Fourier series

A. Complex numbers – a recapitulation

Let $a, b \in \mathbb{R}$ and $z = a + ib \in \mathbb{C}$. The complex number $\overline{z} := a - ib$ is called the *complex conjugate* of z. The *absolute value* $|z|$ is defined as

$$|z| := \sqrt{z\overline{z}} = \sqrt{a^2 + b^2} \quad (\geq 0).$$

We call $a = \mathsf{Re}\,(z)$ the *real part* of z and $b = \mathsf{Im}\,(z)$ the *imaginary part* of z. We have

$$\mathsf{Re}\,(z) = \frac{1}{2}(z + \overline{z})\,, \qquad \mathsf{Im}\,(z) = \frac{1}{2i}(z - \overline{z})$$

Note: $z_1 = z_2 \iff \mathsf{Re}\,(z_1) = \mathsf{Re}\,(z_2)$ **and** $\mathsf{Im}\,(z_1) = \mathsf{Im}\,(z_2)$.

The following calculation rules hold for conjugation and for taking the absolute value. When $z, w \in \mathbb{C}$, then

$$\overline{\overline{z}} = z\,,\ \overline{z + w} = \overline{z} + \overline{w}\,,\ \overline{z \cdot w} = \overline{z} \cdot \overline{w}\,,$$

$$\mathsf{Re}\,(z) \leq |\mathsf{Re}\,(z)| \leq |z|\,, \qquad \mathsf{Im}\,(z) \leq |\mathsf{Im}\,(z)| \leq |z|\,,$$

$$|z| \geq 0\,, \quad |z| = 0 \iff z = 0\,, \qquad |zw| = |z|\,|w|$$

$$|z + w| \leq |z| + |w| \qquad \text{(triangle inequality)}.$$

Definition: *A sequence $(c_n)_{n\in\mathbb{N}}$ converges to $c \in \mathbb{C}$, if to each $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that*

$$|c_n - c| < \varepsilon \qquad \textit{for all } n \geq n_0.$$

Recall: $c_n \to c \iff \mathsf{Re}\,(c_n) \to \mathsf{Re}\,(c)$ and $\mathsf{Im}\,(c_n) \to \mathsf{Im}\,(c)$.

Euler's formula holds

$$\cos x + i \sin x = e^{ix} \quad \Big(= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}\Big)\,,$$

since $i^n = 1$ if $n = 4k$, $i^n = i$ if $n = 4k + 1$, $i^n = -1$ if $n = 4k + 2$, and $i^n = -i$ if $n = 4k + 3$, $k \in \mathbb{Z}$, and, therefore,

$$\begin{aligned} \cos x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \sum_{k=0}^{\infty} \frac{(ix)^{2k}}{(2k)!} && \text{(conv. absolutely)}, \\ i \sin x &= i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(ix)^{2k+1}}{(2k+1)!} && \text{(conv. absolutely)}. \end{aligned}$$

We recall the special case of the functional equation for the exponential function

$$e^{ix}e^{iy} = e^{i(x+y)} \qquad \text{for all } x, y \in \mathbb{R};$$

in particular, e^{ix} is a 2π-periodic function, $e^{i(x+2\pi)} = e^{ix}$ all $x \in \mathbb{R}$, and $\overline{e^{ix}} = e^{-ix}$.

B. Pointwise convergence of Fourier series

A basic problem of analysis consists in controlling/approximating "complicated" quantities by "simple" ones:

a) A real number x, $0 < x < 1$, $x = 0, a_1, a_2, a_3, a_4, \ldots$, $a_i \in \{0, \ldots, 9\}$, is determined by the sequence of the finite decimal fraction $(x_n)_n$, $x_n = 0, a_1, \ldots, a_n$.

b) If $f \in C^\infty(I)$, $a \in I$, one can construct the sequence of Taylor polynomials $\left(T_n(f;a)\right)_n$, $T_n(f;a)(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x-a)^k$, and try to approximate f by this sequence of Taylor polynomials.

c) J. Fourier's (1768 - 1830) vision was: Each continuous, 2π-periodic function can be approximated arbitrarily accurate by the partial sums of the nowadays called Fourier series (in particular, by trigonometric polynomials $t(x) := \sum_{k=-n}^n a_k\, e^{ikx}$).
Now we want to state this more precisely.

