx + \int fg'dx.$$

$\square$

3.4 Examples. a) We have

$$\int_a^b xe^x dx = xe^x\Big|_a^b - \int_a^b e^x dx = be^b - ae^a - [e^b - e^a].$$

b) We identify a recursion formula $I_n = \int \sin^n x dx$ for $n \geq 2$ as follows: We have

$$\begin{aligned} I_n &= \int \sin x \cdot \sin^{n-1} x dx = -\cos x \sin^{n-1}(x) + (n-1) \int \cos x \, \sin^{n-2} x \cos x dx \\ &= -\cos x \sin^{n-1} x + (n-1) \int (1 - \sin^2 x) \sin^{n-2} x dx \\ &= -\cos x \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n \end{aligned}$$

and thus

$$I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n} \cos x \sin^{n-1} x,$$

where $I_0 = \int \sin^0 x = \int 1dx = x$ and $I_1 = \int \sin x = -\cos x$.

c) *Wallis' product* (*Wallissches Produkt* – cf. exercises): We have

$$\frac{\pi}{2} = \prod_{j=1}^{\infty} \frac{4j^2}{4j^2 - 1}.$$

For the proof, consider $A_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$.

3.5 Example. Area of the unit circle

Consider the function $f : [-1, 1] \to \mathbb{R}$ given by $x \mapsto \sqrt{1 - x^2}$. If we define $A = \int_{-1}^1 \sqrt{1 - x^2} \, dx$ and substitute $x = \cos t$, we obtain by the example in (b)

$$A = -\int_\pi^0 \sqrt{1 - \cos^2 t} \, \sin t \, dt = \int_0^\pi \sin^2 t \, dt = \frac{1}{2} \int_0^\pi dt - \frac{1}{2} \sin t \cos t\Big|_0^\pi = \frac{\pi}{2}.$$

Hence, the area of the unit circle is $2 \cdot \frac{\pi}{2} = \pi$.

4 Improper integrals

With our current concept of integral, we can integrate jump continuous functions which are defined on a compact interval $I = [a, b]$.

In this section, we want to extend this concept of integral in order to integrate functions on arbitrary (not necessarily compact) intervals of the real line. This leads to the concept of *improper integrals*.

In the entire section we assume $-\infty \le a < b \le \infty$. We call a function $f : (a, b) \to \mathbb{C}$ *admissible*, , if the restriction of f to each compact subinterval of (a, b) is jump continuous. It is clear that a continuous function $f : (a, b) \to \mathbb{K}$ is admissible; likewise $f \in \mathcal{S}((a, b), \mathbb{K})$ is admissible if $a, b \in \mathbb{R}$, and $|f|$ is admissible if $f : (a, b) \to \mathbb{K}$ is admissible.

4.1 Definition. An admissible function $f : (a, b) \to \mathbb{C}$ is called *improperly integrable*, if there exists a point $c \in (a, b)$ such that the limits

$$\lim_{\alpha \to a+} \int_\alpha^c f \quad \text{and} \quad \lim_{\beta \to b-} \int_c^\beta f$$

exist.

We remark at this point that for an improperly integrable function f the above limits exist for all $c \in (a, b)$.

4.2 Definition. Assume that $f : (a, b) \to \mathbb{K}$ is improperly integrable and $c \in (a, b)$. Then

$$\int_a^b f\,dx := \int_a^b f(x)\,dx := \lim_{\alpha \to\, a+} \int_\alpha^c f\,dx + \lim_{\beta \to\, b-} \int_c^\beta f\,dx$$

is called the *improper integral* of f over (a, b).

4.3 Examples. a) For $\alpha \in \mathbb{R}$ we have

$$\int_1^\infty \frac{1}{x^\alpha}\,dx \;\text{ exists} \quad \Leftrightarrow \quad \alpha > 1.$$

To see this, we choose $\alpha \neq 1$. Then we have

$$\int_1^b \frac{1}{x^\alpha}\,dx = \frac{1}{1-\alpha} x^{1-\alpha}\Big|_1^b = \frac{1}{1-\alpha}(b^{1-\alpha} - 1),$$

and the above integral converges for $b \to \infty$ if and only if $\alpha > 1$.
If $\alpha = 1$, we have $\int_1^b \frac{1}{x}\,dx = \log b$ which means that the limit $\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$ does not

exist.

b) Analogously, one proves the following proposition:

$$\int_0^1 \frac{1}{x^\alpha}\,dx \text{ exists} \quad \Leftrightarrow \quad \alpha < 1.$$

c) We have

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2},$$

because the antiderivative of $x \mapsto \frac{1}{1+x^2}$ is given by $x \mapsto \arctan x$ and we have

$$\lim_{b\to\infty} \arctan x\Big|_0^b = \frac{\pi}{2}.$$

d) For $\alpha > 0$ we have

$$\int_0^\infty e^{-\alpha x} dx = \frac{1}{\alpha},$$

because we have $\int_0^R e^{-\alpha x} dx = \frac{1}{\alpha}(1 - e^{-\alpha R}) \stackrel{R\to\infty}{\longrightarrow} \frac{1}{\alpha}$.

4.4 Theorem. (Comparison of integrals and series). *Let $f : [1,\infty) \to \mathbb{R}_+$ be an admissible and monotone decreasing function. Then we have*

$$\sum_{n=1}^\infty f(n) < \infty \quad \Longleftrightarrow \quad \int_1^\infty f(x)\,dx \quad \textit{exists.}$$

Proof. For $x \in [n-1, n]$ and $n \geq 2$, we have $f(n) \leq f(x) \leq f(n-1)$ by assumption. Therefore, $f(n) \leq \int_{n-1}^n f(x)\,dx \leq f(n-1)$ and thus

$$\sum_{n=2}^N f(n) \leq \int_1^N f(x)\,dx \leq \sum_{n=1}^{N-1} f(n), \quad N \geq 2.$$

Hence,

$$\int_1^N f(x)\,dx \leq \sum_{n=1}^{N-1} f(n) \leq \sum_{n=1}^\infty f(n)$$

and thus $\lim_{N\to\infty} \int_1^N f(x)\,dx$ exists whenever $\sum_{n=1}^\infty f(n)$ converges.
To show the converse direction, we note that

$$\sum_{n=2}^N f(n) \leq \int_1^N f(x)\,dx \leq \int_1^\infty f(x)\,dx < \infty.$$

Thus, $(\sum_{n=1}^{N} f(n))_{N\in\mathbb{N}}$ is a monotone and bounded sequence, and this implies that $\sum_{n=1}^{\infty} f(n)$ converges.

$\square$

As an example, consider the function $f : [1, \infty) \to \mathbb{R}_+$, given by $f(x) = \frac{1}{x^\alpha}$. In this case, the theorem yields

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \text{ is convergent} \quad \Leftrightarrow \quad \int_1^\infty \frac{1}{x^\alpha}\, dx \text{ exists} \quad \overset{4.3}{\Leftrightarrow} \quad \alpha > 1.$$

4.5 Definition. An admissible function $f : (a, b) \to \mathbb{K}$ is called *absolutely integrable* (*absolut integrierbar*), if $\int_a^b |f(x)|\, dx$ exists.

4.6 Lemma. *An absolutely integrable function $f : (a, b) \to \mathbb{K}$ is integrable.*

For the proof we refer to the exercises.

4.7 Theorem. (Comparison test for integrals). *Assume that $f, g : (a, b) \to \mathbb{R}$ are admissible functions, such that we have*

$$|f(x)| \le g(x), \quad x \in (a, b).$$

If g is integrable, then f is absolutely integrable.

For the proof we again refer to the exercises.

4.8 Example. The integral

$$\int_0^\infty \frac{\sin x}{x} dx$$

is convergent, but *not* absolutely convergent
To see this, we first of all observe that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (thus the integrand is continuous on the whole real line). Therefore, it suffices to examine the convergence of the integral $\int_1^\infty \frac{\sin x}{x}\, dx$. An integration by parts gives

$$\int_1^R \frac{\sin x}{x}\, dx = \cos 1 - \frac{\cos R}{R} - \int_1^R \frac{\cos x}{x^2}\, dx.$$

The integral $\int_1^\infty \frac{\cos x}{x^2}\, dx$ exists, since it is dominated by the convergent integral $\int_1^\infty \frac{1}{x^2}\, dx$. This means that the limit

$$\lim_{R\to\infty} \int_1^R \frac{\sin x}{x} dx$$

exists.
On the other hand, the integral $\int_1^\infty \frac{\sin x}{x}\, dx$ does *not* converge absolutely, since for each $k \in \mathbb{N}$ we have

$$\int_{k\pi}^{(k+1)\pi} |\frac{\sin x}{x}|\, dx \geq \frac{1}{(k+1)\pi}\int_{k\pi}^{(k+1)\pi} |\sin x|\, dx = \frac{2}{(k+1)\pi},$$

and therefore we have

$$\int_0^{(k+1)\pi} |\frac{\sin x}{x}| dx \geq \frac{2}{\pi}\sum_{n=0}^{k}\frac{1}{n+1}.$$

The latter expression is the harmonic series, whence the above limit does not exist for $k \to \infty$.

To conclude this section, we consider the gamma function and the beta function. Both functions are defined by improper integrals and represent important functions of analysis.

4.9 Example. (The gamma function).
We begin with the definition of the *gamma function*. For $z \in \mathbb{C}$ with $Re(z) > 0$ we define

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}\, dt.$$

This function was introduced by Euler, whose motivation was to interpolate the factorial function $n \mapsto n!$, defined for $n \in \mathbb{N}$. First of all, we show that the gamma function is well defined.

For $t \in (0,1]$ we have the estimate

$$|t^{z-1}e^{-t}| = t^{Re(z)-1}e^{-t} \leq t^{Re(z)-1},$$

and by Example 4.3b) and Theorem 4.7 it follows that $\int_0^1 t^{z-1}e^{-t}dt$ converges absolutely.

For $t \in [1,\infty)$ we have

$$t^{Re(z)-1}e^{-t} \leq C_z e^{-t/2}$$

for a constant C_z which depends on z. Since the integral $\int_1^\infty e^{-t/2}\, dt$ exists by Example 4.3 d), the integral $\int_1^\infty t^{z-1}e^{-t}\, dt$ is absolutely convergent. The gamma function $\Gamma : \{z \in \mathbb{C} : Re(z) > 0\} \to \mathbb{C}$ defined in this way has the following properties:

a) $\Gamma(z+1) = z\,\Gamma(z), \quad Re(z) > 0,$

b) $\Gamma(1) = 1,$

c) $\Gamma(n+1) = n!, \ \ n \in \mathbb{N}.$

To see property a), we integrate by parts to obtain

$$\underbrace{\int_a^b t^z e^{-t} dt}_{\to\Gamma(z+1) \text{ for } b\to\infty,\, a\to 0+} \quad = \quad \underbrace{-t^z e^{-t} \,|_a^b}_{\to 0 \text{ for } b\to\infty,\, a\to 0+} \quad + \quad \underbrace{z \int\limits_a^b t^{z-1} e^{-t} dt}_{\to z\,\Gamma(z) \text{ for } b\to\infty,\, a\to 0+} \quad , \; 0 < a < b < \infty.$$

Therefore we have $\Gamma(z+1) = z$ for $Re(z) > 0$.
Property b) follows immediately from Example 4.3 d). Similarly, property c) follows by applying a) repeatedly in connection with b).

In many applications, it is important to calculate approximate values of $\Gamma(x)$ or $n!$ of large x and n, respectively. In this context, the *Stirling formula* is of particular interest. It says that for $x > 0$ we have

$$\Gamma(x) = \sqrt{2\pi} x^{x-1/2} e^{-x+\mu(x)}, \quad \text{with } \; 0 < \mu(x) < \frac{1}{12x}.$$

Therefore, $\sqrt{2\pi} x^{x-1/2} e^{-x}$ is often used as an approximation of $\Gamma(x)$. The relative error of the approximation is $e^{-12x} - 1$ and is smaller than one percent already for $x > 10$.

4.10 Example. (The beta function)
Another important function, also defined by an improper integral, is the so-called *beta function*. For $p, q \in \mathbb{C}$ with $Re(p)$, $Re(q) > 0$, it is defined by

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt.$$

The above integral is absolutely convergent (cf. exercises) and thus $B(p,q)$ is well defined. Furthermore, we have the relation

$$\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(p,q), \quad Re(p),\; Re(q) > 0.$$

Chapter VI

Analysis of metric spaces

What are reasons that mathematicians introduce spaces of infinite dimensions, interpret a sequence of real numbers as one point in a space of sequences as well as a function as one point in a function space? Two issues were the moving spirits for the development of these concepts: on the one hand to find a solution of integral equations of type

$$u(t) + \int_0^1 k(t,s)u(s)ds = f(t) \quad , t \in [0,1],$$

where the kernel k and the function f are given.

On the other hand to find a solution u of the variational problem

$$F(u) = \int_0^1 f(s, u(s), u'(s))ds \stackrel{!}{=} \min \quad , u \in X,$$

for a given function f and a given set X of functions, i.e., find some $u \in X$ such that F attains its minimum on X. David Hilbert (1862-1943) realized that the above integral equation could be dealt with as a system of linear equations in the (nowadays called) Hilbert sequence space ℓ^2. A different approach consists of an iterative solution of the above integral equations, i.e., to obtain the solution u as a limit of the sequence

$$u_{j+1}(t) = f(t) - \int_0^1 k(t,s)u_j(s)ds.$$

In the discussion concerning the convergence of the sequence $(u_j)_j$ it is only natural and also very helpful to consider a function as an element of a suitable space. This concept is also quite natural for variational problems: The argument u of f is itself a function.

In his thesis 1906 Maurice Fréchet (1878-1973) introduced the abstract concept of a metric space, an idea which is still of great importance today. It allows to discuss questions concerning convergence and continuity in a consistent and clear way. The theory of convergence then leads to the idea of a complete metric space due to Fréchet and Hausdorff (1868-1942).

A particularly important class of complete metric spaces are the Banach spaces. This concept, due to Stephan Banach (1892-1945), is of enormous importance in today's modern analysis. It is based on the idea of a normed vector space. A significant role is played by a subclass of the Banach spaces, where the norm can be defined by a scalar product. The elements of this subclass are nowadays called Hilbert spaces and were axiomatically introduced 1929 by John von Neumann (1903-1957); in particular, they play a central role in quantum mechanics.

In the following sections we deal with the basic topological ideas such as neighborhood and open sets in metric spaces. Section 2 extends these considerations to convergence of sequences and continuity of functions in metric spaces; here we also introduce the notion of complete metric space and of a normed space.

In Section 3 we introduce compact sets via the covering property and show that the concepts of "cover compactness" and "sequential compactness" coincide in metric spaces. Thus, fundamental properties of continuous functions on compact sets in $\mathbb{R}^n$ can smoothly be carried over to metric spaces.

We close this chapter with a section on continuous functions on connected sets. The latter are, last but not least, of importance since they represent a topological invariant.

1 Topology of metric spaces

Here we extend the concept of a neighborhood in $\mathbb{R}^n$ to the setting of metric spaces.

1.1 Definition (Metric space). Let $M \neq \emptyset$. A function

$$d : M \times M \to \mathbb{R}$$

is called a *metric* on M if the following conditions are valid for all $x, y, z \in M$.

(M1) $d(x,y) = 0 \;\Leftrightarrow\; x = y$ (definiteness)

(M2) $d(x,y) = d(y,x)$ (symmetry)

(M3) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

The pair (M,d) is called a *metric space* . The number $d(x,y)$ is called the *distance* of the points x and y.

$\Diamond$

Remark. For all $x, y \in M$ we have $d(x,y) \geq 0$, since

$$0 = d(x,x) \leq d(x,y) + d(y,x) = 2d(x,y).$$

Let us now illustrate the concept via some examples

1.2 Examples. a) For $x, y \in \mathbb{K}$ we define the *Euclidean metric* by

$$d(x,y) = |x-y|.$$

b) The rule

$$d(x,y) := \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

for $x, y \in M$ defines a metric on M, the so-called *discrete metric*.

c) If $X \subset M$ and if (M,d) is a metric space, then

$$d_X : X \times X \to \mathbb{R}, \qquad d_X(x,y) := d(x,y)$$

defines a metric on X, the so-called *induced metric*.

An important subclass of the metric spaces is formed by the normed spaces. Let us start by explaining a norm on a vector space.

1.3 Definition. Let V be a vector space over $\mathbb{K}$. The map

$$\|\cdot\| : V \to \mathbb{R}$$

is called a *norm* on V, if the following properties hold for all $x, y \in V$ and $\lambda \in \mathbb{K}$.

(N1) $\|x\| = 0 \Leftrightarrow x = 0$ (definiteness)

(N2) $\|\lambda x\| = |\lambda| \, \|x\|$ (homogeneity)

(N3) $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality)

The pair $(V, \|\cdot\|)$ is called a *normed (vector) space*.

◊

1.4 Remarks. a) If $(V, \|\cdot\|)$ is a normed space, then

$$d(x,y) := \|y-x\| \qquad \text{for } x, y \in V$$

defines a metric on V and (V,d) is a metric space. Thus each normed space becomes a metric space when it is equipped with this canonical metric.

b) The following properties follow directly from the definition of the norm.

(i) $\|x\| \geq 0$ for all $x \in V$

(ii) $\|y-x\| \geq |\|y\| - \|x\||$ for all $x, y \in V$

1.5 Examples. a) Let $1 \leq p \leq \infty$ and $n \geq 1$. Define the so-called *p-norm* by

$$\|x\|_p := \|(x_1, \ldots, x_n)\|_p := \begin{cases} \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \max_{1\leq j\leq n} |x_j| & p = \infty. \end{cases}$$

Then $(\mathbb{K}^n, \|\cdot\|_p)$ is a normed space.

In the case $\mathbb{K} = \mathbb{R}$ the norm $\|\cdot\|_2$ is also called *Euclidean norm*, the norm $\|\cdot\|_\infty$ is often called *maximum norm*.

The geometric shape of a ball in a normed space naturally depends on the chosen norm. The figures below show the unit ball in $\mathbb{R}^2$, $B_1(0) := \{x \in \mathbb{R}^2 : \|x\|_p \leq 1\}$, with respect to the p-norms for $p = 1, 2$ and $p = \infty$.

b) Let $[a.b] \subset \mathbb{R}$ be a compact interval. The set

$$V := C([a, b], \mathbb{K}) = \{f : [a, b] \to \mathbb{K} : f \text{ continuous}\}$$

is a vector space which can be normed by

$$\|f\|_p := \begin{cases} \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sup_{x\in[a,b]} |f(x)| & p = \infty. \end{cases}$$

For $p \leq \infty$ this is called the *L^p-norm*, for $p = \infty$ the *supremum norm*. L^p-norms play a central role in harmonic analysis.

c) Let $c := \{(x_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \text{ convergent}\}$, $x_n \in \mathbb{K}$, be the vector space of convergent sequences and $\mathbf{x} := (x_n)_{n\in\mathbb{N}}$. Then

$$\|\mathbf{x}\|_\infty := \|(x_n)_{n\in\mathbb{N}}\|_\infty := sup_{n\in\mathbb{N}} |x_n|$$

is a norm on c.

Let us now carry over the basic ideas of neighborhoods and open/closed sets from $\mathbb{R}^n$ to general metric spaces (M, d).

1.6 Definition. Let (M, d) be a metric space. Then we call

a) $U_\varepsilon(x) := \{y \in M : d(x, y) < \varepsilon\}$, $x \in M, \varepsilon > 0$, an *ε-neighborhood of* x.

b) $U \subset M$ a *neighborhood* of x, if there exists some $\varepsilon > 0$ such that $U_\varepsilon(x) \subset U$.

c) $\mathcal{O} \subset M$ *open*, if to each $x \in \mathcal{O}$ there exists some $\varepsilon_x > 0$ such that $U_{\varepsilon_x}(x) \subset \mathcal{O}$.

d) $A \subset M$ *closed*, if $M \setminus A$ is open.

e) $X \subset M$ *d-bounded*, if there exists some $x \in M$ and $r > 0$ such that $X \subset U_r(x)$.

f) $x \in M$ a *boundary point* of the set $X \subset M$, if each neighborhood of x contains a an element of X as well as an element of $M \setminus X$. If one sets

   (i) $\partial X := \{x \in M : x \text{ is a boundary point of } X\}$, then ∂X is called the *boundary of* X.

   (ii) $\mathring{X} := X \setminus \partial X$, then $\mathring{X}$ is called the *interior of* X; an element $x \in \mathring{X}$ is called an *interior point of* X.

g) $x \in M$ is an *accumulation point* of the set $X \subset M$ if each neighborhood of x contains infinitely many points of X (or equivalently one point of $X \setminus \{x\}$). If one sets

$$\overline{X} := \{x \in M : x \in X \text{ or } x \text{ is an accumulation point of } X\},$$

then $\overline{X}$ is called the *closure* of X.

$\Diamond$

Observe: If $(\mathbb{R}^n, d)$ is a metric space, a d-bounded set $X \subset \mathbb{R}^n$ need not be bounded in the sense of Analysis I; e.g. $\mathbb{R}^n$ is d-bounded if we set $d(x,y) := \frac{\|x-y\|_2}{1+\|x-y\|_2}$.

The above concepts (except d-boundedness) are consistent with the corresponding ones in $\mathbb{K}^n$, which were made in Chap. III, 2 of Analysis I. This implies, in particular, that many proofs on topological assertions in $\mathbb{K}^n$ can directly be carried over to metric spaces while replacing $|x-y|$ by $d(x,y)$.

1.7 Theorem (Hausdorff's separation axiom). *If (M,d) is a metric space and $x, y \in M$ such that $x \neq y$, then there exist neighborhoods U_x and U_y of x and y, resp., such that $U_x \cap U_y = \emptyset$.*

Proof. For $\varepsilon := \frac{1}{2} d(x,y)$ set $U_x := U_\varepsilon(x), U_y := U_\varepsilon(y)$. If there were a $z \in U_x \cap U_y$, we would get

$$2\varepsilon = d(x,y) \leq d(x,z) + d(z,y) < \varepsilon + \varepsilon = 2\varepsilon.$$

Contradiction!

$\square$

Analogously to the $\mathbb{K}^n$-case one proves the following.

1.8 Lemma. *In a metric space (M,d) we have*

- *Arbitrary unions and finite intersections of open sets are open.*
- *Arbitrary intersections and finite unions of closed sets are closed.*

The proof follows that of Theorem 2.5 in Chapter III while replacing $|x-y|$ by $d(x,y)$.

1.9 Definition. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called *equivalent*, if there exist two constants $c, C > 0$ such that

$$c\|x\|_1 \le \|x\|_2 \le C\|x\|_1 \qquad \text{for all } x \in V. \tag{1.1}$$

$\diamondsuit$

Obviously, the above two norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $C([0,1],\mathbb{R})$ are not equivalent. For consider the sequence $(f_n)_n$ of functions where

$$f_n : [0,1] \to \mathbb{R},$$
$$f_n(x) = \begin{cases} 1 - nx & 0 \le x \le 1/n \\ 0 & 1/n < x \le 1 \end{cases}$$

Then $\|f_n\|_\infty = 1 \; \forall n \in \mathbb{N}$, but $\|f_n\|_1 = \frac{1}{2n} \; \forall n \in \mathbb{N}$.

On the other hand, the Euclidean norm and the maximum norm on $\mathbb{R}^n$ are equivalent, since

$$\|x\|_\infty := \max_{1\le k\le n} |x_k| \le \left(\sum_{j=1}^n |x_j|^2\right)^{1/2} =: \|x\|_2 \le \max_{1\le k\le n} |x_k| \left(\sum_{j=1}^n 1^2\right)^{1/2} = \sqrt{n}\|x\|_\infty.$$

We show that all norms on a finite dimensional vector space are equivalent.

1.10 Theorem. *All norms on a finite dimensional $\mathbb{K}$-vector space are equivalent.*

Proof. (a) We first show this assertion for $V = \mathbb{R}^n$. Observe that it is sufficient to show that each norm $\|\cdot\|$ on $\mathbb{R}^n$ is equivalent to the Euclidean norm $\|x\|_2$.

Thus, let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ and e_j be the j-th unit vector. Then $x = (x_1, \ldots, x_n)^T = \sum_{j=1}^n x_j e_j$. The Cauchy-Schwarz inequality implies

$$\|x\| = \|\sum_{j=1}^n x_j e_j\| \le \sum_{j=1}^n |x_j|\,\|e_j\| \le C\|x\|_2, \quad C := \left(\sum_{j=1}^n \|e_j\|^2\right)^{1/2}. \tag{1.2}$$

In order to prove the converse inequality, set

$$c := \inf\{\|x\| : x \in S\}, \quad S := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$$

where S denotes the Euclidean unit sphere. If $c > 0$, then the theorem follows directly from the definition of c, since for $x \ne 0$ we have $\frac{x}{\|x\|_2} \in S$ and hence

$$\|x\| = \left\|\frac{x}{\|x\|_2}\right\| \|x\|_2 \ge c\|x\|_2.$$

If $x = 0$, the inequality is trivial and the assertion true for $\mathbb{R}^n$.
Thus assume $c = 0$ (and try to obtain a contradiction). Then there is a sequence $(x_k)_k$ in S with $\lim_{k\to\infty}\|x_k\| = 0$. By the Bolzano-Weierstrass Theorem from Analysis I (cf. Chapter II, 2) the sequence $(x_k)_k$ has a convergent subsequence with respect to the Euclidean norm, i.e. there exists some $a \in \mathbb{R}^n$ such that $\|x_{k_j} - a\|_2 \stackrel{j\to\infty}{\longrightarrow} 0$. We have $a \in S$ since

$$a_1^2 + \ldots + a_n^2 = \lim_{j\to\infty}(x_{k_j,1}^2 + \ldots + x_{k_j,n}^2) = 1.$$

On the other hand we conclude from (1.2) that for each $k \in \mathbb{N}$

$$\|a\| \leq \|a - x_{k_j}\| + \|x_{k_j}\| \leq C\|a - x_{k_j}\|_2 + \|x_{k_j}\|_2.$$

For $j \to \infty$ this implies $\|a\| = 0$ and therefore $a = 0$ in contradiction to the fact that $a \in S$. Hence $c > 0$.

(b) Let V be an arbitrary $\mathbb{K}$-vector space and $\|\cdot\|$ and $\|\cdot\|^*$ be two norms on V. If $\phi : \mathbb{R}^n \to V$ is an isomorphism from $\mathbb{R}^n$ to V and if one sets

$$\|x\|_\phi := \|\phi(x)\| \quad \text{and} \quad \|x\|_\phi^* := \|\phi(x)\|^*,$$

then the assertion follows by part a).

$\square$

1.11 Remark. The above theorem has important consequences. In particular, it says that the above introduced topological basic concepts like neighborhood, open set, and accumulation point in a finite dimensional normed space do not depend on a particular chosen norm.

2 Convergence and continuity

In the following we will transfer the concepts of convergence of a sequence and of continuity of a mapping from the setting of $\mathbb{K}^n$ to the setting of metric spaces (M, d) in general.

2.1 Definition. Let (M, d) be a metric space.

a) We say that a sequence $(x_j)_{j\in\mathbb{N}} \subset M$ is *convergent* to $x \in M$ if for any neighborhood U of x there exists an $N_0 \in \mathbb{N}$ such that

$$x_j \in U \quad \text{for all} \quad j \geq N_0.$$

In this case we write $x = \lim_{j\to\infty} x_j$, and we say that x is the *limit* of the sequence $(x_j)_{j\in\mathbb{N}} \subset M$.

b) We say that a sequence $(x_j)_{j\in\mathbb{N}} \subset M$ is *d-bounded* if the set $\{x_j : j \in \mathbb{N}\} \subset M$ is d-bounded.

c) We say that an element $x \in M$ is a *cluster point* of the sequence $(x_j)_{j\in\mathbb{N}} \subset M$ if every neighborhood of x contains infinitely many members of the sequence.

d) We say that a sequence $(x_j)_{j\in\mathbb{N}} \subset M$ is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon \quad \text{for all} \quad n, m \geq N_0\,.$$

$\Diamond$

We note that an accumulation point of the set $\{x_k : k \in \mathbb{N}\}$ is a cluster point of the sequence $(x_k)_{k\in\mathbb{N}}$, but that the opposite is not true in general. This was the case already for $\mathbb{R}$.

The following statements can be deduced from the definitions above similarly to the corresponding statements from Analysis I.

2.2 Proposition. *The following statements hold:*

a) The limit of a convergent sequence is unique.

b) Every convergent sequence is a Cauchy sequence.

c) Every Cauchy sequence is d-bounded.

d) A point x is a cluster point of a sequence if and only if the sequence has a subsequence which converges to x.

e) A set $A \subset M$ is closed if and only if for every sequence $(x_j)_{j\in\mathbb{N}} \subset A$ which converges in M we have $\lim_{j\to\infty} x_j \in A$.

We now extend the concept of continuity of functions from the setting of functions defined on subsets of $\mathbb{R}^n$ to the setting of functions from one metric space to another.

2.3 Definition. Let (M, d_M) and (N, d_N) be metric spaces. A mapping $f : M \to N$ is called *continuous* in $x_0 \in M$ if for every neighborhood V of $f(x_0) \in N$ there exists a neighborhood U of x_0 such that $f(U) \subset V$.

$\diamondsuit$

The following theorem characterizes continuous functions in terms of ε-δ-continuity and sequential continuity.

2.4 Theorem (Characterization of continuous functions). *Let $f : M \to N$ be a function between the metric spaces (M, d_M) and (N, d_N). The following statements are equivalent:*

a) f is continuous in $x_0 \in M$.

b) (ε-δ-continuity). To each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_M(x, x_0) < \delta \quad \Rightarrow \quad d_N(f(x), f(x_0)) < \varepsilon.$$

c) (Sequential continuity). For each sequence $(x_j)_{j\in\mathbb{N}} \subset M$ with $\lim_{n\to\infty} x_n = x_0$ we have

$$f(x_0) = \lim_{n\to\infty} f(x_n), \text{ i.e., } \lim_{n\to\infty} d_N(f(x_n), f(x_0)) = 0.$$

Proof. The equivalence of the statements a) and b) follows immediately from the definition of a neighborhood. The equivalence of the statements b) and c) follows as in the proof of Proposition I.1.2 in Analysis I, where we only need to replace $|x - y|$ by $d_M(x, y)$ and $|f(x) - f(y)|$ by $d_N(f(x), f(y))$ throughout.

$\square$

2.5 Theorem. *Let (M, d_M) and (N, d_N) be metric spaces and let $f : M \to N$ be a function. The following statements are equivalent:*

a) f is continuous.

b) $f^{-1}(A)$ is closed in M whenever $A \subset N$ is closed in N.

c) $f^{-1}(O)$ is open in M whenever $O \subset N$ is open in N.

The proof is as the proof of Theorem III.2.17 from Analysis I, where we again replace $|x - y|$ by $d_M(x, y)$ and $|f(x) - f(y)|$ by $d_N(f(x), f(y))$.

We will now consider in more detail continuity of linear maps between normed vector spaces. So let V and W be two vector spaces over $\mathbb{K}$. A mapping $T : V \to W$ is called a *linear map* or a *linear operator* if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \text{for all } \alpha, \beta \in \mathbb{K}, \ \ x, y \in V.$$

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces, and let $T : V \to W$ be a linear operator. If there exists a constant $M \geq 0$ such that

$$\|T(x)\|_W \leq M \quad \text{for all} \quad x \in V \quad \text{with} \quad \|x\|_V \leq 1,$$

then we say that T is *bounded*. We will often write Tx instead of $T(x)$.

2.6 Proposition. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces and let $T : V \to W$ be a linear operator. The following statements are equivalent.*

a) T is continuous.

b) There exists an element $x_0 \in V$ such that T is continuous in x_0.

c) There exists a constant $L > 0$ such that for all $x, y \in V$ we have $\|Tx - Ty\|_W \leq L\|x - y\|_V$.

d) T is bounded.

Proof. We will show the following implications: *a) $\Rightarrow$ b), b) $\Rightarrow$ d), d) $\Rightarrow$ c), c) $\Rightarrow$ a).*

The implication *a) $\Rightarrow$ b)* is clear.

b) $\Rightarrow$ d): By assumption there exists to $\varepsilon = 1$ a $\delta > 0$ such that

$$\|T(x - x_0)\|_W = \|Tx - Tx_0\|_W \leq 1 \quad \text{for all}$$
$$x \in \overline{U}_\delta(x_0) := \{y \in V : \|x_0 - y\|_V \leq \delta\}.$$

Letting $h := (x - x_0) \in \overline{U}_\delta(0)$ we see that this statement is equivalent to saying that $\|Th\|_W \leq 1$ for all $h \in \overline{U}_\delta(0)$. This is again equivalent to saying that we have $\|T(\delta h)\|_W \leq 1$ for all $h \in \overline{U}_1(0)$, i.e., that we have $\|Th\|_W \leq \frac{1}{\delta}$ for all $h \in \overline{U}_1(0)$. This means that T is bounded.

d) $\Rightarrow$ c): Let $M \geq 0$ be such that $\|Tx\|_W \leq M$ for all $x \in \overline{U}_1(0)$. Then we have

$$\|T\Big(\frac{x}{\|x\|_V}\Big)\|_W \leq M \quad \text{for all} \quad x \in V \quad \text{with} \quad x \neq 0$$

and thus $\|Tx\|_W \leq M\|x\|_V$ for all $x \in V$. Thus

$$\|Tx - Ty\|_W \leq M\|x - y\|_V \quad \text{for all} \quad x, y \in V.$$

c) $\Rightarrow$ a): Let $L > 0$ be such that $\|Tx - Ty\|_W \leq L\|x - y\|_V$ for all $x, y \in V$. Hence, if we choose to $\varepsilon > 0$ a $\delta := \varepsilon/L > 0$ we have

$$\|Tx - Ty\|_W \leq L\|x - y\|_V < L \cdot \frac{\varepsilon}{L} = \varepsilon$$

and (a) follows by Theorem 2.4.

□

2.7 Remark. In the above proof of the implication d) $\Rightarrow$ c) we have also proved that

$$\begin{aligned} & \|Tx\|_W \leq M \quad \text{for all} \quad x \in \overline{U}_1(0) \\ \iff \quad & \|Tx\|_W \leq M\|x\|_V \quad \text{for all} \quad x \in V. \end{aligned}$$

The infimum of all such constants M is called the *operator norm* $\|T\|$ of T, i.e.,

$$\|T\| := \inf\{M \geq 0 : \|Tx\|_W \leq M\|x\|_V \text{ for all } x \in V\}.$$

One can show that $\|T\| = \sup\{\|Tx\|_W : \|x\|_V \leq 1\}$, and that $\|Tx\|_W \leq \|T\| \cdot \|x\|_V$ holds for all $x \in V$.

Defining

$$\mathcal{L}(V, W) := \{T : V \to W : T \text{ is linear and bounded}\},$$

makes $(\mathcal{L}(V, W), \|\cdot\|)$ a normed vector space.

2.8 Example (The row-sum norm). If we furnish $\mathbb{K}^n$ with the maximum norm and consider continuous linear mappings T from $\mathbb{K}^n$ into itself, then the associated operator norm is given by

$$\|T\| = \max_i \sum_{j=1}^{n} |a_{ij}|\,,$$

where T is represented by the matrix $(a_{ij})_{1\leq i,j\leq n}$.

For many results from Analysis I the completeness of the real numbers $\mathbb{R}$ was essential. We will now define the concept of completeness of a metric space. We will do this analogously to the formulation of the completeness of $\mathbb{R}$ in terms of the convergence of Cauchy sequences.

2.9 Definition. A metric space (M, d) is called *complete* if all Cauchy sequences in M converge. A complete normed vector space will also be called a *Banach space*.

The above definition of a complete normed vector space traces back to STEFAN BANACH (1892-1945), a Polish mathematician whose contributions to functional analysis are fundamental. He noticed that the proper setting for most of the results which in this connection are important are spaces which are equipped with the structure of a vector space as well as a metric, and where the distance between two objects x and y is derived from the difference $x - y$. The *Banach fixed point theorem*, which we will see below, is still one of the most important fixed point theorems.

2.10 Examples. a) The space $(\mathbb{K}^n, \|\cdot\|_p)$, where we equip $\mathbb{K}^n$ with the $\|\cdot\|_p$-norm, is a Banach space.

b) The function space $C[a, b]$, equipped with the supremum norm $\|\cdot\|_\infty$, is a Banach space for all $-\infty < a < b < \infty$. This follows since uniformly convergent sequences of continuous functions on the compact interval $[a, b]$ converge to continuous functions. (By Theorem IV.4.6 from Analysis I).

c) If we equip the space $C^1[a,b]$ of continuously differentiable functions with the supremum norm, then the result is a normed space which is *not* a Banach space, cf. the exercises.

d) If we give $\mathbb{C}^1[a,b]$ the norm

$$\|f\| := \|f\|_\infty + \|f'\|_\infty,$$

then $(C^1[a,b], \|\cdot\|)$ is complete, i.e. a Banach space. The proof of this fact relies on Theorem IV.4.7 from Analysis I, cf. the exercises.

e) The space ℓ^2 (pronounced "little ell two"), which consists of all sequences $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ which are square-summable, i.e. such that

$$\sum_{j=1}^{\infty} |a_j|^2 < \infty,$$

is a Banach space when equipped with the norm

$$\|a\|_2 := \Big(\sum_{j=1}^{\infty} |a_j|^2\Big)^{1/2}, \qquad a \in \ell^2.$$

For the proof we refer tho the exercises. We can define a scalar product on ℓ^2 by letting

$$\langle a, b\rangle = \sum_{j=1}^{\infty} a_i \overline{b_j}$$

for $a, b \in \ell^2$. Vector spaces equipped with a scalar product are called *Hilbert spaces* if they are complete with respect to the norm induced by the scalar product. So ℓ^2 is an example of a Hilbert space. Hilbert spaces play a decisive role in quantum mechanics. The concept of a Hilbert space traces back to DAVID HILBERT (1862-1943), who recognized that certain integral equations could be translated into systems of linear equations in ℓ^2. The space ℓ^2 is often called the Hilbert sequence space.

Fixed point theorems have many applications in mathematics. The following theorem — the Banach fixed point theorem — says that a strict contraction on a complete metric space has exactly one fixed point.

2.11 Theorem (Banach fixed point theorem). *Let (M, d) be a complete metric space, and let $F : M \to M$ be a strict contraction, i.e. there exists a constant $q < 1$ such that*

$$d(F(x), F(y)) \le q \cdot d(x, y) \quad \text{for all } x, y \in M.$$

Then F has a unique fixed point $r \in M$, i.e. there exists exactly one $r \in M$ such that $F(r) = r$. Furthermore, for any $x_0 \in M$ the sequence $(x_n)_{n\in\mathbb{N}_0}$ defined by

$$x_{n+1} := F(x_n)$$

converges to r, i.e., $\lim_{n\to\infty} d(x_n, r) = 0$.

Proof. The proof is analogous to the proof of II.2.13 from Analysis I, where one replaces any occurrence of $|x - y|$ by $d(x, y)$.

$\square$

3 Compactness

We can define compactness for subsets of metric spaces just as we defined compactness for subsets of $\mathbb{R}^n$ in Section III.3 from Analysis I, namely in terms of open covers. An *open cover* of a subset $K \subset M$ of a metric space (M, d) is a family $\{O_i\}_{i\in I}$ of open subsets of M such that for every $x \in K$ there exists at least one $i \in I$ such that $x \in O_i$, i.e. the family $\{O_i\}_{i\in I}$ should be such that $K \subset \bigcup_{i\in I} O_i$.

3.1 Definition. A subset $K \subset M$ of a metric space (M, d) is called *compact* if every open cover $\{O_i\}_{i\in I}$ of K has a finite subcover, i.e., if there exist $i_1, \ldots, i_r$ such that

$$K \subset O_{i_1} \cup \ldots \cup O_{i_r}$$

3.2 Examples. a) Any finite subset of a metric space is compact.

b) Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in a metric space, and let $x := \lim_{n\to\infty} x_n$. Then

$$K := \{x_n : n \in \mathbb{N}\} \cup \{x\}$$

is compact.

c) A closed subset of a compact set is also compact, i.e. if $K \subset M$ is compact in M and $A \subset K$ is closed, then A is compact. The proof is the same as he proof of Lemma III.3.4.

On the basis of this abstract definition we will now characterize compact sets in metric spaces. First, we recall the situation for $\mathbb{R}^n$ where, by the Heine-Borel theorem, we could characterize the compact subset as the ones which are closed and bounded.

In this section we will see that also for metric spaces in general the compact subsets are closed and bounded. However, the converse is in general false. An example is provided by considering the space $C([0, \pi], \mathbb{C})$ equipped with the supremum norm, cf. Example 3.5 c) below.

However the following theorem shows that a subset of a metric space is compact if and only if it is sequentially compact.

3.3 Theorem. *Let K be a nonempty subset of a metric space M. The following statements are equivalent:*

(i) K is compact (covering compactness)

(ii) Every sequence $(x_n)_{n\in\mathbb{N}} \subset K$ has a convergent subsequence $(x_{n_j})_{j\in\mathbb{N}}$

(iii) Every sequence has a cluster point in K.

Proof. *(i) $\Rightarrow$ (ii)* is proved exactly as in Theorem III.3.11. *(ii) $\Leftrightarrow$ (iii)* was already proved in Proposition 2.2 d).

(ii) $\Rightarrow$ (i): Let $\{O_i\}_{i\in I}$ be an open cover of K. We divide the proof into three steps.

Step 1: There exists one $\delta > 0$ such that for all $x \in K$ there exists $i \in I$ satisfying $U_\delta(x) \subset O_i$.

Assume the contrary. Then to each $n \in \mathbb{N}$ there exists $x_n \in K$ such that $U_{1/n}(x_n) \not\subset O_i$ for any $i \in I$. By the hypothesis, the sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_j})_{j\in\mathbb{N}}$ with $x := \lim_{j\to\infty} x_{n_j} \in K$, hence $x \in O_j$ for some $j \in I$. Since O_j is open, there exists $\varepsilon > 0$ such that $U_\varepsilon(x) \subset O_j$. Now choose $l \in \mathbb{N}$ with $l > 2/\varepsilon$ and $d(x_l, x) < \varepsilon/2$.
Then $U_{1/l}(x_l) \subset U_\varepsilon(x)$ (observe, if $y \in U_{1/l}(x_l)$ then $d(y,x) \le d(y,x_l) + d(x_l,x) < \varepsilon$), thus $U_{1/l}(x_l) \subset U_\varepsilon(x) \subset O_j$, which is a contradiction to the assumption that $U_{1/l} \not\subset O_i$ for any $i \in I$.

Step 2: For any $\delta > 0$ there exist $x_0, \ldots, x_n \in K$ with

$$K \subset \bigcup_{l=0}^{n} U_\delta(x_l).$$

Assume that the claim is false. Then there exists $\delta > 0$ such that $K \not\subset \bigcup_{l=0}^{n} U_\delta(x_l)$ for all $n \in \mathbb{N}$ and for any points $x_0, \ldots, x_n \in K$. Let us choose $x_0 \in K$; then $K \not\subset U_\delta(x_0)$. Therefore, there exists $x_1 \in K \setminus U_\delta(x_0)$ and, moreover,

$$K \not\subset U_\delta(x_0) \cup U_\delta(x_1)\,.$$

We obtain in this way a sequence $(x_n)_{n\in\mathbb{N}} \subset K$ satisfying

$$x_n \in K \setminus \Big(\bigcup_{l=0}^{n-1} U_\delta(x_l)\Big) \quad \text{and} \quad K \not\subset \bigcup_{l=0}^{n} U_\delta(x_l).$$

This sequence satisfies by construction

$$d(x_n, x_m) \ge \delta \qquad \text{for all } n, m \in \mathbb{N} \text{ with } n \ne m.$$

Hence, $(x_n)_{n\in\mathbb{N}}$ cannot have any Cauchy subsequence and, therefore, $(x_n)_{n\in\mathbb{N}}$ does not have convergent subsequences. Contradiction!

Step 3: Let $\delta > 0$ be as in Step 1 and $x_0, \ldots, x_n \in K$ as in Step 2. Then $K \subset \bigcup_{k=0}^{n} U_\delta(x_k) \subset \bigcup_{l=0}^{n} O_{i_l}$ for appropriate $i_0, \ldots, i_n \in I$.

$\square$

The above characterization of compact sets in metric spaces using sequential compactness allows us to prove, exactly as in the case of $\mathbb{R}^n$, that compact sets are always closed and d-bounded. We explicitly state this important remark in the following corollary.

3.4 Corollary. *A compact subset of a metric space (M, d) is closed and d-bounded.*

3.5 Examples. a) Let M be an infinite set endowed with the discrete metric. Then M is closed and d-bounded, but it is not compact.

b) We consider, as in Example 1.5 c), the space c of convergent sequences, endowed with the supremum norm. Then the closed unit ball in c, i.e.

$$\overline{B_1(0)} := \{(x_n)_{n\in\mathbb{N}} : |x_n| \le 1 \text{ for all } n \in \mathbb{N}\}$$

is closed and bounded, but not compact. To see this, let $e_n := (0, 0, \dots, 1, 0, \dots)$ be the n-th unit vector in c, with 1 in the n-th position. Then $\|e_n - e_m\|_\infty = 1$ for all $m \ne n$, from which it follows that the sequence $(e_n)_{n\in\mathbb{N}} \subset \overline{B_1(0)}$ has no convergent susequence. Therefore, $\overline{B_1(0)}$ is not compact.

c) The closed unit ball $B_1(0) := \{f \in C[0, \pi] : \|f\|_\infty \le 1\}$ of the Banach space $(C[0, \pi], \|\cdot\|_\infty)$ is not sequentially compact, hence not compact. Otherwise, the sequence of functions $(f_j)_{j\in\mathbb{N}} \subset \overline{B_1(0)}$ defined by $f_j(x) = \cos(jx) + i \sin(jx)$ would have a convergent subsequence, which can not be, since $\|f_k - f_l\|_\infty = 2$ for all $l \ne k$.

d) One can prove that the closed unit ball $\overline{B_1(0)} := \{x \in V : \|x\| \le 1\}$ of a normed vector space V is compact if and only if $\dim(V) < \infty$.

3.6 Corollary. *A compact metric space (M, d) is complete.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset M$ be a Cauchy sequence. By Theorem 3.3, it has a convergent subsequence $(x_{n_j})_{j\in\mathbb{N}}$ which converges in M. If $x := \lim_{j\to\infty} x_{n_j}$, then for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ with

$$\begin{aligned} d(x_{n_j}, x_m) &\le \varepsilon/2 && \text{for all } m, j \ge N_0, \text{ and} \\ d(x, x_{n_j}) &\le \varepsilon/2 && \text{for all } j \ge N_0. \end{aligned}$$

Hence

$$d(x, x_m) \le d(x, x_{N_0}) + d(x_{N_0}, x_m) \le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $m \ge N_0$, that is $x = \lim_{n\to\infty} x_n \in M$.

□

In Section $III.3$ we studied in detail properties of continuous functions defined on compact sets, as for example the theorem that a real-valued function attains its maximum (cf. Theorem III 3.9) or the theorem on uniform continuity (Theorem III 3.13). In the following we obtain analogous results for continuous functions defined on compact subsets of arbitrary metric spaces. Important results of modern analysis rely on these properties.

3.7 Theorem. *[Continuous images of compact sets are compact] Let (M, d_M), (N, d_N) be metric spaces and $f : M \to N$ be a continuous function. If M is compact, then $f(M) \subset N$ is also compact.*

The proof is completely similar to the proof of Theorem III 3.7.

3.8 Corollary. *If $f : M \to \mathbb{R}$ is continuous and M is compact, then f attains a minimum and a maximum.*

The proof is analogous to the proof of Corollary III 3.8.

3.9 Corollary. *If $f : M \to N$ is continuous and bijective and M is compact, then $f^{-1} : N \to M$ is continuous as well.*

Proof. By Theorem 2.5, it is enough to prove that $f(A)$ is closed for any closed subset $A \subset M$. Since A is a closed subset of a compact set, it follows that A is also compact by using Example 3.2 c). The above Theorem 3.7 implies that $f(A)$ is compact as well, hence closed, by applying Corollary 3.4.

□

We finish this section by considering uniformly continuous functions on metric spaces. In analogy to Analysis I, a function $f : M \to N$ between two metric spaces (M, d_M), (N, d_N) is called *uniformly continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_N(f(x), f(y)) < \varepsilon \quad \text{for all } x, y \in M \text{ with } d_M(x, y) < \varepsilon.$$

The following result can then be proved exactly as Theorem III 3.13.

3.10 Theorem. *Let $f : M \to N$ be a continuous function between two metric spaces (M, d_M), (N, d_N). If M is compact, then f is uniformly continuous.*

4 Connectedness

The intermediate value theorem from Analysis I was an important piece in the construction of analysis. We generalize this theorem, stated in Section III.1 for intervals, to continuous functions defined on connected subsets of metric spaces.

4.1 Definition. A metric space M is called *disconnected* if there are two nonempty and disjoint open subsets X and Y of M such that $M = X \cup Y$. A space that is not disconnected is called *connected*.

4.2 Examples. a) $\mathbb{R}^n$ is connected.

b) A subset $M \subset \mathbb{R}$ with at least two elements is connected if and only if it is an interval. For the proof we refer to the exercises.

c) $\mathbb{Q}$ is disconnected, since

$$\mathbb{Q} = \big((-\infty, \sqrt{2}) \cap \mathbb{Q}\big) \cup \big((\sqrt{2}, \infty) \cap \mathbb{Q}\big).$$

d) The set $M = \{x \in \mathbb{R}^2 : x_1^2 - x_2^2 = 1\}$ is disconnected.

4.3 Theorem. *Let $f : M \to N$ be a continuous function between two metric spaces M and N. If M is connected, then $f(M)$ is also connected.*

Proof. Assume the contrary. Then there exist two nonempty and disjoint open sets X and Y such that $f(M) = X \cup Y$. It follows that $M = f^{-1}(X) \cup f^{-1}(Y)$, that is a contradiction.

□

The Intermediate Value Theorem in metric spaces is the following.

4.4 Theorem. *Let M be a connected metric space, $f : M \to \mathbb{R}$ a continuous function and $a, b \in M$. Then f takes any value between $f(a)$ and $f(b)$.*

The proof is simple: if $f(a) \neq f(b)$, then $f(M)$ is an interval, by Example 4.2 b.

4.5 Examples. Let us consider the group $O(n, \mathbb{R})$ of orthogonal $n \times n$-matrices A, where $A \in O(n, \mathbb{R})$ if and only if $A^{-1} = A^T$. Then $O(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$ is disconnected.

As a matter of fact, $\det : O(n, \mathbb{R}) \to \mathbb{R}$ is a polynomial in n^2 variables, in particular a continuous function. Moreover, $\det(\mathrm{diag}(1, \ldots, 1) = 1$ and $\det(\mathrm{diag}(-1, 1, \ldots, 1)) = -1$. If $O(n, \mathbb{R})$ were connected, then we could obtain a matrix $A \in O(n, \mathbb{R})$ with $\det A = 0$, in contradiction to the fact that A is invertible.

Another concept of connectedness is also of importance.

4.6 Definition. A metric space M is called *path-connected* (also *arcwise connected*), if for any two points $a, b \in M$ there exists a continuous function $\gamma : [\alpha, \beta] \to M$ with $\gamma(\alpha) = a$ and $\gamma(\beta) = b$.

Natural examples of path-connected sets in $\mathbb{R}^n$ for $n \geq 2$ are

a) convex sets,

b) the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$, as well as

c) $\mathbb{R}^n \setminus \{0\}$.

The following theorem gives the relation between connected and path-connected sets.

4.7 Theorem. (a) *Any path-connected metric space is connected.*
(b) *Any connected open subset X of a normed vector space is path-connected.*

The proof is not difficult and it is left as an exercise to the reader.
We finish this short section with an example which is of great importance for the concept of orientation for vector spaces over $\mathbb{R}$.

4.8 Example. a) Let $G = \mathrm{GL}(n, \mathbb{R})$ be the group of real $n \times n$-matrices A with $\det A \neq 0$.

If we consider G as a subspace of $\mathbb{R}^{n^2}$, then G is disconnected.
In fact, assuming that G is connected, we get that the image of the continuous function $\det : G \to \mathbb{R}$ is also connected. But $\det(G) = \mathbb{R} \setminus \{0\}$, which is disconnected. We have got a contradiction.

b) Considerably more difficult to prove is the fact that the subgroup $\mathrm{GL}^+(n, \mathbb{R})$ of real $n \times n$-matrices with $\det A > 0$ is connected.