Let the uniformly convergent trigonometric series

$$\sum_{k=-\infty}^{\infty} c_k\, e^{ikx} =: f(x)\,, \quad c_k \in \mathbb{C}\,, \tag{1}$$

be given. Since all terms of the sum are continuous and 2π-periodic, this series defines a continuous and 2π-periodic function; Notation: $f \in C_{2\pi}$, where

$$C_{2\pi} := \{g \in C(\mathbb{R}, \mathbb{C}) : g(x+2\pi) = g(x) \text{ all } x \in \mathbb{R}\}\,.$$

Problem: What is the relation between f and $(c_k)_k$? How can one obtain the coefficients $(c_k)_k$ from f, and vice versa? Are the coefficients uniquely determined?

Recall that for jump continuous functions $f : [a,b] \subset \mathbb{R} \to \mathbb{C}$ we defined

$$\int_a^b f(x)\,dx := \int_a^b (Re\, f(x))\,dx + i \int_a^b (Im\, f(x))\,dx\,.$$

Therefore, the following definition is reasonable.

Definition. *Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic, jump continuous function,* *notation:* *$f \in \mathcal{S}_{2\pi}$. We call the numbers*

$$\widehat{f}_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx\,, \qquad k \in \mathbb{Z}\,,$$

the **Fourier coefficients** *of f and the series*

$$\sum_{k=-\infty}^{\infty} \widehat{f}_k\, e^{ikx}\,,$$

i.e., the sequence of the *partial sums* *$(s_n(f))_n$, $s_n(f;x) := \sum_{k=-n}^{n} \widehat{f}_k\, e^{ikx}$, the* **Fourier series** *of f.*

If f, like in (1), is given by a uniformly convergent trigonometric series, then

$$\begin{aligned} \widehat{f}_j &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} c_k\, e^{ikx} \right) e^{-ijx}\, dx = \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-j)x}\, dx \\ &= c_j\,, \qquad \text{since } \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-j)x}\, dx = \begin{cases} 1 & , k = j \\ 0 & , k \neq j \end{cases}. \end{aligned}$$

In particular, if f is a trigonometric polynomial,

$$(2) \qquad f(x) = \sum_{k=-N}^{N} c_k\, e^{ikx} \quad \Rightarrow \quad s_n(f) = f,\ n \geq N.$$

Counterexamples show that the partial sums do not converge in general – there exist $f \in C_{2\pi}$, whose partial sums diverge in one point (and hence in countably many points). L. Fejér (1880 - 1959) recognized that, though the partial sums do not converge in the sense of Cauchy, they do converge in a weaker sense, namely, their first arithmetic means converge.
(Recall: If $(s_n)_n \subset \mathbb{C}$, $s_n \to s \quad \Rightarrow \quad \frac{1}{n+1} \sum_{k=0}^{n} s_k \to s$.)

We compute the first arithmetic means of the partial sums of the Fourier series

$$\begin{aligned} \frac{1}{n+1} \sum_{k=0}^{n} s_k(f;x) &= \frac{1}{n+1} \sum_{k=0}^{n} \sum_{j=-k}^{k} \widehat{f}_j\, e^{ijx} = \frac{1}{n+1} \sum_{j=-n}^{n} \widehat{f}_j \Big(\sum_{k=|j|}^{n} 1 \Big)\, e^{ijx} \\ (3) \qquad &= \sum_{j=-n}^{n} \widehat{f}_j \frac{n+1-|j|}{n+1}\, e^{ijx} = \sum_{j=-n}^{n} \Big(1 - \frac{|j|}{n+1} \Big)\, \widehat{f}_j\, e^{ijx}\,. \end{aligned}$$

Theorem 1 (Fejér's Theorem). *Let $f \in \mathcal{S}_{2\pi}$ (i.e., $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic and jump continuous).*

a) If f is continuous at t, then

$$\sigma_n(f;t) := \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \widehat{f}_k \, e^{ikt} \;\to\; f(t) \;\; \text{when } n \to \infty\,.$$

b) If f is continuous on $[a,b] \subseteq [-\pi,\pi]$, then the convergence is uniform on $[a,b]$. In particular, if $f \in C_{2\pi}$, then

$$\lim_{n\to\infty} \|\sigma_n(f) - f\|_\infty := \lim_{n\to\infty} \left[\sup_{-\pi \le t \le \pi} |\sigma_n(f;t) - f(t)| \right] = 0\,.$$

Corollary 1 (Uniqueness Theorem). *If $f, g \in C_{2\pi}$ and $\widehat{f}_k = \widehat{g}_k$ for all $k \in \mathbb{Z}$, then $f = g$.*

Proof. Use the triangle inequality for the $\|\cdot\|_\infty$-norm and the hypothesis $\widehat{f}_k = \widehat{g}_k$ to obtain

$$\|f - g\|_\infty \le \|f - \sigma_n(f)\|_\infty + \|\sigma_n(g) - g\|_\infty \to 0 \;\text{ for }\; n \to \infty.$$

□

Corollary 2 (Approximation Theorem). *To each $f \in C_{2\pi}$ and each $\varepsilon > 0$ there exists a trigonometric polynomial T such that*

$$\|f - T\|_\infty < \varepsilon\,.$$

Proof. Choose, e.g., $T = \sigma_n(f)$. □

Corollary 3 (Lemma of Riemann). If $f \in C_{2\pi}$, then $\lim_{|k|\to\infty} \widehat{f}_k = 0$.

Proof. To given $\varepsilon > 0$ choose $n \in \mathbb{N}$ such that $\|f - \sigma_n(f)\|_\infty < \varepsilon$. Then one obtains for $|k| > n$ by the definition of the Fourier coefficients

$$|\widehat{f}_k| = |\widehat{f}_k - \widehat{\sigma_n(f)}_k| = |(f - \widehat{\sigma_n(f)})_k| \le \|f - \sigma_n(f)\|_\infty < \varepsilon.$$

□

To prove the Theorem of Fejér we need three lemmas.

Lemma 1. *If* $f \in \mathcal{S}_{2\pi}$, *then*

$$\sigma_n(f;t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)\,K_n(s)\,ds, \quad \text{where } K_n(s) = \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{iks}\,.$$

Proof.

$$\begin{aligned}\sigma_n(f;t) &= \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-ikx}\,dx\,e^{ikt} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ik(t-x)}\,dx\end{aligned}$$

$$= \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} f(t-y)\,K_n(y)\,dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-y)\,K_n(y)\,dy\,;$$

here the last equality holds, because $f(t-y)\,K_n(y)$ is a 2π-periodic function in y and the integral of a periodic function, which is integrated over a full period, does not change, if one shifts the integration interval. □

Lemma 2.

$$K_n(y) = \begin{cases} n+1 & , \ y = 2N\pi \\ \dfrac{1}{n+1}\left(\dfrac{\sin\frac{n+1}{2}y}{\sin\frac{1}{2}y}\right)^2 & , \ y \neq 2N\pi\,,\ N \in \mathbb{Z}\,. \end{cases}$$

Proof. For $y = 2N\pi$ we have

$$\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right) = 2n+1-\frac{2}{n+1}\sum_{k=1}^{n} k = n+1.$$