The connectedness of metric space is an important topological invariant. Thus, if M and N are homeomorphic spaces, then M is connected if and only if N is connected. Two spaces M and N are called *homeomorphic* if there exists a bijective, continuous function from M to N such that its inverse is also continuous.

Already in 1878, Cantor proved hat there is a bijection between $\mathbb{R}$ and $\mathbb{R}^2$; in 1890, Peano proved that there exists a continuous surjection from the interval $[0, 1]$ onto the square $[0, 1] \times [0, 1]$. Peano's function is not bijective and Cantor's construction is not continuous, so the question if there exists a homeomorphic mapping from $\mathbb{R}^n \to \mathbb{R}^m$ for $n \neq m$ is natural. Brouwer proved in 1901 that such a mapping can not exist. We give here the proof only for the special case $m = 1$.

4.9 Theorem. *Let $n \geq 2$. Then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$.*

Proof. First we recall: For $n \geq 2$, $\mathbb{R}^n \setminus \{0\}$ is connected. On the other hand, by Example 4.2 b), for any $x \in \mathbb{R}$, the set $\mathbb{R} \setminus \{x\}$ is disconnected.

Now: If we had a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}$, then we would get one between $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R} \setminus \{0\}$ in contradiction to the fact that, by Theorem 4.3, continuous images of connected sets are connected.

□

Chapter VII

Differentiation of functions of several variables

In this chapter the theory of differentiation of a function of one real variable will be extended to the case of functions of several real variables. Since now we cannot form a difference quotient we start with the equivalent definition via linear approximation. Comparing our discussions in the one-variable case with the situation of the multivariable case, the latter one turns out to be distinctly more complicated, since the linear maps have a richer structure than in the one-variable case.

1 Differentiable maps

In this section let $\mathbb{R}^n$ and $\mathbb{R}^m$ be normed spaces and $f : U \to \mathbb{R}^m$, where we all the time assume $U \subset \mathbb{R}^n$ to be open. We recall that $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is the space of all linear maps from $\mathbb{R}^n$ into $\mathbb{R}^m$, which is normed via the operator norm introduced in Section VI.2. Since $\mathbb{R}^n$ and $\mathbb{R}^m$ have finite dimension, the discussion in Section VI.2 implies that

(i) every linear map from $\mathbb{R}^n$ into $\mathbb{R}^m$ is continuous, and

(ii) $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a Banach space.

1.1 Definition. Let $U \subset \mathbb{R}^n$, $x_0 \in U$ (x_0 is an interior point of U), and $f : U \to \mathbb{R}^m$ be a function. f is called *differentiable at* x_0, if there exist maps $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $r : U \to \mathbb{R}^m$ such that

$$f(x) = f(x_0) + A(x - x_0) + r(x), \quad x \in U,$$

with

$$\lim_{x \to x_0} \frac{r(x)}{\|x - x_0\|} = 0.$$

1.2 Remarks. a) As norm on $\mathbb{R}^n$ we choose the Euclidean norm. We recall that the above definition is independent of a particular norm on $\mathbb{R}^n$ on account of Theorem VI.1.10.

b) In the case $n = m = 1$ the above definition is consistent with that from Section IV.1.

c) If $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x_0 \in U$, then the linear map A from Definition 1.1 is uniquely determined (exercise).

d) The unique map A from Definition 1.1 is called *derivative* or *differential* of f at x_0.

Notation: $A = f'(x_0)$ or $A = Df(x_0)$.

e) If one chooses the canonical bases in $\mathbb{R}^n$ and $\mathbb{R}^m$, then $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is represented by a matrix (a_{ij}). In this case we have

$$Ax = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad , x \in \mathbb{R}^n$$

In the following we tacitly identify the linear map A with its matrix representation $(a_{ij})_{m \times n}$.

1.3 Example. Let $B = (b_{ij}) \in M_n(\mathbb{R})$ be a symmetric matrix and $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x) := \langle x, Bx \rangle := \sum_{i,j=1}^{n} b_{ij} x_i x_j \qquad , \langle x, y \rangle = \sum_{j=1}^{n} x_j y_j.$$

Then we have for $x_0, h \in \mathbb{R}^n$, $x = x_0 + h$

$$\begin{aligned} f(x) = f(x_0 + h) = \langle x_0 + h, Bx_0 + Bh \rangle &= \langle x_0, Bx_0 \rangle + \underbrace{\langle x_0, Bh \rangle + \langle h, Bx_0 \rangle}_{=2\langle Bx_0, h \rangle} + \langle h, Bh \rangle \\ &= f(x_0) + 2\langle Bx_0, x - x_0 \rangle + \underbrace{\langle x - x_0, B(x - x_0) \rangle}_{=:r(x)} \\ &= f(x_0) + (2Bx_0)^T \cdot (x - x_0) + r(x). \end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$|r(x)| \le \|x - x_0\| \|B(x - x_0)\| \le \|B\| \|x - x_0\|^2$$

and, therefore,

$$\frac{|r(x)|}{\|x - x_0\|} \le \|B\| \|x - x_0\| \overset{x \to x_0}{\longrightarrow} 0,$$

since B is bounded. We conclude that f is differentiable at $x_0 \in \mathbb{R}^n$ and

$$f'(x_0) \equiv Df(x_0) = (2Bx_0)^T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}).$$

$\diamondsuit$

1.4 Theorem. *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ be differentiable at $x_0 \in U$. Then f is continuous at x_0.*

Proof. For $x \in U$ we have

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + r(x).$$

Since the linear map $Df(x_0)$ is continuous by Theorem VI.2.6, we clearly have $Df(x_0)(x - x_0) \overset{x \to x_0}{\longrightarrow} 0$. Furthermore, by definition, $\lim_{x \to x_0} r(x) = 0$ and, therefore, $\lim_{x \to x_0} f(x) = f(x_0)$.

$\square$

Suppose one knows that $f : U \to \mathbb{R}^m$ is differentiable at $x_0 \in U$. Then one is interested in computing explicitly the derivative $Df(x_0) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

To solve this problem we pursue the following idea: Since $Df(x_0)$ is linear, it is sufficient to know $Df(x_0)$ on a basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$, where we normalize the v_j by assuming $\|v_j\| = 1$

More generally, let us compute $Df(x_0)v$, some $v \in \mathbb{R}^n$, $\|v\| = 1$. Set $x = x_0 + tv$, $t \in \mathbb{R}$. Since U is open, there exists an $\varepsilon > 0$ such that $x \in U$ for all t with $|t| < \delta$. Therefore, with the notation from Definition 1.1

$$Df(x_0)v = \frac{f(x_0 + tv) - f(x_0)}{t} - \frac{r(x)}{t}.$$

Since $\lim_{t \to 0} \frac{r(x)}{t} = 0$ we obtain

$$Df(x_0)v = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}.$$

This motivates the following definition.

1.5 Definition. Let $U \subset \mathbb{R}^n$ be open $f : U \to \mathbb{R}^n$ be a function, $x_0 \in U$ and $v \in \mathbb{R}^n$, $\|v\|_2 = 1$. If

$$D_v f(x_0) := \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} \in \mathbb{R}^m$$

exists, then $D_v f(x_0)$ is called *directional derivative of* f at x_0 in the direction v.

1.6 Theorem. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $x_0 \in U$. Then $D_v f(x_0)$ exists for all $v \in \mathbb{R}^n$, $\|v\|_2 = 1$, and we have*

$$D_v f(x_0) = Df(x_0) \cdot v.$$

Proof. By hypothesis we have for $t \neq 0$

$$f(x_0 + tv) = f(x_0) + Df(x_0)tv + r(x_0 + tv), \quad \text{and} \quad \lim_{t\to 0} \frac{r(x_0 + tv)}{t} = 0.$$

Hence $\frac{f(x_0+tv)-f(x_0)}{t} = Df(x_0)v + \frac{r(x_0+tv)}{t}$ and thus for $t \to 0$ the assertion.

□

Observe, the converse of Theorem 1.6 is not true in general. Consider e.g.

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) := \begin{cases} \frac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

In Definition 1.5 we defined the directional derivative of a function with respect to an arbitrary direction. The derivatives in the direction of the coordinate axes are particularly distinguished.

1.7 Definition. a) For the derivatives in the direction of the coordinate axes e_j, $j = 1, \ldots, n$, we write

$$\partial_j f(x_0) := \frac{\partial f}{\partial x_j}(x_0) := D_{e_j} f(x_0) = \lim_{t\to 0} \frac{f(x_0 + te_j) - f(x_0)}{t} \quad \text{for} \quad 1 \leq j \leq n,$$

and call $\partial_j f(x_0)$ partial derivative of f at x_0 (with respect to x_j).

b) A function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is called *partially differentiable* at x_0, if all partial derivatives

$$\frac{\partial f}{\partial x_j} = \begin{pmatrix} \partial_j f_1(x_0) \\ \vdots \\ \partial_j f_m(x_0) \end{pmatrix}, \quad 1 \leq j \leq n, \text{ exist at } x_0.$$

c) Analogously, f is called *continuously partially differentiable* at x_0 if all $\frac{\partial f_i}{\partial x_j}$, $1 \leq j \leq n, 1 \leq i \leq m$, are continuous at x_0.

If $\partial_j f_i(x_0)$ exists for some i, j, $1 \leq j \leq n, 1 \leq i \leq m$, then

$$\partial_j f_i(x_0) = \lim_{t\to 0} \frac{1}{t}\big(f_i(x_0 + te_j) - f_i(x_0)\big).$$

Thus f is partially differentiable at x_0 with respect to x_j if the map

$$t \mapsto f(x_{0,1}, \ldots, x_{0,j-1}, t, x_{0,j+1}, \ldots, x_{0,n})$$

as a function of the 1-dimensional variable t is differentiable (in the 1-dimensional sense) at $x_{0,j}$, $x_0 = (x_{0,1}, \ldots, x_{0,n})$. If one chooses the canonical bases in $\mathbb{R}^n$ and $\mathbb{R}^m$ and if one identifies $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ with the matrix $(a_{ij})_{m\times n})$, then we get the following representation for the derivative.

1.8 Theorem. *If $f = (f_1, \ldots, f_m) : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x_0 \in U$, then*

$$Df(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}_{m\times n} .$$

Proof. For $f_1 : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $h = (h_1, \ldots, h_n) = \sum_{j=1}^n h_j e_j \in \mathbb{R}^n$ we have the representation

$$Df_1(x_0)h = \sum_{j=1}^n h_j Df_1(x_0)e_j = \sum_{j=1}^n \frac{\partial f_1}{\partial x_j}(x_0)h_j$$

and analogous representations for $Df_j(x_0)$. The assertion now follows, since the function $f = (f_1, \ldots, f_m)$ is differentiable at x_0 if and only if each coordinate function f_j is differentiable at x_0.

□

1.9 Definition. a) The matrix which was defined in Theorem 1.8 is called *Jacobi matrix* or *functional matrix*

$$Df(x_0) := J_f(x_0) := \frac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_n)} := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} .$$

b) If $m = 1$, then

$$\operatorname{grad} f(x_0) := \left(\frac{\partial f}{\partial x_1}(x_0), \ldots, \frac{\partial f}{\partial x_n}(x_0)\right)$$

is called *gradient* (gradient vector) of f at x_0.
Notation: $\nabla f(x_0) := \operatorname{grad} f(x_0)$ (∇ is pronounced 'nabla').

1.10 Remark. If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x_0 \in U$, then $\nabla f(x_0)$ indicates the direction of the steepest slope of f at x_0 and $-\nabla f(x_0)$ the direction of the steepest decay. This follows from the Cauchy-Schwarz inequality because for $v \in \mathbb{R}^n$, $\|v\|_2 = 1$, we have

$$|D_v f(x)| = |Df(x_0)v| = |\langle \operatorname{grad} f(x_0), v\rangle| \le \|\operatorname{grad} f(x_0)\|_2 \|v\|_2$$

Now the equality sign holds if and only i $\operatorname{grad} f(x_0) = \lambda v$ for some $\lambda \ge 0$.

Example. The function $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) := x^2 \sin(\frac{y}{2}) + e^{3z}$, has the gradient

$$\operatorname{grad} f(x, y, z) = \left(2x \sin \frac{y}{2}, \frac{x^2}{2} \cos \frac{y}{2}, 3e^{3z}\right) .$$

In the following we give a sufficient criterion for the differentiability of a function $f = (f_1, \ldots, f_m) : U \subset \mathbb{R}^n \to \mathbb{R}^m$ at $x_0 \in U$ which is easier to handle than Definition 1.1.

Necessarily, all partial derivatives must exist — see Definition 1.9; otherwise f would not be differentiable (cf. Theorem 1.8) at x_0. Also it is necessary that

$$\lim_{x \to x_0} \frac{r(x)}{\|x - x_0\|} = 0, \text{ where } r(x) = f(x) = f(x_0) - Df(x_0)(x - x_0).$$

It is interesting to note that the existence of all directional derivatives $D_v f(x_0)$, $\|v\|_2 = 1$, does *not* imply even the continuity of f at x_0. Consider, e.g., the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then $D_v f(0,0)$ exists for all $v \in \mathbb{R}^2$ with $\|v\|_2 = 1$, but f is not continuous at $(0,0)$ — see exercises.

But we have

1.11 Theorem. *Let $U \subset \mathbb{R}^n$ be open, $x_0 \in U$, and let $f = (f_1, \ldots, f_m) : U \to \mathbb{R}^m$ be continuously partially differentiable in a neighborhood of x_0. Then f is differentiable at x_0.*

Proof. First we note that f is differentiable at x_0 if and only if all functions $f_1, \ldots, f_m$ are differentiable at x_0. Hence, without loss of generality, let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be real-valued. For $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$ set

$$\begin{aligned} z_0 &:= x_0 \\ z_1 &:= z_0 + h_1 e_1 \\ z_2 &:= z_1 + h_2 e_2 \\ &\vdots \\ z_n &:= z_{n-1} + h_n e_n = x_0 + h \end{aligned}$$

Then $\|x_0 - z_j\|_2 \leq \|h\|_2$ for $0, \ldots, n$. Hence $z_j \in U$ for all $j = 0, \ldots, n$ if h is small enough. The mean value theorem from Section IV.2 now implies

$$\begin{aligned} f(x_0 + h) - f(x_0) &= \big(f(z_n) - f(z_{n-1})\big) + \big(f(z_{n-1}) - f(z_{n-2})\big) + \cdots + \big(f(z_1) - f(z_0)\big) \\ &= \frac{\partial f}{\partial x_n}(\xi_n) h_n + \frac{\partial f}{\partial x_{n-1}}(\xi_{n-1}) h_{n-1} + \cdots + \frac{\partial f}{\partial x_1}(\xi_1) h_1 \qquad (*) \end{aligned}$$

for appropriate $\xi_j \in (z_{j-1}, z_j)$. Therefore,

$$|f(x_0 + h) - f(x_0) - \operatorname{grad}(f(x_0) \cdot h| \leq \sum_{j=1}^{n} \left| \frac{\partial f}{\partial x_j}(\xi_j) - \frac{\partial f}{\partial x_j}(x_0) \right| \cdot |h_j|$$

and hence

$$\frac{1}{\|h\|_2}\,|f(x_0+h)-f(x_0)-\operatorname{grad} f(x_0)\cdot h| \le \sum_{j=1}^{n}\left|\frac{\partial f}{\partial x_j}(\xi_j)-\frac{\partial f}{\partial x_j}(x_0)\right| \overset{h\to 0}{\longrightarrow} 0$$

since the partial derivatives $\frac{\partial f}{\partial x_j}, j = 1, \ldots, n$, are continuous at x_0.

□

1.12 Remarks. a) The equation $(*)$ implies immediately:
If the partial derivatives $\partial_j f_k : U \to \mathbb{R}, 1 \le j \le n, 1 \le k \le m$, are bounded in a neighborhood of x_0, then f is continuous at x_0.

b) For $f = (f_1, \ldots, f_m) : U \subset \mathbb{R}^n \to \mathbb{R}^m$ the following statements are equivalent:

(i) f is continuously partially differentiable, i.e., all partial derivatives exist and are continuous.

(ii) f is continuously differentiable, i.e., f is differentiable and $f' = Df : U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is continuous.

If f is continuously differentiable on U or continuously partially differentiable, we use the notation $f \in C^1(U, \mathbb{R}^m)$ which by the above makes sense.

We summarize the results obtained above in the following diagram.

f is continuously differentiable	$\Longleftrightarrow$	f is continuously partially differentiable
$\Downarrow$		$\Downarrow$
f is differentiable		f is partially differentiable with locally bounded partial derivatives
$\Downarrow$		$\Downarrow$
all directional derivatives $D_v f$ exist		f is continuous
$\Downarrow$		$\Downarrow$
f is partially differentiable		f is continuous with respect to each coordinate

2 Differentiation rules

We begin this section with the chain rule for differentiable functions

2.1 Theorem (Chain rule). *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open, and let $f : U \to \mathbb{R}^m$ and $g : V \to \mathbb{R}^l$ be mappings with $f(U) \subset V$. Assume that f is differentiable in $x_0 \in U$ and g is differentiable in $y_0 := f(x_0)$. Then the mapping $g \circ f : U \to R^l$ is differentiable in x_0 and we have*

$$D(g \circ f)(x_0) = Dg(f(x_0)) \cdot Df(x_0).$$

Written in terms of matrices this is

$$J_{g\circ f}(x_0) = J_g(f(x_0)) \cdot J_f(x_0).$$

In the above theorem "$\cdot$" is used once to signify composition of linear transformations and once to signify multiplication of matrices.

Proof. We set $A := Df(x_0) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $B := Dg(f(x_0)) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^l)$. By assumption we have

$$f(x_0 + h) = f(x_0) + Ah + r_f(x)$$

and

$$g(y_0 + k) = g(y_0) + Bk + r_g(y)$$

where $x = x_0 + h \in U$ and $y = y_0 + k \in V$, with r_f and r_g satisfying

$$\lim_{h\to 0} \frac{r_f(x)}{\|h\|} = 0, \quad \lim_{k\to 0} \frac{r_g(y)}{\|k\|} = 0.$$

Choose $k := f(x_0 + h) - f(x_0) = Ah + r_f(x)$ to get

$$\begin{aligned}(g \circ f)(x_0 + h) &= g\big(\overbrace{f(x_0)}^{=y_0} + \overbrace{Ah + r_f(x)}^{=k}\big)\\ &= g(f(x_0)) + \underbrace{BAh + Br_f(x)}_{=Bk} + r_g(y).\end{aligned}$$

We must show that

$$\lim_{h\to 0} \frac{Br_f(x)}{\|h\|} = 0 \qquad (*)$$

and

$$\lim_{h\to 0} \frac{r_g(y)}{\|h\|} = 0. \qquad (**)$$

We know by Theorem VI.2.6 that $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^l)$ is continuous, and so

$$\lim_{h\to 0} \frac{Br_f(x)}{\|h\|} = B \cdot \lim_{h\to 0} \frac{r_f(x)}{\|h\|} = B0 = 0,$$

which gives $(*)$. By Theorem VI.2.6 we also know that $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is continuous, and furthermore that there exists a constant $M > 0$ with

$$\|Ah\|_{\mathbb{R}^m} \leq M\|h\|_{\mathbb{R}^n}$$

for all $h \in \mathbb{R}^n$. Thus

$$\|k\| = \|Ah + r_f(x)\| \leq \big(M + \frac{\|r_f(x)\|}{\|h\|}\big)\|h\|,$$

and so

$$\frac{\|r_g(y)\|}{\|h\|} = \frac{\|r_g(y)\|}{\|k\|} \cdot \frac{\|k\|}{\|h\|} \leq \frac{\|r_g(y)\|}{\|k\|} \cdot \Big(M + \frac{\|r_f(x)\|}{\|h\|}\Big).$$

Note that $k = f(x_0 + h) - f(x_0) \to 0$ when $h \to 0$ and that by hypothesis $\frac{\|r_g(y)\|}{\|k\|} \to 0$ for $k \to 0$, thus

$$\lim_{h\to 0} \frac{r_g(y)}{\|h\|} = 0.$$

$\square$

2.2 Example. We consider the functions $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\begin{aligned} f(x,y) &:= \big(x^2, xy, xy^2\big) \\ g(u,v,w) &:= \big(\sin u, \cos(uvw)\big). \end{aligned}$$

The function $h := g \circ f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $h(x,y) = \big(\sin x^2, \cos(x^4y^3)\big)$ is differentiable, and we have

$$J_h(x,y) = \begin{pmatrix} 2x\cos x^2 & 0 \\ -4x^3y^3\sin(x^4y^3) & -3x^4y^2\sin(x^4y^3) \end{pmatrix}. \tag{2.1}$$

For the derivatives of f and g we have

$$J_g(u,v,w) = \begin{pmatrix} \cos u & 0 & 0 \\ -vw\sin(uvw) & -uw\sin(uvw) & -uv\sin(uvw) \end{pmatrix}$$

and

$$J_f(x,y) = \begin{pmatrix} 2x & 0 \\ y & x \\ y^2 & 2xy \end{pmatrix},$$

and we can verify that the product of these matrices at the point $(u,v,w) = f(x,y)$ fits with (2.1).

From the chain rule we can now relatively easy get differentiation rules for sums and products of differentiable functions.

2.3 Corollary. *Let $U \subset \mathbb{R}^n$ be open and let $f, g : U \to \mathbb{R}^m$ be differentiable in $x_0 \in U$. Then for arbitrary $\alpha, \beta \in \mathbb{R}$ the function $\alpha f + \beta g$ is differentiable in x_0, and we have*

$$D(\alpha f + \beta g)(x_0) = \alpha Df(x_0) + \beta Dg(x_0).$$

Proof. We let $F := \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) : U \to \mathbb{R}^m \times \mathbb{R}^m$ and $G : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $G(u, v) = \alpha u + \beta v$. Then G is linear and hence differentiable with derivative $DG(u, v) = G$ for all $u, v \in \mathbb{R}^m$, and F is differentiable in x_0 with derivative $DF(x_0) = \left(\begin{smallmatrix} Df(x_0) \\ Dg(x_0) \end{smallmatrix}\right)$. The chain rule implies that $G \circ F$ given by $(G \circ F)(x) = \alpha f(x) + \beta g(x)$ is differentiable in x_0, and that

$$\begin{aligned} D(G \circ F)(x_0) &= DG\big(F(x_0)\big) \cdot DF(x_0) \\ &= G \begin{pmatrix} Df(x_0) \\ Dg(x_0) \end{pmatrix} = \alpha \cdot Df(x_0) + \beta \cdot Dg(x_0). \end{aligned}$$

□

2.4 Corollary (Product rule). *Let $U \subset \mathbb{R}^n$ be open and let $f, g : U \to \mathbb{R}$ be differentiable in $x_0 \in U$. Then $f \cdot g$ is differentiable in x_0, and we have*

$$D(f \cdot g)(x_0) = f(x_0)Dg(x_0) + g(x_0)Df(x_0).$$

Proof. We let $F := \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) : U \to \mathbb{R} \times \mathbb{R}$ and $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $G(\alpha, \beta) := \alpha \cdot \beta$. Then $(f \cdot g)(x) = (G \circ F)(x)$, and the result follows from the chain rule as above.

□

2.5 Corollary. *Let $J \subset \mathbb{R}$ be an interval, let $V \subset \mathbb{R}^m$ be open, let $\gamma = (\gamma_1, \dots, \gamma_m) : J \to V$ be differentiable in $t_0 \in J$ and let $f : V \to \mathbb{R}$ be differentiable in $x_0 = \gamma(t_0)$. Then $f \circ \gamma : J \to \mathbb{R}$ is differentiable in t_0, and we have*

$$\begin{aligned} D(f \circ \gamma)(t_0) &= \langle \operatorname{grad} f(\gamma(t_0)), \gamma'(t_0) \rangle \\ &= \sum_{j=1}^{m} \frac{\partial f}{\partial x_j}(\gamma(t_0))\gamma_j'(t_0). \end{aligned}$$

For the proof we refer to the exercises.

2.6 Remark. Let $J \subset \mathbb{R}$ be an interval. Continuous functions $\gamma : J \to \mathbb{R}^m$ are also called *curves*. We will study these in more detail later. At present we remark that if we think of $t \in J$ as the time and $\gamma(t)$ as the position, then γ describes the movement of a point in $\mathbb{R}^m$ in time. Each curve $J \to \mathbb{R}^m$ can be described by an m-tuple $\gamma = (\gamma_1, \dots, \gamma_m)$, and for a differentiable curve γ we have

$$\gamma'(t_0) = \big(\gamma_1'(t_0), \dots, \gamma_m'(t_0)\big)^T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^m) = \mathbb{R}^m.$$

The vector $\gamma'(t_0)$ is called the *tangent vector* to the curve γ in t_0.

Via the above formulation of the chain rule one can interpret the concepts gradient and level set geometrically as follows. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be differentiable and let $\gamma : J \to U$ be a differentiable curve defined on an interval $J \subset \mathbb{R}$. If γ lies on a level set of f, i.e., if there exists a constant $c \in \mathbb{R}$ such that $f(\gamma(t)) = c$ for all $t \in J$, then the gradient of f in the point $\gamma(t)$ is orthogonal to the tangent vector $\gamma'(t)$. That is,

$$\operatorname{grad} f(\gamma(t)) \perp \gamma'(t), \quad t \in J.$$

If $U \subset \mathbb{R}^2$, then we can interpret the graph of f as a "landscape" over U with $f(x)$ as the "elevation" above x. The statement above then says that the gradient of f in x is perpendicular to the curve through x which follows the landscape with constant elevation. Furthermore, $\operatorname{grad} f(x)$ points in the direction of the steepest increase of f, and $-\operatorname{grad} f(x)$ points in the direction of the steepest descent.

2.7 Example. A differentiable function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is called *positively homogeneous of degree $\alpha \in \mathbb{R}$* if

$$f(tx) = t^\alpha f(x)$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ and all $t \in (0, \infty)$. Corollary 2.5 implies that

$$\langle \operatorname{grad} f(x), x\rangle = \alpha f(x)$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. This identity is also called *Euler's relation*. For the proof of this relation we refer to the exercises.