(Note: $2\sum_{k=1}^{n} k = n(n+1)$.) Now let $y \neq 2N\pi$, $N \in \mathbb{Z}$ and choose $f(y) = \sum_{j=-n}^{n} e^{ijy}$ in (3). Then (3) implies

$$\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{iky} = \frac{1}{n+1}\sum_{k=0}^{n}\sum_{j=-k}^{k} e^{ijy} = K_n(y)\,.$$

The inner sum on the right hand side is a finite geometric series. Therefore,

$$\begin{aligned}e^{-iky}\sum_{j=0}^{2k} e^{ijy} &= e^{-iky}\frac{1-e^{i(2k+1)y}}{1-e^{iy}} = \frac{(1-e^{-iy})(e^{-iky}-e^{i(k+1)y})}{2-2\cos y} \\ &= \frac{e^{-iky}-e^{i(k+1)y}-e^{-i(k+1)y}+e^{iky}}{2(1-\cos y)} = \frac{\cos ky-\cos(k+1)y}{1-\cos y}\,.\end{aligned}$$

Hence

$$K_n(y) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{\cos ky - \cos(k+1)y}{1-\cos y} = \frac{1}{n+1} \frac{1-\cos(n+1)y}{1-\cos y}$$

and, since $1 - \cos my = 2\sin^2 \frac{1}{2}my$, $m \in \mathbb{N}_0$, the last display yields the assertion. □

Lemma 3.

(i) $K_n(y) \geq 0$ *for all* $y \in \mathbb{R}$.

(ii) $K_n(y)$ *converges uniformly to* 0 *on* $[-\pi, -\delta] \cup [\delta, \pi]$ *for each* δ *(fixed),* $0 < \delta < \pi$.

(iii) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)\, dy = 1$.

Proof. (i) is obvious on account of Lemma 2. Likewise (ii), since

$$K_n(y) \leq \frac{1}{(n+1)} \frac{1}{\sin^2 \delta/2} \to 0, \quad n \to \infty, \quad \text{for every fixed } \delta,\ 0 < \delta < \pi.$$

(iii) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)\, dy = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iky}\, dy = 1$, since all terms of the sum except for the term with $k = 0$ vanish. □

Idea of proof of Fejér's Theorem: In a certain sense, for small $\delta > 0$ and for large n we may argue as follows:

$$\begin{aligned}
\sigma_n(f;t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)\, f(t-y)\, dy \overset{(ii)}{\approx} \frac{1}{2\pi} \int_{|y|\leq\delta} K_n(y)\, f(t-y)\, dy \\
&\approx f(t) \frac{1}{2\pi} \int_{|y|\leq\delta} K_n(y)\, dy \qquad \text{(since } f \text{ is continuous at } t\text{)} \\
&\overset{(ii)}{\approx} f(t) \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)\, dy \overset{(iii)}{\approx} f(t)\,.
\end{aligned}$$

Proof of Fejér's Theorem. Let $\varepsilon > 0$ be given. Note the following facts:

(α) Since $f \in \mathcal{S}_{2\pi}$, there exists an $M > 0$ such that $|f(y)| \leq M$ for all y.

(β) Due to the continuity of f at t there exists $\delta = \delta_{\varepsilon,t} > 0$ such that $|f(y) - f(t)| < \frac{\varepsilon}{2}$ for all y, $|y - t| < \delta$. This δ will now be fixed.

(γ) $|K_n(y)| \leq \frac{\varepsilon}{4M}$ for all y, $\delta \leq |y| \leq \pi$ and $n \geq N_{t,\varepsilon}$.