A further consequence of the chain rule is the following mean value theorem. As in the case for one variable one can use this to express the difference between function values via the derivative.

2.8 Theorem (Mean Value Theorem). *Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be a differentiable function. Let $a, b \in U$ such that the set $\overline{ab} := \{a + t(b - a) : t \in [0, 1]\}$ lies in U. Then there exists $\xi \in \overline{ab}$ such that*

$$f(b) - f(a) = f'(\xi)(b - a).$$

Proof. Define $g : [0, 1] \to U$ by $g(t) = a + t(b - a)$. Then g is differentiable with $g'(t) = b - a$ for all $t \in (0, 1)$. By Corollary 2.5 we know that also $F = f \circ g : [0, 1] \to \mathbb{R}$ is differentiable with $F'(t) = f'(g(t))(b - a)$ for all $t \in (0, 1)$. By the mean value theorem for real-valued functions of one variable, Theorem IV.2.4 from Analysis I, there exists $\tau \in (0, 1)$ such that

$$f(b) - f(a) = F(1) - F(0) = F'(\tau) = f'(g(\tau))(b - a).$$

Thus we let $\xi := g(\tau) \in \overline{ab}$.

$\square$

For convex sets we have the following version of the mean value theorem.

2.9 Theorem. *Let $U \subset \mathbb{R}^n$ be open and convex. If $f : U \to \mathbb{R}$ is differentiable such that there exists $L \geq 0$ with*

$$\|\operatorname{grad} f(x)\| \leq L$$

for all $x \in U$, then

$$|f(x) - f(y)| \leq L\|x - y\|$$

for all $x, y \in U$. That is, f is Lipschitz continuous with Lipschitz constant L.

Proof. By the mean value theorem and the Cauchy-Schwarz inequality we have for suitable $\xi \in \overline{xy}$

$$\begin{aligned} |f(x) - f(y)| &= \langle \operatorname{grad} f(\xi), x - y \rangle \\ &\leq \|\operatorname{grad} f(\xi)\| \|x - y\| \\ &\leq L\|x - y\| \, . \end{aligned}$$

□

2.10 Corollary. *Let $U \subset \mathbb{R}^n$ be an open set such that for any two elements $x, y \in U$ there exist points $x = z_0, z_1, \ldots, z_l = y$ such that $\overline{z_{k-1} z_k} \subset U$ for all $k = 1, \ldots, l$. Let $f : U \to \mathbb{R}$ be differentiable. Then f is constant if and only if* $\operatorname{grad} f(x) = 0$ *holds for all $x \in U$.*

Proof. See exercises.

□

We end this section with a useful variant of the mean value theorem which requires the stronger assumption of the mapping being continuously differentiable.

2.11 Theorem (Integral form of the mean value theorem). *Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be continuously differentiable. Then*

$$f(y) - f(x) = \int_0^1 Df(x + t(y - x))(y - x)dt$$

for all $x, y \in U$ with $\overline{xy} \subset U$.

Proof. We let $\varphi(t) := f(x+t(y-x))$ for $t \in [0, 1]$. Then φ is continuously differentiable, and the fundamental theorem of differential calculus implies that

$$\begin{aligned} f(y) - f(x) &= \varphi(1) - \varphi(0) \\ &= \int_0^1 \varphi'(t)dt \\ &= \int_0^1 Df(x + t(y - x))(y - x)dt. \end{aligned}$$

□

3 Higher Derivatives

When considering a function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ with partial derivatives $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$, it can be the case that these partial derivatives are again partially differentiable. The functions $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i})\right)$ are called *second order partial derivatives* of f. We often write them as

$$\frac{\partial^2 f}{\partial x_j \partial x_i}.$$

In general a function $f : U \to \mathbb{R}$ is called $(\ell + 1)$-times (continuously) partially differentiable if f is ℓ-times partially differentiable and all ℓ-th order partial derivatives are (continuously) partially differentiable. In the following the vector space

$$C^k(U, \mathbb{R}) := \{f : U \to \mathbb{R} : f \text{ is } k \text{ times continuously partially differentiable}\}$$

will play an important role. If f is k times continuously partially differentiable, we will also write

$$\frac{\partial^k f}{\partial x_{i_k} \cdots \partial x_{i_1}} := f_{x_{i_k} \cdots x_{i_1}} := \partial i_k \cdots \partial i_1 f := \frac{\partial}{\partial x_{i_k}} \cdots \frac{\partial}{\partial x_{i_1}} f$$

and

$$\underbrace{\partial_i \cdots \partial_i}_{k \text{ times}} =: \frac{\partial^k}{\partial x_i^k}$$

In the following example we calculate the second partial derivatives of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := x^2 \sin y$. We have

$$\begin{aligned} f_x(x, y) &= 2x \sin x & f_y(x, y) &= x^2 \cos y \\ f_{xx}(x, y) &= 2 \sin y & f_{xy}(x, y) &= 2x \cos y \\ f_{yx}(x, y) &= 2x \cos y & f_{yy}(x, y) &= -x^2 \sin y \end{aligned}$$

and we observe that in this example, $f_{xy} = f_{yx}$ holds. In general this is not the case, as the following example shows.

3.1 Example. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) := \begin{cases} \frac{x^3 y - x y^3}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

Then $f \in C^1(\mathbb{R}^2, \mathbb{R})$, and the partial derivatives f_{xy} and f_{yx} exist everywhere in $\mathbb{R}^2$ and are continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$, but

$$f_{xy}(0, 0) = 1 \quad \text{and} \quad f_{yx}(0, 0) = -1.$$

We verify this in the exercises.

It can even happen that only one of the two partial derivatives $\partial_{ij} f$ and $\partial_{ji} f$ exists. The following theorem, due to H. A. SCHWARZ (1843-1921) ensures that such phenomena cannot occur if $\partial_{ij} f$ or $\partial_{ji} f$ is continuous.

3.2 Theorem (Schwarz' Theorem). *Let $U \subset \mathbb{R}^n$ be open, let $i, j \in \{1, \ldots, n\}$, and let $f : U \to \mathbb{R}$ have partial derivatives $\partial_i f$, $\partial_j f$, and $\partial_{ij} f$ in a neighborhood of $x_0 \in U$. If $\partial_{ij} f$ is continuous in x_0, then $\partial_{ji} f(x_0)$ exists and*

$$\partial_{ij} f(x_0) = \partial_{ji} f(x_0).$$

Proof. We choose $\delta_i, \delta_j > 0$ so small that $x_0 + se_i + te_j \in U$ for all

$$(s, t) \in (-\delta_i, \delta_i) \times (-\delta_j, \delta_j) =: Q \subset \mathbb{R}^2.$$

Then the function $\varphi : Q \to \mathbb{R}$ given by

$$\varphi(s, t) := f(x_0 + se_i + te_j)$$

is well defined and partially differentiable. Furthermore, the second order partial derivative $\partial_1 \partial_2 \varphi$ exists, and is also continuous in $(0, 0)$. We must show that $\partial_2 \partial_1 \varphi(0, 0)$ exists, and that

$$\partial_1 \partial_2 \varphi(0, 0) = \partial_2 \partial_1 \varphi(0, 0)$$

By definition we have

$$\begin{aligned}\partial_2 \partial_1 \varphi(0, 0) &= \left[\frac{d}{dt} \left(\lim_{s \to 0} \frac{\varphi(s, t) - \varphi(0, t)}{s} \right) \right]_{t=0} \\ &= \lim_{t \to 0} \lim_{s \to 0} \frac{1}{s} \frac{[\varphi(s, t) - \varphi(0, t)] - [\varphi(s, 0) - \varphi(0, 0)]}{t}.\end{aligned}$$

We apply the mean value theorem to the difference quotient with respect to the second variable t and get a $\xi \in (0, 1)$ such that

$$\begin{aligned}&\frac{1}{s} \frac{[\varphi(s, t) - \varphi(0, t)] - [\varphi(s, 0) - \varphi(0, 0)]}{t} \\ &\qquad = \frac{1}{s} \partial_2 [\varphi(s, \xi t) - \varphi(0, \xi t)] = \frac{1}{s} [\partial_2 \varphi(s, \xi t) - \partial_2 \varphi(0, \xi t)].\end{aligned}$$

Now we notice that our result is again a difference quotient for $\partial_2 \varphi$ with respect to the variable s, i.e., the first variable. By assumption $\partial_2 \varphi$ is differentiable with respect to the first variable, and by the mean value theorem there exists $\eta \in (0, 1)$ such that

$$\frac{1}{s} [\partial_2 \varphi(s, \xi t) - \partial_2 \varphi(0, \xi t)] = \partial_1 \partial_2 \varphi(\eta s, \xi t).$$

Our assumptions also imply that $\partial_2 \partial_1 \varphi$ is continuous in $(0, 0)$, and hence

$$\partial_2 \partial_1 \varphi(0, 0) = \lim_{t \to 0} \lim_{s \to 0} \partial_2 \partial_1 \varphi(\eta s, \xi t) = \partial_1 \partial_2 \varphi(0, 0).$$

$\square$

3.3 Corollary. *Let $U \subset \mathbb{R}^n$ be open and let $f \in C^k(U, \mathbb{R})$ for some $k \in \mathbb{N}$. Then*

$$\frac{\partial}{\partial x_{i_k}} \cdots \frac{\partial f}{\partial x_{i_1}} = \frac{\partial}{\partial x_{i_{\pi(k)}}} \cdots \frac{\partial f}{\partial x_{i_{\pi(1)}}}$$

for every permutation $\pi : \{1, \ldots, k\} \to \{1, \ldots, k\}$.

The proof proceeds by induction on k, and is left to the reader.

We will now see how we can view higher derivatives of a k-times continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ as *symmetric multilinear maps*

$$D^k f(x_0) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}.$$

This then provides a generalization of the differential. We start with the case $k = 2$, and consider twice continuously differentiable functions, i.e., differentiable f such that Df is continuously differentiable. We define a bilinear form for $x_0 \in \mathbb{R}^n$ by letting $D^2 f(x_0)$ be the function given by

$$D^2 f(x_0)(u, v) := D_u(D_v f)(x_0)$$

for $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$, $u \neq 0, v \neq 0$, and

$$D^2 f(x_0)(u, v) := 0$$

if $u = 0$ or $v = 0$ (Here, we rely on an extension of the definition of the directional derivative from the case $\|v\|_2 = 1$ to the case $v \neq 0$).

By Theorem 1.6, we know that the directional derivative $D_v f(x_0)$ of f in the direction v is given by

$$D_v f(x_0) = Df(x_0)v.$$

Furthermore, the function $D_v f$ is differentiable in x_0, since D_f is. By Theorem 1.6 we can conclude that $D_v f$ has a directional derivative in the direction u at the point x_0 and that

$$\begin{aligned} D_u(D_v f)(x_0) &= D(D_v f)(x_0) \cdot u \\ &= \sum_{i,j=1}^{n} \partial_{ij} f(x_0) v_i u_j. \end{aligned}$$

The mapping

$$(u, v) \mapsto D_u D_v f(x_0)$$

is linear in u and v, and by Schwarz' theorem it is also symmetric. It is called the *second order differential of f in x_0*. Relative to the canonical basis for $\mathbb{R}^n$ this map can be represented by the matrix

$$H_f(x_0) = \begin{pmatrix} \partial_{11} f(x_0) & \cdots & \partial_{1n} f(x_0) \\ \vdots & \ddots & \vdots \\ \partial_{n1} f(x_0) & \cdots & \partial_{nn} f(x_0) \end{pmatrix}.$$

This matrix is called the *Hessian* of f at x_0. By the theorem of Schwarz the Hessian is a symmetric matrix, and we have

$$D^2 f(x_0)(u,v) = u^T H_f(x_0) v. \tag{3.1}$$

In connection with our later treatment of Taylor's theorem we will study the geometric meaning of the second derivative in more detail.

For arbitrary $k \in \mathbb{N}$ we define $D^k f(x_0)$ analogously to the case $k = 2$ by

$$D^k f(x_0)(v^1, \dots, v^k) := D_{v^1} \dots D_{v^k} f(x_0)$$

for $v^1, \dots, v^k \in \mathbb{R}^n$. This mapping is linear in each of the variables $v^1, \dots, v^k$.

In this connection it is useful to introduce the concept of a *multi-index*. By this we mean an n-tuple $\alpha_1, \dots, \alpha_n \in \mathbb{N}_0^n$. The non-negative integer

$$|\alpha| := \alpha_1 + \dots + \alpha_n$$

is called the *order* of α. We further define

$$\alpha! := \alpha_1! \alpha_2! \dots \alpha_n!,$$

and for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we let

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}, \qquad D^0 f := f$$

$$D^\alpha f := D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} f := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}} f.$$

(In our definition of x^α, we use the convention $0^0 = 1$.)

We let P be a polynomial of degree $m \in \mathbb{N}$ with n variables $\xi_1, \dots, \xi_n$, i.e.

$$P(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha.$$

If we replace the variables ξ_i by differential operators ∂_i we get a so-called *linear differential operator* $P(D)$ *of the form*

$$P(D) : C^m(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R}), \quad P(D) := \sum_{|\alpha| \le m} a_\alpha D^\alpha$$

with coefficients a_α.

3.4 Examples. a) A very important example of a differential operator is the *Laplace operator* defined by

$$\Delta := \partial_1^2 + \dots + \partial_n^2.$$

The corresponding polynomial P is in this case given by $P(\xi) = \xi_1^2 + \dots + \xi_n^2$.

b) For $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$ we consider the polynomial $p(\xi) = h_1\xi_1 + \cdots + h_n\xi_n$ and put

$$\nabla h := P(D) = h_1\partial_1 + \cdots + h_n\partial_n.$$

c) For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $l \in \mathbb{N}$ we have

$$(a_1 + \cdots + a_n)^l = \sum_{|\alpha|=l} \frac{l!a^\alpha}{\alpha!}.$$

For a proof of this fact (by induction) we refer to the lecture.

d) For $h = (h_1, \ldots, h_n)$ and $l \in \mathbb{N}$ we have

$$\begin{aligned}(\nabla h)^l &= (h_1\partial_1 + \cdots + h_n\partial_n)^l \\ &= l! \sum_{|\alpha|=l} \frac{h^\alpha \partial^\alpha}{\alpha!}.\end{aligned}$$

One can also identify the Laplace operator with the trace of the Hessian $H_f(x)$ of a twice continuously differentiable function, i.e., we have

$$\operatorname{tr}(H_f(x)) = \sum_{i=1}^n \partial_i^2 f(x) = \Delta f(x). \tag{3.2}$$

Because of this connection we can transfer the rotation invariance of the trace of a matrix to rotation invariance of the Laplace operator. Specifically, we have the following theorem.

3.5 Theorem (Rotation invariance of the Laplace operator). *For each orthonormal basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ we have*

$$\Delta = \partial_{v_1}^2 + \cdots + \partial_{v_n}^2.$$

Proof. By equation (3.1) we have

$$\partial_{v_i}\partial_{v_i} f(x) = v_i^T H_f(x) v_i = e_i^T \tilde{H} e_i,$$

with $\tilde{H} = V^T H_f V$, $V = (v_1, \ldots, v_n)$, and with $e_1, \ldots, e_n$ the vectors in the canonical basis. Hence

$$\sum_{i=1}^n \partial_{v_i}^2 f = \operatorname{tr}(\tilde{H}).$$

Since V by assumption is orthogonal we know that the matrices H_f and $\tilde{H}$ have the same trace, i.e.

$$\operatorname{tr}(\tilde{H}) = \operatorname{tr}(H_f).$$

Thus the claim follows by (3.2).

$\square$

The Laplace operator (also called *Laplacian*) appears in many differential equations in analysis and in physics. We mention the following examples:

a) The Laplace equation
$$\Delta u = 0.$$
This equation describes diffusion processes and makes frequent appearances in probability theory. The solutions of this equation are called *harmonic functions*. In dimension 2, these form the starting point of the subject of complex analysis.

b) The wave equation
$$u_{tt} = c^2 \Delta u$$
describes the oscillation of an elastic body.

c) The heat equation
$$u_t = c\Delta u$$
describes the propagation of heat in a homogeneous medium.

d) The Schrödinger equation
$$u_t = i\Delta u$$
is the central equation of quantum mechanics.

In equations b), c) and d) above, u is a function of $n+1$ variables $x_1, \dots, x_n, t$, with
$$\Delta = \partial_1^2 + \cdots + \partial_n^2$$
and with t meant to be interpreted as the time. c is a constant.

We round off this section with the calculation of Δf for a rotation symmetric function. Let $F \in C^2(J, \mathbb{R})$, with $J \subset (0, \infty)$ an interval. We let
$$f(x) := F(\|x\|_2)$$
and $r := \|x\|_2$.

Then
$$\partial_i f(x) = F'(r)\frac{x_i}{r}$$
and
$$\partial_i^2 f(x) = F''(r)\frac{x_i^2}{r^2} + F'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right).$$
Thus
$$\Delta f(x) = F''(r) + \frac{n-1}{r}F'(r),$$

and we have $\Delta f = 0$ if and only if the equation

$$F''(r) + \frac{n-1}{r} F'(r) = 0$$

is satisfied. We can easily verify that for $n > 2$ the function F given by $F(r) := r^{2-n}$ is a solution for this equation. Thus the function $N : \mathbb{R}^n \setminus \{0\} \to \mathbb{R},\ n > 2$, given by

$$N(x) := \frac{1}{\|x\|_2^{n-2}}$$

is a solution of the Laplace equation $\Delta f = 0$. The function N coincides, modulo a constant, with the so-called *Newton potential* on $\mathbb{R}^n \setminus \{0\}$.

4 Taylor's Theorem

Let us recall Taylor's theorem in the 1-dimensional situation:

Taylor's theorem in one variable. *If $f \in C^{m+1}(J,\mathbb{R})$, $J \subset \mathbb{R}$ being an interval, and if $0, x \in J$, then*

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(m)}(0)}{m!}x^m + R_{m+1}f(x,0) \qquad (*)$$

where $R_{m+1}f(x,0)$ denotes the remainder term of the Taylor approximation. Its Lagrange form reads as follows

$$R_{m+1}f(x,0) = \frac{f^{m+1}(\xi)}{(m+1)!}x^{m+1} \qquad \text{for some } \xi \in (0,x) \quad (\text{if } x > 0).$$

The aim of this formula is to approximate "nicely" a given smooth function by a polynomial in a neighborhood of $x = 0$. In the following we consider the analogous problem for functions of several variables, more precisely for $f \in C^{m+1}(U,\mathbb{R})$, where $U \subset \mathbb{R}^n$ denotes an open set. Thus we look for a polynomial p in n variables, which "nicely" approximates f in a neighborhood of $x = 0$.

4.1 Theorem (Taylor's theorem in n variables). *Let $U \subset \mathbb{R}^n$ be an open set, $a, x \in U$, with $\overline{ax} \subset U$, and let $f \in C^{m+1}(U,\mathbb{R})$. Then there exists some $\xi \in \overline{ax}$ such that*

$$\begin{aligned} f(x) &= \sum_{j=0}^{m} \frac{(\nabla(x-a))^j f(a)}{j!} + \frac{(\nabla(x-a))^{m+1} f(\xi)}{(m+1)!} \\ &= \sum_{|\alpha| \le m} \frac{D^\alpha f(a)}{\alpha!}(x-a)^\alpha + \sum_{|\alpha| = m+1} \frac{D^\alpha f(\xi)}{\alpha!}(x-a)^\alpha \\ &=: T_m f(x,a) + R_{m+1}f(x,a). \end{aligned}$$

Proof. We divide the proof into two steps.

Step 1: calculate the directional derivatives of f at a.
Set $h := (h_1, \ldots, h_n) := (x-a) \in \mathbb{R}^n$. Then the function

$$F : [0,1] \to \mathbb{R}, \qquad F(t) := f(a+th)$$

is $(m+1)$-times continuously differentiable, since by the chain rule we obtain

$$F'(t) = \frac{d}{dt}f(a+th) = \sum_{i=1}^{n} \partial_i f(a+th)h_i = (\nabla h)f(a+th).$$

The same argument applied upon $g := (\nabla h)f$ yields

$$F''(t) = \frac{d}{dt}g(a+th) = (\nabla h)^2 f(a+th).$$

By induction we see $F \in C^{m+1}[0,1]$ with

$$F^{(\ell)}(t) = (\nabla h)^\ell f(a+th), \quad t \in [0,1], \;\; \ell \in \{0,1,\dots,m+1\}. \tag{$**$}$$

Step 2: Apply the 1-dim. version of Taylor's formula $(*)$.
Observe that $f(x) = F(1)$ and $f(a) = F(0)$; then it follows by $(*)$ and $(**)$ that

$$\begin{aligned}
f(x) = F(1) &= \sum_{l=0}^{m} \frac{F^{(\ell)}(0)}{\ell!} + \frac{F^{(m+1)}(\tau)}{(m+1)!} \qquad\qquad \text{some } \tau \in [0,1] \\
&= \sum_{l=0}^{m} \frac{(\nabla h)^\ell f(a)}{\ell!} + \frac{(\nabla h)^{m+1} f(a+\tau h)}{(m+1)!} \\
&= \sum_{\ell=0}^{m} \sum_{|\alpha|=\ell} \frac{h^\alpha D^\alpha f(a)}{\alpha!} + \sum_{|\alpha|=m+1} \frac{h^\alpha D^\alpha f(a+\tau h)}{\alpha!} \\
&= \sum_{|\alpha|\le m} \frac{D^\alpha f(a)}{\alpha!}(x-a)^\alpha + \sum_{|\alpha|=m+1} \frac{D^\alpha f(\xi)}{\alpha!}(x-a)^\alpha
\end{aligned}$$

for a suitable $\tau \in [0,1]$ and hence $\xi := a + \tau h \in \overline{ax}$.

□

4.2 Remarks. a) Analogously to the case of one variable one calls

$$T_m f(x,a) = \sum_{|\alpha|\le m} \frac{D^\alpha f(a)}{\alpha!}(x-a)^\alpha$$

the *Taylor polynomial of degree* m of f around a. Further

$$R_{m+1} f(x,a) = \sum_{|\alpha|=m+1} \frac{D^\alpha f(\xi)}{\alpha!}(x-a)^\alpha$$

is called *remainder term*, here in Lagrange's form.

b) For $m = 0$ Taylor's theorem is identical to the mean value theorem.

c) Let us explicitly specify the Taylor polynomial for $m = 2$ in the case of n real variables:

$$\begin{aligned}
&T_2 f(x,a) = f(a) + (\nabla(x-a))f(a) + \tfrac{1}{2!}(\nabla(x-a))^2 f(a) \\
&= f(a) + \sum_{i=1}^{n} \partial_i f(a)(x_i - a_i) + \frac{1}{2}\sum_{i=1}^{n} \partial_i^2 f(a)(x_i-a_i)^2 \\
&\qquad\qquad\qquad\qquad + \sum_{\substack{i,j=1 \\ i\ne j}}^{n} \partial_{ij} f(a)(x_i-a_i)(x_j-a_j).
\end{aligned}$$

d) In the special case $n = 2$, $m = 3$ we have for $f \in C^3$

$$
\begin{aligned}
T_3 f\big((x,y),a_1,a_2)\big) &= f(a_1,a_2) + f_x(a_1,a_2)(x-a_1) + f_y(a_1,a_2)(y-a_2)\\
&+ \frac{1}{2} f_{xx}(a_1,a_2)(x-a_1)^2 + f_{xy}(a_1,a_2)(x-a_1)(y-a_2) + \frac{1}{2} f_{yy}(a_1,a_2)(y-a_2)^2\\
&+ \frac{1}{6} f_{xxx}(a_1,a_2)(x-a_1)^3 + \frac{1}{6} f_{yyy}(a_1,a_2)(y-a_2)^3\\
&+ \frac{1}{2} f_{xxy}(a_1,a_2)(x-a_1)^2(y-a_2) + \frac{1}{2} f_{xyy}(a_1,a_2)(x-a_1)(y-a_2)^2
\end{aligned}
$$

e) If $U \subset \mathbb{R}^n$ is (as usual) open, $f \in C^m(U,\mathbb{R})$ and $a \in U$, then

$$\lim_{x \to a} \frac{f(x) - T_m f(x,a)}{\|x-a\|^m} = 0.$$

To see this, first choose $\delta > 0$ so small that $U_\delta(a) \subset U$. By Taylor's theorem there exists to each $x \in U_\delta(a)$ a $\tau \in [0,1]$ such that

$$
\begin{aligned}
f(x) - T_m f(x,a) &= T_{m-1} f(x,a) + \sum_{|\alpha|=m} \frac{D^\alpha f(a+\tau(x-a))}{\alpha!}(x-a)^\alpha - T_m f(x,a)\\
&= \sum_{|\alpha|=m} \frac{(x-a)^\alpha}{\alpha!}\big(D^\alpha f(a+\tau(x-a)\big) - D^\alpha f(a)\big).
\end{aligned}
$$

Since $\frac{|(x-a)^\alpha|}{\|x-a\|^m} \le 1$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = m$ and since $D^\alpha f$ is continuous for all such α we finally get

$$0 \le \frac{|f(x) - T_m(f(x,a)|}{\|x-a\|^m} \le \sum_{|\alpha| \le m} \frac{1}{\alpha!} |D^\alpha f(a+\tau(x-a)) - D^\alpha f(a)| \overset{x \to a}{\longrightarrow} 0$$

4.3 Example. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) := \sin\big(\frac{x+2y}{2}\big) + \cos\big(\frac{2x-y}{2}\big)$ and $a = (0,0)$. According to Remark 4.2 c), we have

$$
\begin{aligned}
T_2 f\big((x,y),(0,0)\big) &= f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{f_{xx}(0,0)}{2!}x^2\\
&\quad + f_{xy}(0,0)xy + \frac{f_{yy}(0,0)}{2!}y^2\\
&= 1 + \frac{x}{2} + y - \frac{x^2}{2} + \frac{1}{2}xy - \frac{y^2}{8}
\end{aligned}
$$

after the calculation of the required partial derivatives at $(0,0)$.

Starting with the mean value theorem 2.11 in integral form, i.e.,

$$f(a+h) = f(a) + \int_0^1 Df(x+th)h\,dt,$$

we now want to represent the remainder term in integral form.

4.4 Corollary (Taylor's theorem with remainder in integral form). *Let* $U \subset \mathbb{R}^n$ *be an open set,* $a, x \in U$ *with* $\overline{ax} \subset U$ *and* $f \in C^{m+1}(U, \mathbb{R})$*. Then*

$$f(x) = \sum_{|\alpha| \le m} \frac{D^\alpha f(a)}{\alpha!}(x-a)^\alpha + \int_0^1 \frac{(1-t)^m}{m!}\big((\nabla(x-a))^{m+1} f\big)\big(a + t(x-a)\big)\, dt.$$

Proof. (by induction): For $m = 0$ the assertion coincides with the above mean value theorem in integral form. If $m = 1$ define two functions

$$\begin{aligned} &u : [0,1] \to \mathbb{R}, \qquad && u(t) := Df(a+th) \cdot h = (\nabla h) f(a+th) \\ &v : [0,1] \to \mathbb{R}, \qquad && v(t) := t - 1 \end{aligned}$$

By the product rule $(u(t)v(t))' = u'(t)(t-1) + u(t)$ we obtain for the integral in the mean value theorem (note $u'(t) = (\nabla h)^2 f(a+th)$)

$$\begin{aligned} \int_0^1 (\nabla h) f(a+th)\, dt = \int_0^1 u(t)\, dt &= \int_0^1 \frac{d}{dt}(u(t)v(t)\, dt - \int_0^1 u'(t)(t-1)\, dt \\ &= u(1)v(1) - u(0)v(0) + \int_0^1 (1-t)((\nabla h)^2 f)(a+th)\, dt \\ &= (\nabla h) f(a) + \int_0^1 (1-t)((\nabla h)^2 f)(a+th)\, dt. \end{aligned}$$

The general induction step follows the same pattern and is left as an exercise.

4.5 Remark. In this connection let us discuss the idea of the tangent plane. If $U \subset \mathbb{R}^n$ is open and $f \in C^1(U, \mathbb{R})$, we say that a surface F in the $(n+1)$-dimensional space is given by the equation $z = f(x), x \in U$. It is described by the *graph* of f,

$$\operatorname{graph} f = \{(x, z) \in \mathbb{R}^{n+1} : z = f(x), x \in U\}.$$

We have seen that the Taylor polynomial $T_1 f(x, a)$ is an affine function, which approximates f near $a \in U$. The hyperplane

$$z = T_1 f(x, a) = f(a) + Df(a)(x - a)$$

is called *tangent plane* to the surface $z = f(x)$ at the point $(a, f(a))$. The vector

$$v = (\nabla f(a), -1)$$

is called *normal vector* of the tangent plane at the point $(a, f(a))$.

In the case $n = 1$ one speaks of a *curve* instead of a surface and of a *tangent* instead of a tangent plane. In this case we have as equation of the tangent to the curve $z = f(x)$

$$t_z(x) = f(a) + f'(a)(x - a)$$

which we already met in Chapter IV. In the case $n = 2$ the equation of the tangent plane at the point (a_1, a_2) reads

$$t_z(x, y) = f(a_1, a_2) + f_x(a_1, a_2)(x - a_1) + f_y(a_1, a_2)(y - a_2).$$

5 Local extreme points

In this paragraph we discuss sufficient criteria for local maxima and minima of real valued functions of several real variables. We start with the definition of local extrema.

5.1 Definition. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be a function. A point $a \in U$ is called a *local maximum (minimum)* if there exists a neighborhood $\tilde{U} \subset U$ of a such that

$$f(x) \le f(a) \quad \forall\, x \in \tilde{U} \qquad (f(x) \ge f(a) \;\; \forall\, x \in \tilde{U} \text{ for minima}).$$

If $f(x) = f(a)$ only for $x = a$, then f is said to have an *isolated maximum (minimum)* at a.

A *local extremum* is a local maximum or minimum.

5.2 Theorem. *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be partially differentiable at $a \in U$. If f exhibits a local extremum at a, then*

$$\operatorname{grad} f(a) = 0.$$

Proof. Choose $\delta > 0$ so small that the functions g_i, defined by

$$g_i : (-\delta, \delta) \to \mathbb{R},\; g_i(t) := f(a + te_i),\; i \in \{1, \dots, n\},$$

are well defined and differentiable at $t = 0$. Since each g_i exhibits a local extremum at $t = 0$, Theorem IV.3.8 (Analysis I) yields

$$0 = g_i'(0) = \partial_i f(a), \quad i \in \{1, \dots, n\}.$$

$\square$

If $f : U \to \mathbb{R}$ is differentiable at $a \in U$ and if $\operatorname{grad} f(a) = 0$, then a is called a *critical point* of f.

5.3 Examples. a) Consider $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) := x^2 + y^2$. Then $\operatorname{grad} f(x) = (2x, 2y) \stackrel{!}{=} (0, 0)$ if and only if $x = 0$ and $y = 0$. Further $f(x, y) > 0$ for all $(x, y) \ne (0, 0)$. Hence f has at $(0, 0)$ an isolated minimum.

b) Consider $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^2$. Then $\operatorname{grad} f(x) = (2x, -2y) \stackrel{!}{=} (0, 0)$ if and only if $x = 0$ and $y = 0$. Further $f(x, 0) > 0$ for all $x \ne 0$ and $f(0, y) < 0$ for all $y \ne 0$. This implies that f has no local extremum at $(0, 0)$, but a so-called saddle point.

Let us now pursue the question: How can one distinguish the above sketched cases systematically? First recall the sufficient criterion for extremal values of functions of one real variable from Analysis I:

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function and let a be a critical point of f. Then a is

- *a local minimum of f if $f''(a) > 0$*
- *a local maximum of f if $f''(a) < 0$.*

For functions of n real variables we replace $f''(a)$ by the Hessian matrix $(\partial_i \partial_j f(a))_{n\times n}$. We recall its definition: Let $U \subset \mathbb{R}^n$ be open (as is always assumed), $f \in C^2(U, \mathbb{R})$. The Hessian matrix of f at $a \in U$ is given by

$$H_f(a) = \begin{pmatrix} \partial_1\partial_1 f(a) & \partial_1\partial_2 f(a) & \cdots & \partial_1\partial_n f(a) \\ \partial_2\partial_1 f(a) & \partial_2\partial_2 f(a) & \cdots & \partial_2\partial_n f(a) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_n\partial_1 f(a) & \partial_n\partial_2 f(a) & \cdots & \partial_n\partial_n f(a) \end{pmatrix}_{n\times n}.$$

5.4 Remarks.

a) $H_f(a)$ is a symmetric matrix by the Theorem of Schwarz 3.2.

b) The polynomial formula in Section 3 implies for $h = (h_1, \dots, h_n) \in \mathbb{R}^n$ and $f \in C^2(U, \mathbb{R})$

$$\begin{aligned}(h_1\partial_1 + \cdots + h_n\partial_n)^2 f(a) &= 2\sum_{|\alpha|=2} \frac{h^\alpha}{\alpha!} D^\alpha f(a) \\ &= \sum_{i,j=1}^{n} D_i D_j f(a) h_i h_j = \langle h, H_f(a) h\rangle.\end{aligned}$$

Thus, by Taylor's theorem and Remark 4.2 c)

$$f(x) = f(a) + \langle \operatorname{grad} f(a), x-a\rangle + \frac{1}{2}\langle (x-a), H_f(a)(x-a)\rangle + r(x), \quad x \in U,$$

with

$$\lim_{x\to a} \frac{r(x)}{\|x-a\|^2} = 0.$$

For the identification of local extrema we also need the following concept of definiteness of a matrix from linear algebra.

5.5 Definition. A symmetric matrix $T \in M_n(\mathbb{R})$ is called

a) *positive definite* if $\langle x, Tx\rangle > 0 \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}$

b) *negative definite* if $\langle x, Tx\rangle < 0 \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}$

c) *indefinite* if there exist $x, y \in \mathbb{R}^n$ such that $\langle x, Tx\rangle > 0$ and $\langle y, Ty\rangle < 0$.

The property of a symmetric matrix to be positive or negative definite can be characterized by the signs of the eigenvalues of T.

5.6 Theorem. *Let $T \in M_n(\mathbb{R})$ be symmetric. Then*

a) the following are equivalent

- *T is positive definite*
- *all eigenvalues of T are positive*
- *for all $k \in \{1, \ldots, n\}$,* $\det \begin{pmatrix} t_{11} & \cdots & t_{1k} \\ \vdots & \ddots & \vdots \\ t_{k1} & \cdots & t_{kk} \end{pmatrix}_{k\times k} > 0.$

b) the following are equivalent

- *T is negative definite*
- *all eigenvalues of T are negative*
- *for all $k \in \{1, \ldots, n\}$,* $(-1)^k \det \begin{pmatrix} t_{11} & \cdots & t_{1k} \\ \vdots & \ddots & \vdots \\ t_{k1} & \cdots & t_{kk} \end{pmatrix}_{k\times k} > 0.$

c) the following are equivalent

- *T is indefinite*
- *T has positive and negative eigenvalues.*

For a proof we refer to Linear Algebra.

5.7 Theorem (Sufficient criterion for local extrema). *Let $U \subset \mathbb{R}^n$ be open, $f \in C^2(U, \mathbb{R})$ and $a \in U$ a critical point of f. Then we have*

a) If $H_f(a)$ is positive definite, then f has an isolated minimum at a.

b) If $H_f(a)$ is negative definite, then f has an isolated maximum at a.

c) If $H_f(a)$ is indefinite, then f has no local extremum at a.

Proof. By Remark 5.4.b) and 4.2.e) we have for $x \in U$, $h := (x - a)$

$$f(x) = f(a) + \frac{1}{2}\langle h, H_f(a)h\rangle + r(x), \quad \lim_{x\to a} \frac{r(x)}{\|h\|^2} = 0.$$

This implies that to each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$0 \le |r(x)| < \varepsilon\|h\|^2 \quad \text{for all} \quad h \in U_\delta(0) := \{v \in \mathbb{R}^n : \|v\| < \delta\}.$$

Ad a). By hypothesis, $H_f(a)$ is positive definite. Since the continuous function $h \mapsto \langle h, H_f(a)h\rangle$ attains its minimum on the compact unit sphere

$$\mathbb{S}^{n-1} := \{v \in \mathbb{R}^n : \|v\|_2 = 1\},$$

there exists a $v_0 \in \mathbb{S}^{n-1}$ with

$$\langle v, H_f(a)v\rangle \geq \langle v_0, H_f(a)v_0\rangle =: m > 0 \quad \forall\, v \in \mathbb{S}^{n-1}.$$

If one sets $v := \frac{h}{\|h\|}$ for $h \in \mathbb{R}^n \setminus \{0\}$, one obtains

$$\langle h, H_f(a)h\rangle \geq m\|h\|^2 \quad \forall\, h \in \mathbb{R}^n.$$

Now choose $\delta > 0$ so small that

$$\frac{m}{4}\|h\|^2 > |r(x)| \geq 0 \quad \forall\, h = (x-a) \in U_\delta(0),\ x, a \in U.$$

Therefore, by the above Taylor formula,

$$\begin{aligned} f(x) = f(a) + \frac{1}{2}\langle h, H_f(a)h\rangle + r(x) &> f(a) + \frac{m}{2}\|h\|^2 - \frac{m}{4}\|h\|^2 \\ &> f(a), \quad h \in U_\delta(0). \end{aligned}$$

Thus f possesses an isolated minimum at a.

Ad b). Apply a) upon $-f$ to get the assertion.

Ad c). By hypothesis we know: There exist $v, w \in \mathbb{R}^n$ such that

$$\langle v, H_f(a)v\rangle > 0 \quad \text{and} \quad \langle w, H_f(a)w\rangle < 0.$$

Now choose $\delta > 0$ such that $a + tv \in U$ and $a + tw \in U$ $\forall\, t \in (-\delta, \delta)$ and set

$$\begin{aligned} F_v &: (-\delta, \delta) \to \mathbb{R}, & F_v(t) &:= f(a + tv), \\ F_w &: (-\delta, \delta) \to \mathbb{R}, & F_w(t) &:= f(a + tw). \end{aligned}$$

The functions F_v and F_w are twice continuously differentiable and by the chain rule, by a being a critical point,

$$F_v'(0) = (\nabla v)f(a) = 0, \qquad F_w'(0) = (\nabla w)f(a) = 0.$$

Further

$$F_v''(0) = (\nabla v)^2 f(a) = \langle v, H_f(a)v\rangle > 0, \qquad F_w''(0) = (\nabla w)^2 f(a) = \langle w, H_f(a)w\rangle < 0.$$

Thus, F_v has a local minimum at $t = 0$, F_w a local maximum at $t = 0$. Therefore, f cannot have a local extremum at a.

$\square$

5.8 Examples. First consider the examples from 5.3.

a) For $f(x,y) = x^2+y^2$ we have $H_f(0,0) = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, which implies that the eigenvalues are positive, i.e., $H_f(0,0)$ is positive definite and f has a local minimum at $(0,0)$.

b) For $f(x,y) = x^2 - y^2$ we have $H_f(0,0) = \left(\begin{smallmatrix} 2 & 0 \\ 0 & -2 \end{smallmatrix}\right)$, the eigenvalues have different signs, $H_f(0,0)$ is indefinite, f has no local extremum at $(0,0)$.

c) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) := x^3 + y^3 - 3xy$. Then $\operatorname{grad} f(x) = (3x^2 - 3y, 3y^2 - 3x)$, and the necessary condition $\operatorname{grad} f(x) \stackrel{!}{=} 0$ leads to the critical points $(0,0)$ and $(1,1)$ of f. The Hessian reads

$$H_f(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

so that we have to test $H_f(0,0)$ and $H_f(1,1)$ for definiteness.
The two eigenvalues of $H_f(0,0) = \left(\begin{smallmatrix} 0 & -3 \\ -3 & 0 \end{smallmatrix}\right)$ are ± 3. Therefore, $H_f(0,0)$ is indefinite which implies that f has no local extremum at $(0,0)$.
For $(1,1)$, we have $H_f(1,1) = \left(\begin{smallmatrix} 6 & -3 \\ -3 & 6 \end{smallmatrix}\right)$, thus $\det H_f(1,1) = 36 - 9 > 0$ and $\det(6)_{1\times 1} > 0$, which entails that $H_f(1,1)$ is positive definite. Hence f attains an isolated minimum at $(1,1)$.

6 Differentiation of integrals with parameters

In physics, variational principles are frequently used, for example to determine the path of a moving system as the extremum of a certain variational problem. For a rigorous treatment of such variational problems we need differentiability properties of integrals with parameters. We consider in this section only integrals with a compact domain of integration; the general case of non-compact domains of integration will be first considered in the context of Lebesgue integration.

Let $U \subset \mathbb{R}^n$ be open and $J \subset \mathbb{R}$ be a compact interval. We consider a function $f : U \times J \to \mathbb{R}$ with the property that for every $x \in U$ the mapping $t \mapsto f(x,t)$ is continuous. Let us define

$$F : U \to \mathbb{R}, \qquad F(x) := \int_J f(x,t)\,dt.$$

The following theorem holds.

6.1 Theorem (Differentiation of integrals with parameters).

a) If f is continuous on $U \times J$, then F is continuous on U.

b) If, moreover, f is continuously partially differentiable with respect to x_i, then F is also continuously partially differentiable with respect to x_i and one can "differentiate under the integral sign", that is

$$\frac{\partial F}{\partial x_i}(x) = \int_J \frac{\partial f}{\partial x_i}(x,t)\,dt.$$

Proof. Let us remark first that it suffices to consider the case $U \subset \mathbb{R}$.

Ad a). Let $x \in U$ and $(x_k) \subset U$ be a sequence with $\lim_{k\to\infty} x_k = x$ and $x_k \neq x$ for all $k \in \mathbb{N}$. Then f is uniformly continuous on the compact set $J \times \{x, x_1, x_2, x_3, \dots\}$, so for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $k \in \mathbb{N}$,

$$|x_k - x| < \delta \quad \text{implies} \quad |f(x_k,t) - f(x,t)| < \varepsilon \quad \text{for all} \quad t \in J.$$

Hence, $(f(x_k,\cdot))_k$ converges uniformly to $f(x,\cdot)$. By Theorem V.2.13, we get that

$$\lim_{k\to\infty} \int_J f(x_k,t)\,dt = \int_J f(x,t)\,dt.$$

Thus, F is continuous at x and, since $x \in U$ was taken arbitrary, it follows that f is continuous on U.

Ad b). As in a), let $x \in U$ and (x_k) be a sequence in U such that $\lim_{k\to\infty} x_k = x$ and $x_k \neq x$ for all $k \in \mathbb{N}$. Moreover, let $K \subset U$ be a compact interval with $x \in K$, namely $K = [c, d]$, $c \neq d$. The function $\frac{\partial f}{\partial x_i}$ is continuous on the compact set $K \times J$, hence for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\tilde{x} \in K$,

$$|x - \tilde{x}| < \delta \quad \text{implies} \quad \left| \frac{\partial f}{\partial x_i}(\tilde{x}, t) - \frac{\partial f}{\partial x_i}(x, t) \right| < \varepsilon \quad \text{for all} \quad t \in J.$$

Applying the classical Mean Value Theorem IV.2.4, we obtain the existence of ξ_k between x and x_k satisfying for all $t \in J$

$$\frac{f(x_k, t) - f(x, t)}{x_k - x} = \frac{\partial f}{\partial x}(\xi_k, t).$$

Since $\lim_{k \to \infty} x_k = x$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $|x_k - x| < \delta$ and $x_k \in U$. It follows that $|\xi_k - x| < \delta$ for all $k \geq N$, hence

$$\left| \frac{\partial f}{\partial x}(x, t) - \frac{f(x_k, t) - f(x, t)}{x_k - x} \right| = \left| \frac{\partial f}{\partial x}(x, t) - \frac{\partial f}{\partial x}(\xi_k, t) \right| < \varepsilon$$

for all $k \geq \mathbb{N}$ and $t \in J$.

Thus, $\left(\frac{f(x_k, \cdot) - f(x, \cdot)}{x_k - x} \right)$ converges uniformly on J to $\frac{\partial f}{\partial x}(x, \cdot)$. Applying again Theorem V.2.13, we obtain that

$$\lim_{k \to \infty} \frac{F(x_k) - F(x)}{x_k - x} = \lim_{k \to \infty} \int_J \frac{f(x_k, t) - f(x, t)}{x_k - x} \, dt = \int_J \frac{\partial f}{\partial x}(x, t) \, dt.$$

It follows that F is partially differentiable with respect to x and

$$F'(x) = \int_J \frac{\partial f}{\partial x}(x, t) \, dt.$$

Since $\frac{\partial f}{\partial x}$ is continuous on $U \times J$ by hypotheses, we can apply a) to conclude that F' is continuous.

□

As a first application of the above theorem we prove the permutability of the order of integration for iterated integrals.

6.2 Theorem. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous. Then*

$$\int_a^b \left(\int_c^d f(x, t) \, dt \right) dx = \int_c^d \left(\int_a^b f(x, t) \, dx \right) dt.$$

Proof. Let us define the functions $F_1, F_2 : [a, b] \to \mathbb{R}$ by

$$F_1(\xi) := \int_a^\xi \left(\int_c^d f(x, t) \, dt \right) dx, \qquad F_2(\xi) := \int_c^d \left(\int_a^\xi f(x, t) \, dx \right) dt.$$

Since the integrand of F_1 is continuous on $[a,b]$, we can apply the Fundamental Theorem of Differential and Integral Calculus to conclude that F_1 is differentiable and $F_1'(\xi) = \int_c^d f(\xi,t)\,dt$. By the above Theorem 6.1, F_2 is differentiable with $F_2'(\xi) = \int_c^d f(\xi,t)\,dt$. It follows that $F_1' = F_2'$ and thus, since $F_1(a) = F_2(a) = 0$, that $F_1 = F_2$. In particular, $F_1(b) = F_2(b)$, hence the conclusion.

□

By a repeated application of the above procedure, one can define the iterated integral of a continuous function on a rectangle $Q := [a_1,b_1] \times \cdots \times [a_k,b_k] \subset \mathbb{R}^k$ as

$$\int_Q f(x) := \int_{a_k}^{b_k} \left(\dots \int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} f(x_1,\dots,x_k)\,dx_1 \right] dx_2 \dots \right) dx_k.$$

In the following, we consider the so-called Euler differential equation of the variational calculus. Let us study the following problem: Determine the surface of revolution between to coaxial circular lines with minimal surface area. More exactly, we try to find, given two points (a,α) and (b,β) with $a < b$, a continuously differentiable function $f : [a,b] \to \mathbb{R}_+$ with $f(a) = \alpha$ and $f(b) = \beta$ such that the surface obtained by rotating its graph around the x-axis has minimal surface area. The surface area is given by

$$F(f) = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx,$$

as we shall prove later.

Based on this example, we consider more generally a twice continuously differentiable function

$$L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (t,y,p) \mapsto L(t,y,p).$$

For $\alpha, \beta \in \mathbb{R}$ let

$$V := \{\varphi \in C^2([a,b],\mathbb{R}) :\ \varphi(a) = \alpha,\ \varphi(b) = \beta\}$$

and define

$$J : V \to \mathbb{R}, \qquad J(\varphi) := \int_a^b L(t,\varphi(t),\varphi'(t))\,dt.$$

We are looking for $\varphi \in V$ at which an extremum of J is attained. In the above example, $L(t,y,p) = y\sqrt{1+p^2}$. The extremal problem formulated here is of a special kind, since the domain of J is (a subset of) the infinite-dimensional vector space V.

The following theorem gives a necessary condition for the existence of an extremum of J.

6.3 Theorem (Euler differential equation). *If $\varphi \in V$ is such that $J(\varphi) = \inf_{\psi \in V} J(\psi)$, then the Euler differential equation*

$$\frac{d}{dt}\frac{\partial L}{\partial p}(t,\varphi(t),\varphi'(t)) = \frac{\partial L}{\partial y}(t,\varphi(t),\varphi'(t))$$

holds.

Proof. Let $\varphi \in V$ be such that $J(\varphi) = \inf_{\psi \in V} J(\psi)$ and $g \in C^2([a,b],\mathbb{R})$ be a function with $g(a) = g(b) = 0$. Then $\varphi + \varepsilon g \in V$ for all $\varepsilon \in \mathbb{R}$, hence $J(\varphi) \leq J(\varphi + \varepsilon g)$ for all $\varepsilon \in \mathbb{R}$. Let us define

$$F : \mathbb{R} \to \mathbb{R}, \quad F(\varepsilon) := J(\varphi + \varepsilon g).$$

Since F has a minimum at $\varepsilon = 0$, we must have $\frac{dF}{d\varepsilon}(0) = 0$. Applying Theorem 6.1 we differentiate under the integral sign and obtain

$$\begin{aligned} \frac{dF}{d\varepsilon}(\varepsilon) &= \int_a^b \frac{d}{d\varepsilon} L(t, (\varphi + \varepsilon g)(t), (\varphi' + \varepsilon g')(t))\, dt \\ &= \int_a^b \frac{\partial L}{\partial y}(t, (\varphi + \varepsilon g)(t), (\varphi' + \varepsilon g')(t)) \cdot g(t)\, dt \\ &\quad + \int_a^b \frac{\partial L}{\partial p}(t, (\varphi + \varepsilon g)(t), (\varphi' + \varepsilon g')(t)) \cdot g'(t)\, dt. \end{aligned}$$

Applying integration by parts to the second integral, we get that

$$\int_a^b \frac{\partial L}{\partial p} \cdot g' = \big(\frac{\partial L}{\partial p} \cdot g\big)\Big|_a^b - \int_a^b g(t) \cdot \frac{d}{dt}\frac{\partial L}{\partial p}(t, (\varphi + \varepsilon g)(t), (\varphi' + \varepsilon g')(t))\, dt.$$

Hence,

$$0 = \frac{dF}{d\varepsilon}(0) = \int_a^b \left[\frac{\partial L}{\partial y}(t, \varphi(t), \varphi'(t)) - \frac{d}{dt}\frac{\partial L}{\partial p}(t, \varphi(t), \varphi'(t))\right] \cdot g(t)\, dt$$

for any function $g \in C^2([a,b],\mathbb{R})$ with $g(a) = g(b) = 0$. The conclusion follows by an application of the following lemma.

$\square$

6.4 Lemma. *If $f : [a,b] \to \mathbb{R}$ is a continuous function such that*

$$\int_a^b f(t)g(t)\, dt = 0$$

for any function $g \in C^2([a,b],\mathbb{R})$ with $g(a) = g(b) = 0$, then $f \equiv 0$ on $[a,b]$.

Proof. Since f is continuous, it suffices to show that $f \equiv 0$ on (a,b). Assume that $f(x) \neq 0$ for some $x \in (a,b)$. Without loss of generality, let $f(x) = \varepsilon > 0$. The continuity of f implies the existence of a neighborhood $U_\delta(x)$ of x such that $f(t) \geq \frac{\varepsilon}{2}$ for all $t \in U_\delta(x)$. Let us choose now $g \in C^2([a,b],\mathbb{R})$ satisfying $g \geq 0$, $g(x) > 0$ and $g(t) = 0$ for all $t \in [a,b] \setminus U_\delta(x)$. It follows that

$$0 = \int_a^b f(t) \cdot g(t)\, dt = \int_{x-\delta}^{x+\delta} f(t)g(t)\, dt \geq \frac{\varepsilon}{2} \underbrace{\int_{x-\delta}^{x+\delta} g(t)\, dt}_{>0} > 0,$$

and we obtain a contradiction.

$\square$

6.5 Example. As above, let $V = \{\varphi \in C^2([a,b],\mathbb{R}) : \ \varphi(a) = \alpha, \varphi(b) = \beta\}$. Motivated by the arclength of curves, let us consider

$$J(\varphi) = \int_a^b \sqrt{1 + \varphi'(t)^2}\, dt.$$

Then $L(t,y,p) = \sqrt{1+p^2}$, so $\frac{\partial L}{\partial y} = 0$, $\frac{\partial L}{\partial p}(t,y,p) = \frac{p}{\sqrt{1+p^2}}$, and the Euler differential equation becomes

$$\frac{d}{dt}\frac{\varphi'(t)}{\sqrt{1+\varphi'(t)^2}} = \frac{\partial L}{\partial y} = 0.$$

We get that

$$\frac{\varphi''(t)}{\sqrt{1+\varphi'(t)^2}} - \varphi'(t)\frac{\varphi'(t)\cdot\varphi''(t)}{\sqrt{(1+\varphi'(t)^2)^3}} = 0 \quad \Rightarrow \quad \varphi''(t)\frac{1}{(1+\varphi'(t)^2)^{3/2}} = 0$$

for all $t \in [a,b]$, hence $\varphi''(t) = 0$ for all $t \in [a,b]$. It follows that

$$\varphi(t) = mt + m$$

where m, n are determined from $\varphi(a) = \alpha$ and $\varphi(b) = \beta$:

$$m = \frac{\beta - \alpha}{b-a}, \qquad n = \frac{\alpha b - \beta a}{b - a}.$$

Hence, we have proved that the shortest distance between two points is the straight line.

In the following, we generalize the above strategy to the situation with n functions. The mapping L has now the form

$$L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad (t, y_1, \ldots, y_n, p_1, \ldots, p_n) \mapsto L(t, y_1, \ldots, y_n, p_1, \ldots, p_n)$$

and for $\alpha, \beta \in \mathbb{R}^n$,

$$V = \{f \in C^2([a,b],\mathbb{R}^n) : \ f(a) = \alpha, \ f(b) = \beta\}.$$

One defines $J : V \to \mathbb{R}$ via

$$J(\varphi) = \int_a^b L(t, \varphi_1(t), \ldots, \varphi_n(t), \varphi_1'(t), \ldots, \varphi_n'(t))\, dt.$$

If J attains a minimum at $\varphi = (\varphi_1, \ldots, \varphi_n) \in V$, then

$$\frac{d}{dt} L_{p_i}(t, \varphi, \varphi') - L_{y_i}(t, \varphi, \varphi') = 0, \qquad i = 1, \ldots, n$$

holds.

Let us consider a physical system described by the time coordinate t and the spatial coordinates $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$. Then

a) $L(t, \varphi, \varphi')$ is called the *Lagrangian* or the *Lagrange function*;

b) $L = T - U$, where $T = T(\varphi, \varphi')$ is the *kinetic energy* and $U = U(\varphi)$ is the *potential energy* of the system;

c) moreover; $J(\varphi) = \int_a^b L(t, \varphi(t), \varphi'(t))\, dt$ is called in physics the *action integral.*

Hamilton's principle in mechanics states that the motion of the system from time t_0 to time t_1 is such that the integral

$$J(\varphi) = \int_{t_0}^{t_1} \Big(T(\varphi, \varphi') - U(\varphi)\Big)\, dt$$

has a minimum for the actual path $\varphi(t)$ of the motion. The Euler differential equation implies that

$$\frac{d}{dt}\frac{\partial T}{\partial \varphi_i'} - \frac{\partial}{\partial \varphi_i}(T - U) = 0, \qquad i = 1, \dots, n.$$

In mechanics, these equations are called the *Lagrange motion equations.* Let us consider the special case of the motion of a mass point with a time-independent potential U. We get for $x = (x_1, x_2, x_3)$ and $v = (x_1', x_2', x_3')$ that

$$\begin{aligned} T &= \frac{m}{2}\sum_{i=1}^{3} v_i^2 = \frac{m}{2} \cdot \sum_{i=1}^{3}(x_i')^2(t) \quad \text{and} \\ L(x, v) &= \frac{m}{2}(v_1^2 + v_2^2 + v_3^2) - U(x_1, x_2, x_3). \end{aligned}$$

Since the Lagrange function does not depend explicitly on t,

$$\frac{\partial L}{\partial x_i} = -\frac{\partial U}{\partial x_i} \quad \text{and} \quad \frac{\partial L}{\partial v_i} = m v_i$$

hold and the Euler differential equation becomes

$$\frac{d}{dt}(m x_i'(t)) + \frac{\partial U}{\partial x_i}(x(t)) = 0.$$

Hence, the motion equation in this case is

$$m x_i'' = -\frac{\partial U}{\partial x_i}(x), \quad i = 1, 2, 3.$$

6.6 Example. We consider again the example with the surface of revolution from the beginning and define J by

$$J(\varphi) = \int_a^b \varphi(t)\sqrt{1 + \varphi'(t)^2}\, dt. \tag{6.1}$$

The Euler differential equation is

$$L_{pp}\varphi'' + L_{py}\varphi' + L_{pt} - L_y = 0.$$

Since L is independent from t, we get for $E_\varphi := L_p(\varphi, \varphi')\varphi' - L(\varphi, \varphi')$ that

$$\begin{aligned}\frac{d}{dt}E_\varphi &= (L_{py}\varphi'^2 + L_{pp}\varphi'\varphi'' + L_p\varphi'') - L_y\varphi' - L_p\varphi'' \\ &= \varphi'(L_{py}\varphi' + L_{pp}\varphi'' - L_y) = 0,\end{aligned}$$

so every solution φ of the Euler differential equation satisfies in this case

$$E_\varphi = L_p(\varphi, \varphi') \cdot \varphi' - L(\varphi, \varphi') = \text{constant}.$$

In physics, E_φ is interpreted as the energy of the system. Let us now consider the special J defined by (6.1). It follows that

$$L(t, y, p) = y\sqrt{1+p^2}, \qquad \frac{\partial L}{\partial p}(t, y, p) = \frac{yp}{\sqrt{1+p^2}}, \quad \text{and} \quad \frac{\partial L}{\partial y}(t, y, p) = \sqrt{1+p^2}.$$

The Euler equation becomes

$$\frac{d}{dt}\left(\frac{\varphi\,\varphi'}{\sqrt{1+\varphi'^2}}\right) = \sqrt{1+\varphi'^2} \tag{6.2}$$

and, since L is independent of t, there exists $c \in \mathbb{R}$ such that $L_p(\varphi, \varphi')\cdot\varphi' - L(\varphi, \varphi') = -c$ holds. It follows that

$$\frac{\varphi \cdot \varphi'^2}{\sqrt{1+\varphi'^2}} - \varphi \cdot \sqrt{1+\varphi'^2} = -c \qquad \text{or} \qquad \frac{\varphi}{\sqrt{1+\varphi'^2}} = c.$$

Using this relation in (6.2) yields

$$\frac{d}{dt}(c\varphi') = \frac{\varphi}{c} \qquad \text{or} \qquad \varphi'' - \frac{1}{c^2}\varphi = 0.$$

Now a solution of (6.2) is the function

$$\varphi(t) = c \cdot \cosh\left(\frac{1}{c}(t - t_0)\right).$$

These functions are called *catenaries* and the surface of revolution which they generate is called a catenoid of revolution.

Finally, we determine the constants c and t_0 for the case $\alpha = \beta$ and $a = -b$. Due to symmetry reasons, $t_0 = 0$, so we have the equation

$$\frac{\cosh b/c}{b/c} = \frac{\alpha}{b}.$$

There exists $c \in \mathbb{R}$ such that for $\alpha/b = c$ the above equation has exactly one solution.

As a conclusion, we have proved that the problem of minimal surface of revolution possesses at most one solution for this c and $\alpha/b = c$.

Chapter VIII

Inverse mappings and implicit functions

In this chapter we shall study the following subjects:

- the question when a continuously differentiable function has a continuously differentiable inverse;
- the solving of equations in $\mathbb{R}^n$ and, connected with it, the Implicit Function Theorem;
- submanifolds in $\mathbb{R}^n$ and, connected with it, local extrema with constraints

Answering the first question turns out to be much more complicated than in the one-dimensional case, since the Mean Value Theorem from Analysis I has no n-dimensional analogue. Our main ingredient to obtain the theorem on the local invertibility of a continuously differentiable mapping is the Banach Fixed Point Theorem, which is used to prove that, under certain hypotheses, a continuous inverse exists locally. Then the Chain Rule implies that this inverse is again continuously differentiable.

The Implicit Function Theorem is concerned with the question under which conditions an equation $f(x, y) = 0$ has a differentiable solution $y = g(x)$ in the neighborhood of a zero of f. This theorem leads to the concept of submanifolds of $\mathbb{R}^n$; these are those subsets of $\mathbb{R}^n$ which locally look like an open subset of $\mathbb{R}^d$. The geometrically motivated introduction of the concepts of tangent space and normal space in a submanifold allows an elegant proof of the theorem on Lagrange multipliers, a necessary condition for the existence of local extrema with constraints.

1 Inverse Mapping Theorem

We begin this section with the definition of the notion of a diffeomorphism.

1.1 Definition. (diffeomorphism). A bijective and continuously differentiable function $f : U \to V$ between two open sets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$ is called a *diffeomorphism* if the inverse mapping $f^{-1} : V \to U$ is also continuously differentiable.

1.2 Remark. a) If $I \subseteq \mathbb{R}$ is an open interval and $f : I \to J$ a continuously differentiable function with $f'(x) \neq 0$ for all $x \in I$, then f is strictly monotone, the inverse function exists on the interval $J = f(I)$ and it is also differentiable, by the Inverse Mapping Theorem IV.1.8 from Analysis I.

b) A result similar to (a) does not hold for functions in $\mathbb{R}^n$ with $n \geq 2$. As a counterexample, we consider the mapping defining polar coordinates:

$$f : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2 \setminus \{(0, 0)\}, \qquad f(r, \varphi) := (r \cos \varphi, r \sin \varphi).$$

This is surjective and continuously differentiable, $J_f(r, \varphi)$ is invertible for all $(r, \varphi) \in (0, \infty) \times \mathbb{R}$, but f is not injective. Hence, f is not invertible.

c) Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open and $f : U \to V$ be a diffeomorphism with the inverse $g := f^{-1} : V \to U$. By the Chain Rule we get

$$g'(f(x))f'(x) = (g \circ f)'(x) = (id_U)'(x) = id_{\mathbb{R}^n}, \quad x \in U$$

$$f'(g(y))g'(y) = (f \circ g)'(y) = (id_V)'(y) = id_{\mathbb{R}^m}, \quad y \in V$$

Using Linear Algebra, we get that $n = m$ and that for $y = f(x)$ the mappings $g'(y)^{-1} \in \mathcal{L}(\mathbb{R}^n)$ and $f'(x)^{-1} \in \mathcal{L}(\mathbb{R}^n)$ are isomorphisms inverse to each other

d) The assertion (c) implies that $Df(x)$ is invertible for all $x \in U$ if $f : U \to V$ is a diffeomorphism. As we have seen in (b), the converse is not true in general.

1.3 Theorem. (Inverse Mapping Theorem) *Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ be a continuously differentiable function. Assume that $Df(a) \in \mathcal{L}(\mathbb{R}^n)$ is invertible for some $a \in U$. Then there exists an open neighborhood V of $b = f(a)$ and an open neighborhood $\tilde{U} \subset U$ of a such that the restriction $\tilde{f} : \tilde{U} \to V$ of f to $\tilde{U}$ is a diffeomorphism. Moreover*

$$D(\widetilde{f})^{-1}(b) = (Df(a))^{-1}.$$

For the proof of the above theorem, we need some auxiliary results.

1.4 Lemma. *Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ be a differentiable function and $a, x \in U$ with $\overline{ax} \subset U$. Assume that*

$$\sup_{t \in [0,1]} \|Df(a + t(x - a)\| := L < \infty.$$

Then $\|f(x) - f(a)\| \leq L\|x - a\|$.

For the proof we refer to the exercises.

1.5 Lemma. (Continuity of the inversion)
The set $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A \text{ invertible}\}$ *is open in* $\mathbb{R}^{n^2}$ *and the mapping*

$$Inv : GL_n(\mathbb{R}) \to M_n(\mathbb{R}), \quad A \longmapsto A^{-1},$$

is continuous.

Proof. Since $GL_n(\mathbb{R}) = \det^{-1}(\{x \in \mathbb{R} : x \neq 0\})$, the set $GL_n(\mathbb{R})$ is the inverse image of the open set $\{x \in \mathbb{R} : x \neq 0\}$ under the continuous function $\det : M_n(\mathbb{R}) \to \mathbb{R}$. For the continuity of Inv we refer to the exercises. □

Proof of Theorem 1.3. We shall prove the theorem in a number of steps and begin with the following remark.

By considering instead of f the function $Df(a)^{-1} \circ f \; : \; U \to \mathbb{R}$, it follows that without loss of generality we can assume that

$$Df(a) = Id_{\mathbb{R}^n}. \tag{1.1}$$

Moreover, we can assume also w.l.o.g. that

$$a = 0 \text{ and } f(a) = 0,$$

(consider the function $x \mapsto f(x+a) - f(a)$ for $x \in \{z \in \mathbb{R}^n : z + a \in U\}$).

Step 1:
Our target is to solve the equation $y = f(x)$ for "small" $y \in \mathbb{R}^n$ with respect to x. For $y \in \mathbb{R}^n$ we define

$$\varphi_y : U \to \mathbb{R}^n, \qquad \varphi_y(x) := y + x - f(x).$$

Then $y = f(x)$ holds if and only if $\varphi_y(x) = x$, i.e. if and only if x is a fixed point of the mapping φ_y.

Step 2:
We shall apply Banach's Fixed Point Theorem to the above equation.

First, let us remark that $\varphi_0 : U \to \mathbb{R}^n$ is continuously differentiable and that the scaling 1.1 implies that

$$D\varphi_0(0) = Id_{\mathbb{R}^n} - Id_{\mathbb{R}^n} = 0.$$

The continuity of $D\varphi_0$ implies the existence of some $r > 0$ such that $\overline{U}_{2r}(0) = \{z \in \mathbb{R}^n : |z| \leq 2r\} \subset U$ and

$$\|D\varphi_0(x) - D\varphi_0(0)\| = \|D\varphi_0(x)\| \leq \frac{1}{2}, \quad x \in \overline{U}_{2r}(0). \tag{1.2}$$

Since $D\varphi_y = D\varphi_0$, by applying Lemma 1.4, we get that

$$\|\varphi_y(x_1) - \varphi_y(x_0)\| \leq \frac{1}{2}\|x_1 - x_0\|, \quad x_0, x_1 \in \overline{U}_{2r}(0), \tag{1.3}$$

hence

$$\|\varphi_y(x)\| \leq \|\varphi_y(x) - \varphi_y(0)\| + \|\underbrace{\varphi_y(0)}_{=y}\| \leq \frac{1}{2}\|x\| + \|y\| < 2r \tag{1.4}$$

in the case $\|x\| \leq 2r$ and $\|y\| < r$, i.e., if $\|y\| < r$ then φ_y maps $\overline{U}_{2r}(0)$ into itself. Furthermore, by (1.3), for each $y \in U_r(0)$ the mapping φ_y is a contraction on the complete metric space $\overline{U}_{2r}(0)$ (as a closed subset of $\mathbb{R}^n$). Therefore, we may apply Banach's Fixed Point Theorem to conclude that for all $y \in U_r(0)$ there exists a unique fixed point $x \in \overline{U}_{2r}(0)$ of φ_y, which satisfies $\|x\| < 2r$ by (1.4).

Let

$$V := U_r(0) \quad \text{and} \quad \tilde{U} := f^{-1}(V) \cap U_{2r}(0).$$

Then the restriction $\tilde{f}$ of f to $\tilde{U}$ is bijective.

Step 3:

Let us now prove that $g := (\tilde{f})^{-1} : V \to \tilde{U},\ g(y) = x$, is continuous. For all $x \in \tilde{U}$ we have that $x = \varphi_0(x) + f(x)$. Using this and (1.3) we obtain for all $y_0, y_1 \in V$

$$\begin{aligned}\|g(y_1) - g(y_0)\| = \|x_1 - x_0\| &\leq \|\varphi_0(g(y_1)) - \varphi_0(g(y_0))\| + \|f(g(y_1)) - f(g(y_0))\| \\ &\leq \tfrac{1}{2}\|g(y_1) - g(y_0)\| + \|f(g(y_1)) - f(g(y_0))\|,\end{aligned}$$

hence

$$\|g(y_1) - g(y_0)\| \leq 2\|f(g(y_1)) - f(g(y_0))\| = 2\|y_1 - y_0\|. \tag{1.5}$$

Therefore, g is Lipschitz continuous.

Step 4:

We shall show that for all $x \in \tilde{U}$, $Df(x) \in \mathcal{L}(\mathbb{R}^n)$ is invertible. First, let us remark that

$$f(x) = x - \varphi_0(x) \quad \text{and} \quad Df(x) = Id_{\mathbb{R}^n} - D\varphi_0(x)$$

for all $x \in \tilde{U}$. If $Df(x)v = 0$ for some $v \in \mathbb{R}^n$, then $v = [D\varphi_0(x)]v$ and (1.2) implies $\|v\| \leq \|D\varphi_0(x)\| \cdot \|v\| \leq \frac{1}{2}\|v\|$, so $v = 0$.

It follows that $Df(x)$ is injective and the Dimension formula from Linear Algebra implies that $Df(x)$ is also surjective. Thus we have proved that $Df(x)$ is invertible.

Step 5:

We prove that g is differentiable. To this end, let $y_0 \in V$ and $k \in \mathbb{R}^n$ with $y_0 + k \in V$. If we set $x_0 := g(y_0)$ and $h := g(y_0 + k) - g(y_0)$, then $k = f(x_0 + h) - f(x_0)$ and

$$g(y_0 + k) - g(y_0) - [Df(x_0)]^{-1}k = h - [Df(x_0)]^{-1}\big(f(x_0 + h) - f(x_0)\big).$$

Since f is differentiable at x_0,

$$f(x_0 + h) - f(x_0) = Df(x_0)h + r(x_0 + h)$$

with

$$\lim_{h\to 0} \frac{r(x_0 + h)}{\|h\|} = 0.$$

It follows that

$$g(y_0 + k) - g(y_0) = [Df(x_0)]^{-1}k - [Df(x_0]^{-1}r(x_0 + h)$$

and it remains to prove that

$$\lim_{k\to 0} \frac{[Df(x_0)]^{-1}r(x_0 + h)}{\|k\|} = 0. \tag{1.6}$$

Using (1.5), we get that

$$\|h\| = \|(x_0 + h) - x_0\| = \|g(y_0 + k) - g(y_0)\| \leq 2\|k\|,$$

and for $k \to 0$, so also $h \to 0$, we obtain that

$$\frac{\|r(x_0) + h\|}{\|k\|} \leq \frac{2}{\|h\|}\|r(x_0 + h)\| \underset{h\to 0}{\longrightarrow} 0,$$

since f is differentiable. Since $[Df(x_0)]^{-1} \in \mathcal{L}(\mathbb{R}^n)$, (1.6) follows.

Step 6:
An application of the Chain Rule similar to Remark 1.2 (c) gives us

$$Dg(y) = [Df(g(y))]^{-1}$$

and Lemma 1.5 implies that $Dg : V \to \mathcal{L}(\mathbb{R}^n)$ is continuous. Hence $\tilde{f} : \tilde{U} \to V$ is a diffeomorphism and the theorem is completely proved. □

The above theorem has numerous impacts. As immediate consequences we get the following corollaries.

1.6 Corollary. (Theorem of the open mapping) *Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ be a continuously differentiable function such that $Df(x) \in \mathcal{L}(\mathbb{R}^n)$ is invertible for all $x \in U$. Then $f(U)$ is open in $\mathbb{R}^n$.*

Proof. By the Inverse Mapping Theorem, for every $x \in U$ there exists an open neighborhood $U_x \subseteq U$ of x such that $f(U_x) \subseteq \mathbb{R}^n$ is open. Since $f(U) = \cup_{x\in U} f(U_x)$ and arbitrary unions of open sets are again open, it follows that $f(U)$ is open. □

Mappings with the property that $f(O)$ is open for all open sets $O \subseteq U$ are called *open mappings* .

1.7 Corollary. *In the hypothesis of Corollary 1.6 assume moreover that f is injective. Then f is a diffeomorphism from U onto the open set $f(U) \subseteq \mathbb{R}^n$.*

Proof. The inverse mapping $g : f(U) \to U$ is continuous, since for every open set $O \subseteq U$, the inverse image $g^{-1}(O) = f(O)$ is open by Corollary 1.6. Applying the Inverse Mapping Theorem, we get that f is a diffeomorphism. □

1.8 Remark. By applying the Inverse Mapping Theorem to solving nonlinear systems of equations, we get the following statement:
If $\det Df(x_0) \neq 0$ for some $x_0 \in U$, then there exist neighborhoods U of x_0 and V of $f(x_0)$ such that the system of equations

$$\begin{aligned} f_1(x_1, ..., x_n) &= y_1 \\ \vdots \qquad & \quad \vdots \\ f_n(x_1, ..., x_n) &= y_n \end{aligned}$$

has a unique solution $x_1 = x_1(y_1, ..., y_n), ..., x_n = x_n(y_1, ..., y_n)$ in U for every $(y_1, ..., y_n) \in V$. Moreover the functions $x_1, ..., x_n$ have the same regularity as $f_1, ..., f_n$.

1.9 Example. a) **Plane polar coordinates**

Every point $(x, y) \in \mathbb{R}^2$ has a presentation using polar coordinates:

$$x = r\cos\varphi, \quad y = r\sin\varphi \quad \text{with } r = \sqrt{x^2 + y^2}.$$

For the mapping $f : (r, \varphi) \longmapsto (x, y)$ we get that

$$\det Df(r, \varphi) = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r \neq 0 \text{ when } \ (x, y) \neq (0, 0)$$

For every point $(x, y) \neq (0, 0)$ there exist infinitely many inverse images $(r, \varphi + 2k\pi)$; however, using the Inverse Mapping Theorem, we get the existence in a neighborhood of $(x_0, y_0) = (r_0 \cos\varphi_0, r_0 \sin\varphi_0) \neq (0, 0)$ of an infinitely often differentiable inverse function given by

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \varphi = \arctan\tfrac{y}{x}, \quad x_0 \neq 0,$$

where the branch of arctangent which gives the value φ_0 for (x_0, y_0) is chosen. If $x_0 = 0$, then we take $\varphi = \operatorname{arccot}\frac{x}{y}$.

Thus, the presentation using polar coordinates maps the strip

$$S = \{(r, \varphi) \in \mathbb{R}^2 : r > 0, \ |\varphi| < \pi\}$$

diffeomorph to $\mathbb{R}^2_- := \mathbb{R}^2 \setminus \{(x, y) : x \leq 0, \ y = 0\}$.

b) **The complex logarithm** The algebraic form of a complex number $z \in \mathbb{C}$ is $z = x+iy$. If we restrict z to the strip $S = \mathbb{R} \times (-\pi, \pi)$ and consider $w = e^z$, it follows that $|w| = e^x$ and $\arg w = y$. Hence, the exponential function maps the strip S bijective onto $\mathbb{C} \setminus \{0\}$. Its inverse function is given by $(x, y) \longmapsto (\log |w|, \arg w)$ with $y \in (-\pi, \pi)$. The periodicity of the exponential (that is, $e^z = e^{z+2\pi i}$) implies that every strip shifted by $2k\pi i$, i.e. every $S_k = 2\pi ki + S$ $(k \in \mathbb{Z})$ is mapped in a bijective way onto $\mathbb{C} \setminus \{0\}$. The above formula for the inverse remains true, with the difference that now $(2k-1)\pi < y \le (2k+1)\pi$ is required.

For any $w \neq 0$, every complex number z satisfying the equation $e^z = w$ is called a *logarithm of* w. It follows that in every strip S_k there exists exactly one logarithm of w. The different logarithms differ by multiples of $2\pi i$ and are given by

$$\log w = \log |w| + i \arg w.$$

If $\arg w \in (-\pi, \pi)$, then we call that the *principal branch of the logarithm* .

In order to apply the Inverse Mapping Theorem to $w = e^z$, let us consider the function $f = (f_1, f_2)$ given by

$$f_1(x, y) = e^x \cos y, \quad f_2(x, y) = e^x \sin y.$$

Then

$$\det Df(x, y) = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix} = e^{2x} \neq 0.$$

If we restrict (x, y) to the strip $S = \mathbb{R} \times (-\pi, \pi)$, then $f = (f_1, f_2)$ is a diffeomorphism with range $\mathbb{R}^2_-$. The inverse mapping is then the principal branch of the logarithm, given by

$$x = \tfrac{1}{2} \log(f_1^2 + f_2^2), \quad y = \arg(f_1, f_2) \in (-\pi, \pi).$$

2 The Implicit Function Theorem

In the previous paragraph we dealt with the question whether or not a system of nonlinear equations is solvable; it was assumed that the number of equations coincides with the number of the variables. In the following we consider the solvability of such systems, which have more variables than equations. More precisely, we consider m equations for $m + k$ variables, i.e.,

$$\begin{aligned} f_1(x_1, ..., x_k, y_1, ..., y_m) &= 0 \\ \vdots \qquad\qquad & \quad \vdots \\ f_m(x_1, ..., x_k, y_1, ..., y_m) &= 0 \end{aligned}$$

and ask: Under which conditions can we express $y_1, ..., y_m$ by functions of $x_1, ..., x_k$? In different words: Can the above system of equations be solved for y?

If we examine the special case of systems of linear equations of the type

$$Ax + By = 0$$

with $A \in M_{m,k}(\mathbb{R}), x = (x_1, ..., x_k)^T$ and $B \in M_m(\mathbb{R}), y = (y_1, ..., y_m)^T$, then this is possible if B is invertible. In this case we have

$$y = -B^{-1}Ax.$$

In the following we will derive a local analogous result for continuously differentiable functions $f = (f_1, ..., f_m) : U \subseteq \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^m$.
To this end we define

$$\begin{aligned} a :=& (a_1, ..., a_k), \quad b := (b_1, ..., b_m), \quad (a, b) := (a_1, ..., a_k, b_1, ..., b_m) \\ x :=& (x_1, ...x_k), \quad y := (y_1, ..., y_m), \quad (x, y) := (x_1, ..., x_k, y_1, ..., y_m) \\ f(a, b) :=& f(a_1, ..., a_k, b_1, ..., b_m), \quad f(x, y) := f(x_1, ..., x_k, y_1, ...y_m), \end{aligned}$$

$$D_x f(a, b) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a, b) & ... & \frac{\partial f_1}{\partial x_k}(a, b) \\ . & . & . \\ . & . & . \\ . & . & . \\ \frac{\partial f_m}{\partial x_1}(a, b) & ... & \frac{\partial f_m}{\partial x_k}(a, b) \end{pmatrix}_{m \times k,}$$

$$D_y f(a, b) := \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(a, b) & ... & \frac{\partial f_1}{\partial y_m}(a, b) \\ . & . & . \\ . & . & . \\ . & . & . \\ \frac{\partial f_m}{\partial y_1}(a, b) & ... & \frac{\partial f_m}{\partial y_m}(a, b) \end{pmatrix}_{m \times m.}$$

The following theorem on implicit functions is the main result of this paragraph.

2.1 Theorem. (Implicit Function Theorem) *Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^m$ be an open set and $f : U \to \mathbb{R}^m$ be a continuously differentiable function. Further, let $(a, b) \in U$ be such that $f(a, b) = 0$ and*

$$\det(D_y f(a, b)) \neq 0.$$

Then there exists an open neighborhood $W \subseteq \mathbb{R}^k$ of a and one, and only one, vector-valued continuously differentiable map $\varphi : W \to \mathbb{R}^m$ with $\varphi(a) = b$, $(x, \varphi(x)) \in U$ for all $x \in W$ and

$$f(x, \varphi(x)) = 0 \quad \text{for all } x \in W,$$

Furthermore,

$$D\varphi(x) = -[D_y f(x, \varphi(x))]^{-1} D_x f(x, \varphi(x)), \quad x \in W.$$

Let us consider the special case $m = 1 = k$. Let $f : U \to \mathbb{R}, U \subseteq \mathbb{R}^2$, be continuously differentiable in a neighborhood of $(a, b) \in \mathbb{R}^2$ with $f(a, b) = 0$, $\frac{\partial f}{\partial y}(a, b) \neq 0$. Then the implicit function theorem tells us: There exists a $\delta > 0$ and a unique function $\varphi : (a - \delta, a + \delta) \to \mathbb{R}$ with $\varphi(a) = b$ and $f(x, \varphi(x)) = 0$ for all $x \in (a - \delta, a + \delta)$.
For the validity of the theorem it is essential that U is reduced, for otherwise, to a given x, there may exist several y-values (or even none).

Proof. Consider the map

$$F : U \subseteq \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k \times \mathbb{R}^m, \qquad F(x, y) = (x, f(x, y)).$$

Then F is continuously differentiable and we have

$$DF(x, y)(h, k) = (h, D_x f(x, y)h + D_y f(x, y)k), \ \ (x, y) \in U, \ (h, k) \in \mathbb{R}^k \times \mathbb{R}^m. \quad (2.1)$$

Further, $DF(a, b) \in \mathcal{L}(\mathbb{R}^k \times \mathbb{R}^m)$ is invertible, since (2.1) implies that $DF(a, b)$ is injective:
If $DF(a, b)(h, k) = (0, 0) \ \Rightarrow \ h = 0$ and, by the hypothesis that $D_y f(a, b)$ is invertible, we have that $D_y f(a, b)k = 0 \ \Rightarrow \ k = 0$.
Thus, $DF(a, b)$ is injective, hence also bijective and, therefore, invertible. Hence the Inverse Function Theorem can be applied upon F at the point (a, b) and we obtain an open neighborhood $\tilde{U} \subseteq U$ of (a, b) with the property that $\tilde{F} := F|_{\tilde{U}} : \tilde{U} \to F(\tilde{U}) =: V$ has a continuously differentiable inverse function $G : V \to \tilde{U}$.
Since F behaves like the identity map in the first k coordinates, this is also true for G, i.e., there exists a continuously differentiable map $g : V \to \mathbb{R}^m$ with

$$G(\xi, \eta) = (\xi, g(\xi, \eta)) \quad \text{for all} \ \ (\xi, \eta) \in V.$$

Hence we have for $(x, y) \in \tilde{U}$ that

$$f(x, y) = 0 \Leftrightarrow F(x, y) = (x, 0) \Leftrightarrow (x, y) = G(x, 0) = (x, g(x, 0)) \Leftrightarrow y = g(x, 0) \quad (2.2)$$

and, in particular, $b = g(a, 0)$. If we set

$$W := \left\{ x \in \mathbb{R}^k : \begin{pmatrix} x \\ 0_{\mathbb{R}^m} \end{pmatrix} \in V \right\},$$

then V open implies W open and

$$V \ni F(a, b) = (a, f(a, b)) = (a, 0),$$

i.e., W is a neighborhood of a. Finally we define the function

$$\varphi \, : \, W \to \mathbb{R}^m, \qquad \varphi(x) := g(x, 0).$$

Then, by (2.2), φ explicitly solves the equation $f(x,y)=0$ in a unique way in a neighborhood of the point (a,b). The chain rule gives the derivative of the function φ :

$$f(x,\varphi(x)) = f(x_1,\dots,x_k,\varphi_1(x_1,\dots,x_k),\dots,\varphi_m(x_1,\dots,x_k)) = 0$$

$$\Rightarrow \qquad D_x f(x,\varphi(x)) Id_{\mathbb{R}^k} + D_y f(x,\varphi(x)) D\varphi(x) = 0.$$

In the case $(x,y)=(a,b)$ we have, since $\varphi(a)=b$,

$$D\varphi(a) = -[D_y f(a,b)]^{-1} D_x f(a,b).$$

□

We note, that analogously to the inverse theorem the derivative $D\varphi(a)$ can be determined without *explicit* knowledge of φ.

2.2 Example. a) **Level curves.** Let $U \subseteq \mathbb{R}^2$ be an open set, $f : U \to \mathbb{R}$ be a continuously differentiable function and $c \in \mathbb{R}$. Then we call

$$N_f(c) = \{(x,y) \in U : \, f(x,y) = c\}$$

the *level set* of f. The level set is also denoted by level curve though, in general, the level set is not necessarily a curve. The implicit function theorem in the case $m=k=1$ applied to the function

$$f_c : U \to \mathbb{R}, \qquad f_c(x,y) := f(x,y) - c,$$

yields the following result.

If $(a,b) \in U$ and $f(a,b)=c$ and if

$$(\#) \qquad \operatorname{grad} f_c(a,b) = (f_x(a,b),\ f_y(a,b)) \neq (0,0),$$

then the equation $f(x,y)=c$ can be solved in the form

1. $y=\varphi(x)$ in a neighborhood of a, if $f_y(a,b)\neq 0$,
2. $x=\psi(y)$ in a neighborhood of b, if $f_x(a,b)\neq 0$.

In other words: The level sets through the point (a,b), satisfying (#) locally, can be represented via a continuously differentiable map of the form $x \mapsto (x,\varphi(x))$ in the case 1 and $y \mapsto (\psi(y),y)$ in the case 2 locally. Thus, in fact, the level sets are described locally by level curves if (#) holds.

b) **Systems of equations.** If $k=1$ and $m=2$ consider the system

$$\begin{aligned} f_1(x,y_1,y_2) &= x^3 + y_1^3 + y_2^3 - 7 = 0 \\ f_2(x,y_1,y_2) &= xy_1 + y_1y_2 + xy_2 + 2 = 0 \qquad (\Rightarrow\ f_1,f_2 \in C^\infty) \end{aligned}$$

at the point $(2,-1,0)$. Then $f_1(2,-1,0)=0$ and $f_2(2,-1,0)=0$.
With $f=(f_1,f_2)$ we have

$$D_y f(2,-1,0) = \begin{pmatrix} 3y_1^2 & 3y_2^2 \\ x+y_2 & x+y_1 \end{pmatrix}\Bigg|_{(2,-1,0)} = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}.$$

Since this matrix is invertible, there exist two continuously differentiable functions $\varphi_1, \varphi_2 : W \to \mathbb{R}$, where $W \subseteq \mathbb{R}$ is a neighborhood of $a=2$ with the following property

$$(\varphi_1(2), \varphi_2(2)) = (-1, 0)$$

and

$$f_1(x,\varphi_1(x),\varphi_2(x)) = 0, \quad f_2(x,\varphi_1(x),\varphi_2(x)) = 0 \text{ on } W.$$

For the derivatives of φ_1 and φ_2 at $a=2$ we obtain

$$\begin{aligned}\begin{pmatrix} \varphi_1'(2) \\ \varphi_2'(2) \end{pmatrix} &= -\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial f_1}{\partial x}(2,-1,0) \\ \frac{\partial f_2}{\partial x}(2,-1,0) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{3} & 0 \\ \frac{2}{3} & -1 \end{pmatrix} \begin{pmatrix} 3x^2 \\ y_1+y_2 \end{pmatrix}\Bigg|_{(2,-1,0)} = \begin{pmatrix} -4 \\ 9 \end{pmatrix}.\end{aligned}$$

3 Submanifolds and extremal problems with constraints

From a geometrical point of view, the above implicit function theorem leads to the idea of differentiable manifolds. This concept plays an important role in modern mathematics. Here first properties are presented.

3.1 Definition. A subset $M \subseteq \mathbb{R}^n$ is called a *d-dimensional differentiable submanifold* of $\mathbb{R}^n$, if to each $a \in M$ there exist a neighborhood U of a, which is open in $\mathbb{R}^n$, and a diffeomorphism $\varphi : U \to V$ onto an open subset $V \subseteq \mathbb{R}^n$ such that

$$\varphi(U \cap M) = V \cap (\mathbb{R}^d \times \{0\}).$$

One- and two-dimensional submanifolds of $\mathbb{R}^n$ are also called curves and surfaces embedded (submersed) in $\mathbb{R}^n$; resp.; differentiable submanifolds of $\mathbb{R}^n$ of dimension $(n-1)$ are called *hypersurfaces*. In the following, when we speak about a submanifold, we understand this in the sense of a differentiable submanifold of $\mathbb{R}^n$. Examples of manifolds can easily be obtained via graphs of differentiable functions as the following theorem shows.

3.2 Theorem. *Let $O \subseteq \mathbb{R}^d$ be open and $f : O \to \mathbb{R}^m$ be a continuously differentiable function. Then graph(f) is a d-dimensional submanifold of $\mathbb{R}^{d+m}$.*

Proof. Set $U = O \times \mathbb{R}^m$; then U is open and

$$\varphi : U \to \mathbb{R}^{d+m}, \qquad (x,y) \longmapsto (x, y - f(x)),$$

is continuously differentiable with $\varphi(U) = U$. Further $\varphi : U \to U$ is bijective with $\varphi^{-1}(x,z) = (x, z + f(x))$, hence a diffeomorphism of U onto U with

$$\varphi(U \cap \text{graph}(f)) = O \times \{0\} = U \cap (\mathbb{R}^d \times \{0\}).$$

$\square$

The following theorem on the regular value is one of the most useful criteria for manifolds. Let us recall the following theorem from Linear Algebra: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a regular, i.e. a surjective and linear map, then, for each $c \in \mathbb{R}^m$, the set of all solutions of the equation $f(x) = c$ is an affine subspace of $\mathbb{R}^n$ of dimension $n - m$.

In the following we generalize this result to the case when f is a continuously differentiable function and c a regular value.

3.3 Definition. A point $x \in U \subseteq \mathbb{R}^d$ is called a *regular point* of the continuously differentiable function $f : U \to \mathbb{R}^n$, if the derivative $Df(x) \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$ is surjective. Further, a point $y \in \mathbb{R}^n$ is called a *regular value* of f, if all $x \in f^{-1}(y)$ are regular points of f.

3.4 Remark. a) If $d < n$, then f has no regular points.

b) If $d \geq n$, then $x \in U$ is a regular point of f, if the derivative $Df(x)$ has rank n.

c) If $n = 1$, then $x \in U$ is a regular point of $f \quad \Leftrightarrow \quad \nabla f(x) \neq 0$.

d) A point $y \in \mathbb{R}^n$ is a regular value of f if and only if the matrix $Df(x)$ has rank n in all points $x \in f^{-1}(y)$.

e) If $a \in U$ is a regular point of the continuously differentiable function $f : U \to \mathbb{R}^n$, then, by the implicit function theorem (possibly after a suitable orthogonal transformation), there exist n variables such that the system of equations

$$\begin{aligned} f_1(x_1, ..., x_d) &= 0 \\ \vdots \quad & \quad \vdots \\ f_n(x_1, ..., x_d) &= 0 \end{aligned}$$

can uniquely be solved for these variables in a neighborhood of a as a function of the remaining $(d - n)$ variables.

f) If $0 \in f(U)$ is a regular value of the cont. differentiable function $f : U \to \mathbb{R}^n$, then, to each $a \in f^{-1}(0)$, there exists a neighborhood $V \subseteq \mathbb{R}^d$ of a such that $f^{-1}(0) \cap V$ can be represented as a graph of a continuously differentiable function in $(d-n)$ variables.

The announced theorem on the regular value now reads as follows

3.5 Theorem. *If $U \subseteq \mathbb{R}^d$ is an open set and c a regular value of $f : U \to \mathbb{R}^n$, f being continuously differentiable, then $f^{-1}(c)$ is a $(d-n)$-dimensional submanifold of $\mathbb{R}^d$.*

The proof follows directly from Theorem 3.2 and Remark 3.4(e).

3.6 Example. a) The Euclidean $(n-1)$-sphere $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ is a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. To realize this consider the continuously differentiable function

$$f \,:\, \mathbb{R}^n \to \mathbb{R}, \quad x \longmapsto \|x\|_2^2.$$

Since $\nabla f(x) = 2x$ for all $x \in \mathbb{R}^n$, we have that 1 is a regular value. Since $f^{-1}(1) = \mathbb{S}$, the assertion follows by the theorem on the regular value.

b) Consider on $U = \mathbb{R}^3 \setminus (\{0\} \times \{0\} \times \mathbb{R})$ the continuously differentiable function

$$f \,:\, U \to \mathbb{R}, \quad f(x_1, x_2, x_3) := \left(\sqrt{x_1^2 + x_2^2} - 2\right)^2 + x_3^2.$$

Since 1 is a regular value of f we have that $\mathbb{T}^2 := f^{-1}(1)$ is a 2-dimensional submanifold of $\mathbb{R}^3$. $\mathbb{T}^2$ is the 2-dimensional torus which results by the revolution of the circle $(x_1 - 2)^2 + x_3^2 = 1$, lying in the (x_1, x_3)-plane, around the x_3-axis.

c) If A is a real, symmetric $(n \times n)$-matrix with $\det A \neq 0$, then the quadric

$$Q := \{x \in \mathbb{R}^n : \langle x, Ax\rangle = 1\}$$

is a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.
For the proof choose $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) := \langle x, Ax\rangle$ and note that $Q = f^{-1}(1)$. Since $Df(x) = 2x^T A \neq 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$, we have that 1 is a regular value of f and the assertion again follows by the theorem on the regular value.

If one wants to carry over the concepts of the differential calculus to maps between submanifolds, it is very useful to introduce linear structures like tangent space and normal space on manifolds in the Euclidean space $\mathbb{R}^n$. We start with the idea of a tangent space.

3.7 Definition. Let $M \subseteq \mathbb{R}^n$; a vector $v \in \mathbb{R}^n$ is called *tangent vector* on M at the point $a \in M$, if there exists a continuously differentiable curve $\gamma : (-\varepsilon, \varepsilon) \to M$, $\varepsilon > 0$, in M with $\gamma(0) = a$ and $\gamma'(0) = v$.
The set of all tangent vectors on M at a is called the *tangent cone* of M at a and is denoted by T_aM. If T_aM is a vector space, then T_aM is also called *tangent space*.

3.8 Theorem. *Let $M \subseteq \mathbb{R}^n$ be a d-dimensional submanifold. Then we have for each $a \in M$*

a) T_aM is a vector space of dimension d

b) If $f : U \to \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$ open, is a continuously diff. function, c a regular value of f and $M = f^{-1}(c)$, then

$$T_aM = \mathit{kern}\, Df(a) = \{v \in \mathbb{R}^n : Df(a)v = 0\}.$$

Proof.

a) The assertion is easy to see for the d-dimensional submanifold $M = V \cap (\mathbb{R}^d \times \{0\})$, where $V \subset \mathbb{R}^n$ is open. In this case

$$T_a(V \cap (\mathbb{R}^d \times \{0\})) = \mathbb{R}^d \times \{0\}.$$

In the general case we consider the map $\varphi : U \to V$ from Definition 3.1 on the submanifold M which associates to each curve $\gamma : (-\varepsilon, \varepsilon) \to M \cap U$ the image curve $\tilde{\gamma} := \varphi \circ \gamma$ in $V \cap (\mathbb{R}^d \times \{0\})$. Each curve in $V \cap (\mathbb{R}^d \times \{0\})$ is such an image curve and for two curves γ and $\tilde{\gamma}$ we have

$$\gamma'(0) = [D\varphi(a)]^{-1}\tilde{\gamma}'(0).$$

Hence

$$T_a(M \cap U) = [D\varphi(a)]^{-1}(\mathbb{R}^d \times \{0\})$$

and, since $T_aM = T_a(M \cap U)$, the assertion (a) follows.

b) For $\gamma : (-\epsilon, \epsilon) \to M$ we have $f \circ \gamma = c$ and hence $Df(a)\gamma'(0) = 0$. Therefore, $T_aM \subseteq \mathrm{kern}\, Df(a)$. Since $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ is surjective by hypothesis we get from the dimension formula of Linear Algebra

$$\dim \text{ kern } Df(a) = n - m = \dim\, M = d.$$

Therefore, T_aM is no genuine subvector space of $\mathrm{kern}\, Df(a)$ and hence the assertion.

□

We define a *normal vector* of a submanifold $M \subseteq \mathbb{R}^n$ at the point $a \in M$ as that vector $v \in \mathbb{R}^n$, which is perpendicular to the tangent space T_aM; the *normal space* N_aM is the orthogonal complement to T_aM, i.e.

$$N_aM := (T_aM)^{\perp}.$$

The above theorem on the tangent space gives the following Corollary.

3.9 Corollary. *Let $M = f^{-1}(c)$ be the level set of a continuously differentiable function $f = (f_1, ..., f_{n-d}) : U \to \mathbb{R}^{n-d}$, where $U \subseteq \mathbb{R}^n$ is open, for the regular value $c \in \mathbb{R}^{n-d}$. Then the gradients $\nabla f_1(a), \dots, \nabla f_{n-d}(a)$ at $a \in M$ form a basis of the normal space, i.e., we have*

$$N_a M = \operatorname{span}\{\nabla f_1(a), ..., \nabla f_{n-d}(a)\}.$$

Proof. The rows of the matrix $Df(a)$ are described by the above gradients. By Theorem 3.8 we have for $v \in \mathbb{R}^n$ that

$$v \in T_a M \iff \langle \nabla f_i(a), v\rangle = 0 \text{ for } i = 1, ..., n-d.$$

Hence the vectors $\nabla f_i(a)$, $i \in \{1, ..., n-d\}$, are perpendicular to $T_a M$. By hypothesis, rank $Df(a) = n-d$. Therefore, the vectors $\nabla f_i(a)$, $i \in \{1, ..., n-d\}$, are linear independent and form a basis for $N_a M$. □

3.10 Example. a) For the sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ we have $\mathbb{S}^{n-1} = f^{-1}(1)$, where $f : \mathbb{R}^n \to \mathbb{R}$, $x \longmapsto \|x\|_2^2$. Since $\nabla f(a) = 2a$, it follows that

$$N_a S^{n-1} = \{\lambda a : \lambda \in \mathbb{R}\}, \quad a \in \mathbb{S}^{n-1}.$$

b) A normal vector to the torus (surface) $\mathbb{T}^2$, described in Example 3.6 (b) at the point (a_1, a_2, a_3) is given by $2(a-h)$, where

$$h = \left(\frac{2a_1}{\sqrt{a_1^2 + a_2^2}}, \frac{2a_2}{\sqrt{a_1^2 + a_2^2}}, 0\right).$$

c) If $U \subseteq \mathbb{R}^n$ is open, $f : U \to \mathbb{R}$ is continuously differentiable, then the *unit normal* $\nu(x)$, i.e. a normal vector of length 1, to the manifold $M = \operatorname{graph} f$ at the point $a = (x, f(x))$ is given by

$$\nu(x) = \frac{(-\nabla f(x), 1)}{(1 + |\nabla f(x)|^2)^{1/2}}.$$

In many applications one is not only looking for the mere extremum of a function $f : \mathbb{R}^n \to \mathbb{R}$, but one is interested in extremal values under side conditions.

More precisely: Given $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ and a manifold M defined via $g = (g_1, ..., g_m) : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $M := (x \in U : g(x) = 0)$.
Look for points $x_0 \in M$ such that

$$f(x_0) \leq f(x) \quad \text{for all} \quad x \in M \cap U_{x_0}$$

or

$$f(x_0) \geq f(x) \quad \text{for all} \quad x \in M \cap U_{x_0}$$

for some neighborhood $U_{x_0} \subseteq U$ of x_0. Such a point is called a *local extremum* of f under the *side condition* or *constraint* $g = 0$.

The following Lagrange multiplier rule gives a necessary condition for the existence of such extremal values under constraints.

3.11 Theorem. (Lagrange multiplier rule). *Let $U \subseteq \mathbb{R}^n$ be open, $m < n$, $f : U \to \mathbb{R}$ and $g = (g_1, ..., g_m) : U \to \mathbb{R}^m$ be continuously differentiable functions. Let 0 be a regular value of g and $M = g^{-1}(0) \neq \emptyset$ be nonempty. If f has a local extremum under the constraint $g = 0$ at x_0, then there exist $\lambda_1, ..., \lambda_m \in \mathbb{R}$ such that*

$$\textit{grad}\ f(x_0) = \sum_{j=1}^{m} \lambda_j\ \textit{grad}\ g_j(x_0).$$

Proof. By the theorem on the regular value, M is a $(n-m)$-dimensional submanifold of $\mathbb{R}^n$. Hence, to each $v \in T_{x_0}M$ there exists a continuously differentiable curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = x_0$, $\gamma'(0) = v$. Then the function

$$F\ :\ (-\varepsilon, \varepsilon) \to \mathbb{R},\ F(t) := f(\gamma(t))$$

has a local extremum at $t = 0$. Therefore, $F'(0) = 0$, which means on account of the chain rule

$$0 = F'(0) = \sum_{i=1}^{n} \frac{\partial f(x_0)}{\partial x_i} \gamma_i'(0) = \langle \text{grad } f(x_0), v \rangle,$$

i.e., grad $f(x_0) \in N_{x_0}M$. By Corollary 3.9 this implies that there exist unique $\lambda_1, ..., \lambda_m \in \mathbb{R}$ such that grad $f(x_0) = \sum_{i=1}^{m} \lambda_j \nabla g_j(x_0)$. □

3.12 Remark. a) The numbers $\lambda_1, ..., \lambda_m$ are called *Lagrange multipliers*

b) In the special case $m = 1$, we have: Let $U \subseteq \mathbb{R}^n$ be open, $f, g : U \to \mathbb{R}$ continuously differentiable. If $\nabla g(x_0) \neq 0$ and if f has a local extremum at x_0 under the side condition $g = 0$, then there exists some $\lambda \in \mathbb{R}$ such that grad $f(x_0) = \lambda$ grad $g(x_0)$.

3.13 Example. Let $A = M_n(\mathbb{R})$ be a symmetric matrix, $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) := \langle x, Ax \rangle$ and $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ the $(n-1)$-sphere. Since $\mathbb{S}^{n-1}$ is compact, f continuous on $\mathbb{S}^{n-1}$, the function f attains a maximum on $\mathbb{S}^{n-1}$. We determine this point x_0 via the Lagrange multiplier rule. To this end we observe that $\mathbb{S}^{n-1} = g^{-1}(0)$ for $g : \mathbb{R}^n \to \mathbb{R}$, $g(x) = \langle x, x \rangle - 1$. Thus we have to maximize f under the constraint $g = 0$. By Example 1.3 in Chapter VII, the functions f, g are continuously differentiable with

$$\text{grad}\, f(x_0) = 2Ax_0\,, \quad \text{grad}\, g(x_0) = 2x_0 \neq 0.$$

Hence, by the Lagrange multiplier rule, there exists $\lambda \in \mathbb{R}$ such that

$$Ax_0 = \lambda x_0\,,$$

i.e., x_0 is an eigenvector of the matrix A. Since $\|x_0\|_2 = 1$ it follows that

$$\langle x_0, Ax_0\rangle = \langle x_0, \lambda x_0\rangle = \lambda.$$

Thus we have proved the following theorem from Linear Algebra.

3.14 Theorem. *If $A \in M_n(\mathbb{R})$ is a symmetric matrix, then*

$$M := \max_{\|x\|_2=1} \langle x, Ax\rangle \in \mathbb{R}$$

is an eigenvalue of A and each $x_0 \in \mathbb{R}^n$ with $\|x_0\|_2 = 1$ and $\langle x_0, Ax_0\rangle = M$ is an eigenvector associated to the eigenvalue M.

As a further application we prove the following spectral theorem for symmetric matrices from Linear Algebra.

3.15 Theorem. (Spectral theorem for symmetric matrices). *If $A \in M_n(\mathbb{R})$ is a symmetric matrix, then there exist $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, $\lambda_j \in \mathbb{R}$, and $x_1, ..., x_n \in \mathbb{S}^{n-1}$ with $Ax_j = \lambda_j x_j$ for all $j \in \{1, ..., n\}$. Furthermore, the vectors $x_1, ..., x_n$ form an orthonormal basis of $\mathbb{R}^n$ and A is of diagonal type w.r.t. this basis.*

Proof. By Theorem 3.14 there exist $x_1 \in \mathbb{S}^{n-1}$ and $\lambda_1 \in \mathbb{R}$ such that $Ax_1 = \lambda_1 x_1$. We construct a second vector $x_2 \in \mathbb{S}^{n-1}$ as follows: Define $g := (g_1, g_2)$, $g_0(x) := \|x\|_2^2 - 1$, $g_1(x) := 2\langle x_1, x\rangle$. Then $g^{-1}(0) = \mathbb{S}^{n-1} \cap \{x_1\}^\perp =: K$ and K is compact. Hence the continuous function $f : K \to \mathbb{R}$, $x \longmapsto \langle x, Ax\rangle$, attains a maximum at some $x_2 \in K$ (cf. Corollary 3.9 in Chapter VI) with $f(x) \leq f(x_2)$ for all $x \in K$. Further, $Dg(x)$ has rank 2 for all $x \in K$. By the Lagrange multiplier rule (Theorem 3.11), there exist $\mu_0, \mu_1 \in \mathbb{R}$ such that

$$(\#) \qquad \nabla f(x_2) = 2Ax_2 = \mu_0 \nabla g_0(x_2) + \mu_1 \nabla g_1(x_2) = \mu_0\, 2x_2 + \mu_1\, 2x_1.$$

By Theorem 3.14 and since by construction $\langle x_2, x_1\rangle = 0$, we obtain

$$\langle Ax_2, x_1\rangle = \langle x_2, Ax_1\rangle = \langle x_2, \lambda_1 x_1\rangle = \lambda_1\langle x_2, x_1\rangle = 0.$$

Together with (#) this latter equation yields

$$0 = \langle Ax_2, x_1\rangle = \mu_0\langle x_2, x_1\rangle + \mu_1\langle x_1, x_1\rangle = \mu_1\,.$$

Thus (#) implies $Ax_2 = \mu_0\, x_2$, which means that μ_0 is an eigenvalue of A associated to the eigenvector x_2. The value of μ_0 can be computed from

$$\mu_0 = \mu_0\langle x_2, x_2\rangle = \langle Ax_2, x_2\rangle = f(x_2).$$

If we iterate this procedure, the assertion follows. □

Chapter IX

Paths and vector fields

We begin this chapter with the concept of a path in $\mathbb{R}^n$. This concept has been influenced by the need in physics – more precisely in kinematics – to consider the movement of a point in space by letting $\gamma(t)$ denote the position at time t. The origin of the concept goes back to the French mathematician C. JORDAN (1838-1922). Paths in this sense can have very surprising properties. For example, G. PEANO (1858-1932) constructed a path which completely covers a square.

A basic task in the theory of paths is the study of rectifiability, i.e. , the task of finding the lengths of paths. Closely connected with this problem are the so-called functions of bounded variation.

In the second section we discuss vector fields and line integrals. The latter are integrals which are taken over images of intervals under continuously differentiable maps – examples of paths – rather than over intervals. This extension of the concept of integration has important consequences. One can for example use this to characterize those vector fields which are gradients of so-called potential functions. The basic concepts of the divergence and curl of a vector field also play an important role in this connection.

1 Paths

We begin this section with the definition of a path

1.1 Definition. A continuous map $\gamma : I \to \mathbb{R}^n$ defined on an interval $I \subseteq \mathbb{R}$ is called a *path in* $\mathbb{R}^n$. In place of the word "path" we will also use "curve" and "arc".
A path is called *(continuously) differentiable*, if γ is *(continuously) differentiable.*
The image $\gamma(I)$ is called the *trace* of γ.

So according to the definition above a path is not simply a set of points in $\mathbb{R}^n$, but the knowledge of γ also includes the information in which direction a point on γ runs through the trace.

1.2 Example. a) Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function. Then

$$\gamma : I \to \mathbb{R}^2, \quad \gamma(t) := (t, f(t)),$$

is a curve. The trace of γ is the graph of f. Furthermore, if f is differentiable then also γ is differentiable, and we have

$$\gamma'(t) = (1, f'(t)).$$

b) For $r > 0$ the curve

$$\gamma : [0, 2\pi] \to \mathbb{R}^2, \quad \gamma(t) = (r \cos t, r \sin t),$$

describes a circular movement around $0 \in \mathbb{R}^2$ with radius r.

Since γ is differentiable and

$$\gamma'(t) = (-r \sin t, r \cos t),$$

the (Euclidean) norm of the velocity vector equals r.

c) For $a \in \mathbb{R}^m$ and $v \in \mathbb{R}^m \setminus \{0\}$ the curve

$$\gamma : \mathbb{R} \to \mathbb{R}^m, \quad \gamma(t) = a + vt,$$

describes a movement in a straight line in the direction of v with velocity $\gamma'(t) = v$.

d) For $r > 0$ and $c \neq 0$ the curve

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \quad \gamma(t) = (r \cos t, r \sin t, ct)$$

describes a *helix*. The trace lies on the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r^2\}$, and $2\pi c$ is the *lead* of the helix.

e) *Neil's parabola* is given by

$$\gamma : [-1, 1] \to \mathbb{R}^2, \quad \gamma(t) = (t^2, t^3).$$

Aside from the circle this was historically the first (nonlinear) curve for which one was able to calculate the arc length.

f) The *cycloid* is given by the curve

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \quad \gamma(t) = (t - \sin t, 1 - \cos t).$$

It describes the movement of a point on the boundary of the unit disc as the latter rolls on the x-axis.

The cycloid is interesting also from another perspective. It is the solution of the so-called *Brachistochrone proplem* – the variational problem of finding the curve between two fixed points which minimizes the time used by a point mass starting at the first point with zero speed and travelling along the curve to the second point, under the action of a constant gravity and assuming no friction. Bernoulli, Huygens and Leibnitz found that the solution to this problem is the cycloid.

1.3 Definition. Let $\gamma : I \to \mathbb{R}^n$ be a differentiable path. The vector

$$\gamma'(t) = (\gamma_1'(t), ..., \gamma_n'(t)) \in \mathbb{R}^n$$

is called the *tangent vector* of the curve γ in t.

One can interpret $\gamma'(t)$ as the velocity of the curve γ in the point t. The norm of the velocity vector is then

$$\|\gamma'(t)\| = \sqrt{|\gamma_1'(t)|^2 + ... + |\gamma_n'(t)|^2}.$$

In the following we will investigate the arc length of a given curve. The basic idea consists of approximating the arc length via suitable polygonal lines. Thus we consider a partition P of the interval $[a, b]$, i.e.

$$a = t_0 \le t_1 \le ... \le t_k = b,$$

and define the length of a polygonal line with the vertices $\gamma(t_0), \ldots, \gamma(t_k)$ by

$$L_{P,\gamma} := \sum_{j=1}^{k} \|\gamma(t_j) - \gamma(t_{j-1})\|.$$

1.4 Definition. A path $\gamma : [a, b] \to \mathbb{R}^n$ is called *rectifiable* with length L_γ in case

$$L_\gamma := \sup_P L_{P,\gamma} < \infty.$$

L_γ is called the *arc length* of γ.

1.5 Theorem. *Let $\gamma : [a, b] \to \mathbb{R}^n$ be a continuously differentiable curve. Then γ is rectifiable and*

$$L_\gamma = \int_a^b \|\gamma'(t)\|_2 \, dt = \int_a^b (|\gamma_1'(t)|^2 + \ldots + |\gamma_n'(t)|^2)^{1/2} \, dt.$$

In particular, the graph of a C^1-function $f : [a, b] \to \mathbb{R}$ has the length

$$L_f = \int_a^b \sqrt{1 + |f'(t)|^2} \; dt.$$

For the proof of this theorem we will use the notion of the integral of an n-tuple of continuous functions. If we set

$$\int_a^b \gamma(t) \, dt := \left(\int_a^b \gamma_1(t) \, dt, \ldots, \int_a^b \gamma_n(t) \, dt \right),$$

then
$$\left\|\int_a^b \gamma(t)\,dt\right\|_2 \leq \int_a^b \|\gamma(t)\|_2\,dt.$$
For, with the notation $v = \int_a^b \gamma(t)dt \in \mathbb{R}^n$ we have
$$\begin{aligned}\left\|\int_a^b \gamma(t)\,dt\right\|_2^2 = \left\langle \int_a^b \gamma(t)\,dt, v\right\rangle = \int_a^b \langle\gamma(t), v\rangle\,dt \\ \leq \int_a^b \|\gamma(t)\|_2\,dt \cdot \|v\|_2\end{aligned}$$
by the Cauchy-Schwarz inequality.

Proof. Let P be a partition of $[a,b]$ given by $P\ :\ a = t_0 < t_1 < ... < t_k = b$. The fundamental theorem of calculus implies that
$$\begin{aligned}L_{P,\gamma} &= \sum_{j=1}^k \|\gamma(t_j) - \gamma(t_{j-1})\|_2 = \sum_{j=1}^k \left\|\int_{t_{j-1}}^{t_j} \gamma'(t)\,dt\right\|_2 \\ &\leq \sum_{j=1}^k \int_{t_{j-1}}^{t_j} \|\gamma'(t)\|_2\,dt = \int_a^b \|\gamma'(t)\|_2\,dt =: L.\end{aligned} \tag{1.1}$$
Thus γ is rectifiable and $L_\gamma \leq L$.
We will now show that $L_\gamma = L$. It is enough to show that for all $\varepsilon > 0$ there exists a partition $P : a = t_0 < ... < t_k = b$ of $[a,b]$ with
$$L_{P,\gamma} \geq L - \varepsilon.$$

We need a polygonal line which lies sufficiently close to the curve, and with vertices on the curve. We write $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$ for $t \in [a,b]$. Let $\varepsilon > 0$.

Since γ_i' is uniformly continuous on $[a,b]$ for each $1 \leq i \leq n$, we know that there exists a partition $P_i : a = t_0 < ... < t_{k_i} = b$ such that
$$|\gamma_i'(s) - \gamma_i'(t)| < \varepsilon$$
for all $s,t \in [t_{j-1}, t_j]$ and for all $1 \leq j \leq k_i$. We let the partition $P : a = t_0 < ... < t_k = b$ be a common refinement of all the P_i's for $1 \leq i \leq n$. Then for all $1 \leq i \leq n$, all $1 \leq j \leq k$ and all $s,t \in [t_{j-1}, t_j]$ we have
$$|\gamma_i'(s) - \gamma_i'(t)| < \varepsilon.$$
Let $1 \leq i \leq n, 1 \leq j \leq k$. By the mean value theorem from Analysis I there exists $\tau_{ij} \in (t_{j-1}, t_j)$ such that
$$\frac{\gamma_i(t_j) - \gamma_i(t_{j-1})}{t_j - t_{j-1}} = \gamma_i'(\tau_{ij})$$

or

$$\gamma_i(t_j) - \gamma_i(t_{j-1}) = \gamma_i'(\tau_{ij})(t_j - t_{j-1}).$$

We write $c_{ij} := \gamma_i'(\tau_{ij})$ and define an n-tuple $\varphi = (\varphi_1, ..., \varphi_n)$ of step functions by letting $\varphi_i(a) = \gamma_i'(a)$ and

$$\varphi_i(t) = c_{ij}$$

for $t \in (t_{j-1}, t_j]$. Then

$$|\gamma_i'(t) - \varphi_i(t)| < \varepsilon$$

and

$$\|\gamma'(t) - \varphi(t)\|_2 < \sqrt{n}\varepsilon$$

for $t \in [a, b]$.

Now $t \longmapsto \int_a^t \varphi_i(s)ds + \gamma_i(a), t \in [a, b]$, describes the i-th component of the polygonal line we use to approximate γ. We have

$$\begin{aligned} L_{P,\gamma} &= \sum_{j=1}^{k} \|\gamma(t_j) - \gamma(t_{j-1}\|_2 = \sum_{j=1}^{k} \left\| \int_{t_{j-1}}^{t_j} \varphi(s)\, ds \right\|_2 \\ &= \sum_{j=1}^{k} \|(c_{1j}\,(t_j - t_{j-1}), \ldots, c_{nj}\,(t_j - t_{j-1})\|_2 \\ &= \sum_{j=1}^{k} \sqrt{c_{1j}^2 + ... + c_{1n}^2}\,(t_j - t_{j-1}) = \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \|\varphi(s)\|_2\, ds = \int_a^b \|\varphi(s)\|_2\, ds. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_a^b \|\varphi(s)\|_2\, ds &\geq \int_a^b \|\gamma'(s)\|_2\, ds - \int_a^b \|\varphi(s) - \gamma'(s)\|_2\, ds \\ &\geq \int_a^b \|\gamma'(s)\|_2\, ds - \varepsilon\sqrt{n}(b-a) \\ &= L - \varepsilon\sqrt{n}(b-a). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary the result follows. $\square$

1.6 Example. a) We calculate the length of one arc of the cycloid

$$\gamma\,:\,\mathbb{R} \to \mathbb{R}^2, \quad \gamma(t) = (t - \sin t, 1 - \cos t),$$

as follows. We note that γ is differentiable with $\gamma'(t) = (1 - \cos t, \sin t)$. Thus

$$\begin{aligned} \|\gamma'(t)\|_2^2 &= 1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{=1} \\ &= 2 - 2\cos t = 4\sin^2\left(\frac{t}{2}\right). \end{aligned}$$

Thus

$$\begin{aligned} L_{\gamma_{|[0,2\pi]}} &= \int_0^{2\pi} 2\left|\sin\left(\frac{t}{2}\right)\right| dt \\ &= -4\cos\left(\frac{t}{2}\right)\Bigg|_0^{2\pi} = 8. \end{aligned}$$

b) We define $\gamma :\ [0,1] \to \mathbb{R}$ by $\gamma(t) = (t, f(t))$,

$$f(t) = \begin{cases} t^2 \cos\frac{\pi}{t^2} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Then γ is continuous, and provides an example of a curve which is not rectifiable. To see this we define the partition P of $[0,1]$ by

$$P : 0 = t_0 < k^{-\frac{1}{2}} < (k-1)^{-\frac{1}{2}} < ... < (2)^{-\frac{1}{2}} < 1,$$

and note that $f(j^{-1/2}) = \frac{(-1)^j}{j}$, $1 \leq j \leq k.$
So we have

$$L_{P,\gamma} > 1 + \frac{1}{2} + ... + \frac{1}{n},$$

and so $L_\gamma = \infty$.

The concept of a rectifiable path is closely connected with the concept of a function of bounded variation.

Let $f : I = [a,b] \to \mathbb{R}$ and let P be a partition of I. The *variation* of f with respect to P is given by

$$\mathrm{var}_{P,f} := \sum_{j=1}^{k} |f(t_j) - f(t_{j-1})|\,.$$

The *total variation* of f on I is given by the supremum taken over all partitions of I, i.e.,

$$V_a^b(f) := \sup_P \mathrm{var}_{P,f}\,.$$

If $V_a^b(f) < \infty$, then we say that f is of *bounded variation* on I. We denote the class of all such functions by $BV(I)$. If $f : [a,b] \to \mathbb{R}$ is a path, then $L_f = V_a^b(f)$. In the following lemma we provide some basic properties of functions $f \in BV[a,b]$.

1.7 Lemma. *For functions $f \in BV[a,b]$ the following statements hold:*

(a) $BV[a,b] \subset B[a,b]$, *and*

$$|f(a) - f(b)| \leq V_a^b(f).$$

(b) *$BV[a,b]$ is a vector space and even an algebra. We have the following inequalities:*

(i) $V_a^b(\lambda f + \mu g) \le |\lambda| V_a^b(f) + |\mu| V_a^b(g)$ *for* $\lambda, \mu \in \mathbb{R}$, $f, g \in BV[a,b]$.

(ii) $V_a^b(fg) \le \|f\|_\infty V_a^b(g) + \|g\|_\infty V_a^b(f)$.

(c) *For $a < c < b$ we have*

$$V_a^b(f) = V_a^c(f) + V_c^b(f).$$

(d) *If f is monotone on $[a,b]$, then $V_a^b(f) = |f(b) - f(a)|$.*

(e) *If $f \in C^1([a,b], \mathbb{R})$, then $V_a^b(f) = \int_a^b |f'(t)|\,dt$.*

Proof. See the Exercises. □

In the following theorem we characterize the functions of bounded variation as the functions which can be given as the difference of two monotone functions.

1.8 Theorem. *A function $f : [a,b] \to \mathbb{R}$ is of bounded variation if and only if there exist increasing functions $g, h : [a,b] \to \mathbb{R}$ such that $f = g - h$.*

Proof. Let $f \in BV[a,b]$ and, for $t \in [a,b]$, set $g(t) := V_a^t(f)$. Then by Lemma 1.7 (c) we have

$$0 \le V_c^d(f) = V_a^d(f) - V_a^c(f) = g(d) - g(c)$$

for all $a \le c < d \le b$. So g is increasing.
Furthermore, by Lemma 1.7 (a) we have

$$f(d) - f(c) \le V_c^d(f) = g(d) - g(c),$$

and so for $h := g - f$ we have $h(c) \le h(d)$. Thus also h is increasing.
The opposite direction follows immediately from Lemma 1.7 (d) and (b). □

If $f = (f_1, \ldots, f_n) : I \to \mathbb{R}^n$ is a path, then

$$V_a^b(f_i) \le L_f \le V_a^b(f_1) + \ldots + V_a^b(f_n).$$

Thus we obtain the connection between functions of bounded variation and rectifiable paths which was mentioned earlier.

1.9 Theorem. *A path $f : I \to \mathbb{R}^n$ is rectifiable if and only if all component functions f_i are of bounded variation on I.*

2 Vector fields and line integrals

Let $U \subset \mathbb{R}^n$ be an open set and $F = (F_1, \ldots, F_n) : U \to \mathbb{R}^n$ be a continuous function. Then this map is often called a *vector field*. Vector fields can be visualized if one brings to mind that at each $x \in U$ there is attached the vector $F(x)$. Important examples of vector fields in physics are so-called force fields or velocity fields. In the following we want to discuss the following classes of vector fields in more detail.

2.1 Example. a) *Constant vector fields.* These are defined by $F(x) := y$ for some fixed $y \in \mathbb{R}^n$.

b) *Central fields.* Let $I = [a, b] \subset \mathbb{R}$ be an interval, $K := \{x \in \mathbb{R}^n : a < |x| < b\}$ a spherical shell in $\mathbb{R}^n$ and $g : I \to \mathbb{R}$ a continuous function. Then

$$F : K \to \mathbb{R}^n, \qquad F(x) := g(\|x\|)\, x$$

is called a *central field.*

c) *Rotational fields.* Let $I = [a, b]$ be an interval, $K := \{x \in \mathbb{R}^2 : a < |x| < b\}$ an annulus in $\mathbb{R}^2$ and $g : I \to \mathbb{R}$ a continuous function. Then

$$F : K \to \mathbb{R}^2, \qquad F(x) := g(\|x\|)(-x_2, x_1)^T$$

is called a *rotational field.*

d) A vector field F is called a *gradient field*, if there exists a continuously differentiable function $V : U \to \mathbb{R}$ such that

$$\operatorname{grad} V = F.$$

Of particular importance are the notions of divergence and rotation of vector fields, defined as follows.

2.2 Definition. If $U \subset \mathbb{R}^n$ is open and $F : U \to \mathbb{R}^n$ a continuously differentiable vector field, then the function

$$\operatorname{div} F(x) := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$

is called the *divergence of* F at the point $x \in U$.

One easily checks that for C^1-vector fields F and G as well as for scalar C^1-functions $h : \mathbb{R}^n \to \mathbb{R}$ one has

$$\operatorname{div}(F + G) = \operatorname{div} F + \operatorname{div} G, \qquad \operatorname{div}(hF) = \nabla h \cdot F + h \operatorname{div} F.$$

2.3 Definition. If $U \subset \mathbb{R}^3$ is open and $F : U \to \mathbb{R}^3$ a continuously differentiable vector field, then one defines $\operatorname{rot} F : U \to \mathbb{R}^3$, the *rotation* of F, by

$$\operatorname{rot} F := \Big(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\Big).$$

If one uses the vector product from Linear Algebra, one can formally write

$$\operatorname{rot} F = \nabla \times F \qquad \text{and} \qquad \operatorname{div} F = \nabla \cdot F.$$

For continuously differentiable vector fields $F, G \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ and scalar functions $h \in C^1(\mathbb{R}^3, \mathbb{R})$ we have the following relations

a) $\operatorname{rot}(hF) = h \operatorname{rot} F - F \times \nabla h$

b) $\operatorname{div}(F \times G) = G \cdot \operatorname{rot} F - F \cdot \operatorname{rot} G$

c) $\operatorname{rot}(\nabla h) = 0$

d) $\operatorname{div}(\operatorname{rot} F) = 0$

e) $\operatorname{rot}(\operatorname{rot} F) = \nabla(\operatorname{div} F) - \Delta F$.

As a motivation for the concept of a *line integral* let us consider a particle that moves along the image of a path $\gamma : [a, b] \to \mathbb{R}^3$ while being acted upon by a force field F. Then the work done by F is given by the integral $\int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$, since only the tangential part of F along γ will yield a contribution. By analogy, we define the line integral as follows.

2.4 Definition. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a piecewise continuously differentiable path and f a continuous, real-valued function defined on $\gamma([a, b])$. Then

$$\int_\gamma f(x)\, dx_j := \int_a^b f(\gamma(t))\gamma_j'(t)\, dt\,, \quad j = 1, \dots, n,$$

is called the *line integral of f along γ with respect to x_j*.
If $f = (f_1, \dots, f_n) : \gamma([a, b]) \to \mathbb{R}^n$ is continuous, then we define the *line integral of f along γ* by

$$\int_\gamma f(x)\, dx := \int_\gamma f_1(x)\, dx_1 + \cdots + \int_\gamma f_n(x)\, dx_n = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt.$$

2.5 Remark. The value of the line integral $\int_\gamma f(x)\, dx$ does not depend on the parametrization of the path.

2.6 Example. Given the path γ by

$$\gamma : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}^2, \quad \gamma(t) = (\cos t, \sin t),$$

i.e. a semi-circle in the plane. Let us integrate the function $f(x_1, x_2) := (x_2, -x_1)^T$ along γ. Then, with $x = (x_1, x_2)^T$,

$$\begin{aligned}
\int_\gamma f(x)\, dx &= \int_\gamma f_1(x)\, dx_1 + \int_\gamma f_2(x)\, dx_2 \\
&= \int_{-\pi/2}^{\pi/2} \sin t\, (-\sin t)\, dt + \int_{-\pi/2}^{\pi/2} (-\cos t) \cos t\, dt \\
&= -\int_{-\pi/2}^{\pi/2} (\underbrace{\sin^2 t + \cos^2 t}_{=1})\, dt = -\pi\,.
\end{aligned}$$

2.7 Definition. Let $U \subset \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^n$ be a vector field.

(a) If F is a gradient field, then there exists a function $V \in C^1(U, \mathbb{R})$ with $\nabla V = F$ on U. Such a function V is called a *potential of* F.

(b) Let $\gamma : [a, b] \to U \subset \mathbb{R}^n$ be a piecewise continuously differentiable path with $\gamma(a) = x_0$ and $\gamma(b) = y_0$. If for every piecewise continuously differentiable path σ, connecting x_0 with y_0, we have that $\int_\sigma f(x)\, dx = \int_\gamma f(x)\, dx$, i.e., the value of the line integral $\int_\gamma f(x)\, dx$ does not depend on γ itself, but only on the endpoints of γ, then we say that $\int_\gamma f(x)\, dx$ is *independent of the path.*

The following theorem characterizes those vector fields having the line integral independent of the path.

2.8 Theorem. *Let $U \subset \mathbb{R}^n$ be an open, convex set and $F : U \to \mathbb{R}^n$ be a continuous vector field. Then the line integral $\int_\gamma F(x)\, dx$ is independent of the (piecewise continuously differentiable) path if and only if F is a gradient field. In this case*

$$\int_\gamma F(x)\, dx = V(\gamma(b)) - V(\gamma(a))\,,$$

where V is a potential of F and $\gamma : [a, b] \to U \subset \mathbb{R}^n$ is piecewise cont. differentiable.

Proof. "$\Leftarrow$": If V is a potential of F, then

$$\begin{aligned}
\int_\gamma F(x)\, dx &= \int_a^b \sum_{j=1}^n F_j(\gamma(t))\gamma_j'(t)\, dt \\
&= \int_a^b \frac{dV(\gamma(t))}{dt}\, dt = V(\gamma(b)) - V(\gamma(a))\,.
\end{aligned}$$

The converse implication " $\Rightarrow$ " is left to the reader as an exercise. $\square$

In the following we are looking for simple criteria to prove that a given vector field is a gradient field. A necessary condition can immediately be given. If F is a continuously differentiable gradient field, then the potential $V \in C^2(U, \mathbb{R})$ satisfies

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i}\,,$$

that is

$$\frac{\partial F_j}{\partial x_i} = \frac{\partial F_i}{\partial x_j} \quad \text{on } U, \quad i, j = 1, \dots, n.$$

We would like now to dicuss the converse question if the above condition implies the independence of path of the line integral. It is not surprising that the answer to this question depends on the geometry of the domain.

A set $U \subset \mathbb{R}^n$ is called *star-shaped*, if there exists some $x_0 \in U$ such that every segment $\overline{x_0 x} \subset U$ for every $x \in U$. We also say that U is star-shaped with respect to x_0.

Note that every convex set U is star-shaped (with respect to every $x_0 \in U$), but that a star-shaped set is not necessarily convex.

2.9 Theorem. *Let $U \subset \mathbb{R}^n$ be open and star-shaped with respect to $x_0 \in U$ and $F = (F_1, \dots, F_n) : U \to \mathbb{R}^n$ be a continuously differentiable vector field satisfying*

$$\frac{\partial F_j}{\partial x_i} = \frac{\partial F_i}{\partial x_j} \quad \textit{on } U, \quad i, j = 1, \dots, n.$$

Then F is a gradient field.

Proof. Without loss of generality we can assume that $x_0 = 0$. Consider

$$V(x) := \sum_{i=1}^n \Big(\int_0^1 F_i(tx)\, dt \Big)\, x_i\,.$$

Then

$$\begin{aligned} \frac{\partial V}{\partial x_j}(x) &= \sum_{i=1}^n \frac{\partial}{\partial x_j} \Big(\int_0^1 F_i(tx)\, dt \Big)\, x_i + \sum_{i=1}^n \Big(\int_0^1 F_i(tx)\, dt \Big) \frac{\partial x_i}{\partial x_j} \\ &= \sum_{i=1}^n \Big(\int_0^1 t\Big(\frac{\partial}{\partial x_j} F_i \Big)(tx)\, dt \Big)\, x_i + \int_0^1 F_j(tx)\, dt\,. \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{d}{dt}\big(tF_j(tx)\big) &= F_j(tx) + t\frac{d}{dt} F_j(tx) = F_j(tx) + t \sum_{i=1}^n \Big(\frac{\partial}{\partial x_i} F_j \Big)(tx) x_i \\ &= F_j(tx) + t \sum_{i=1}^n \Big(\frac{\partial}{\partial x_j} F_i \Big)(tx) x_i \end{aligned}$$

by hypothesis. Hence

$$\frac{\partial V}{\partial x_j}(x) = \int_0^1 \frac{d}{dt}\big(tF_j(tx)\big)\,dt = tF_j(tx)\Big|_{t=0}^{t=1} = F_j(x).$$

$\square$

2.10 Remark. If $U \subset \mathbb{R}^3$ is an open ball and $F : U \to \mathbb{R}^3$ a continuously differentiable vector field, then Theorem 2.9 and the discussion preceding it implies that F is a gradient field if and only if $\operatorname{rot} F = 0$.