Thus, by Lemma 1,

$$\begin{aligned}
|\sigma_n(f;t) - f(t)| &\overset{(iii)}{=} \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-y)\,K_n(y)\,dy - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,K_n(y)\,dy\right| \\
&\leq \frac{1}{2\pi}\left(\int_{|y|\leq\delta} + \int_{\delta\leq|y|\leq\pi}\right) |f(t-y)-f(t)|\,K_n(y)\,dy \\
&\overset{(\beta,\alpha)}{\leq} \frac{\varepsilon}{2}\,\frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(y)\,dy + \frac{2M}{2\pi}\int_{\delta\leq|y|\leq\pi} K_n(y)\,dy \\
&\overset{(\gamma)}{\leq} \frac{\varepsilon}{2} + \frac{M}{\pi}\int_{\delta\leq|y|\leq\pi} \frac{\varepsilon}{4M}\,dy < \varepsilon \qquad \text{for all } n \geq N_{\varepsilon,t}\,.
\end{aligned}$$

If f is continuous on $[a,b]$, then f is uniformly continuous on $[a,b]$, and the previous arguments hold uniformly w.r.t. $t \in [a,b]$. □

C. Parseval's formula

We introduce a scalar product and a related norm on $\mathcal{S}_{2\pi}$:

$$\langle f,g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx\,, \quad \|f\|_2 := \langle f,f\rangle^{1/2}, \quad f,g\in\mathcal{S}_{2\pi}\,.$$

then there hold the inequality of Cauchy-Schwarz (cf. Chap. V, Cor. 2.19 or see the Appendix) and the triangle inequality, i.e.,

$$|\langle f,g\rangle| \leq \|f\|_2\|g\|_2\,, \qquad \|f+g\|_2 \leq \|f\|_2 + \|g\|_2\,.$$

Theorem 2. *For* $f \in \mathcal{S}_{2\pi}$

$$\|f - s_n(f)\,\|_2^2 = \|f\,\|_2^2 - \sum_{k=-n}^{n} |\widehat{f}_k|^2\,,$$

where $s_n(f;x) = \sum_{k=-n}^{n} \widehat{f}_k\, e^{ikx}$, $n\in\mathbb{N}$, *denote the partial sums of the Fourier series. In particular, one gets the following results:*

(a) *If* f *is a trigonometric polynomial of degree* N, *i.e.,* $f(x) = \sum_{k=-N}^{N} \widehat{f}_k e^{ikx}$, *then, by* (2),

$$\|f\,\|_2^2 = \sum_{k=-n}^{n} |\widehat{f}_k|^2 \qquad \textit{for every } n \geq N.$$

(b) *If* $f \in \mathcal{S}_{2\pi}$, *then*

$$\sum_{k=-\infty}^{\infty} |\widehat{f}_k|^2 \le \|f\|_2^2 , \qquad \text{(}\textit{Bessel's inequality}\text{)}.$$

Proof. Let $g := s_n(f)$. Then $\widehat{g}_k = \widehat{f}_k$ for $|k| \le n$, and even

$$\begin{aligned} \langle f, g\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{\sum_{k=-n}^{n} \widehat{f}_k\, e^{ikx}}\, dx = \sum_{k=-n}^{n} \overline{\widehat{f}_k}\, \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx \\ &= \sum_{k=-n}^{n} \overline{\widehat{f}_k}\,\widehat{f}_k = \sum_{k=-n}^{n} \overline{\widehat{g}_k}\, \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\, e^{-ikx}\, dx = \langle g, g\rangle\ . \end{aligned}$$

Hence

$$\begin{aligned} 0 &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - g(x)|^2\, dx = \langle f - g, f - g\rangle \\ &= \langle f, f\rangle - \langle f, g\rangle - \langle g, f\rangle + \langle g, g\rangle \\ &= \|f\|_2^2 - 2\sum_{k=-n}^{n} |\widehat{f}_k|^2 + \sum_{k=-n}^{n} |\widehat{f}_k|^2 = \|f\|_2^2 - \sum_{k=-n}^{n} |\widehat{f}_k|^2\ . \end{aligned}$$

In particular,

$$\sum_{k=-n}^{n} |\widehat{f}_k|^2 \le \|f\|_2^2 \qquad \text{for all } n \in \mathbb{N}\,,$$

yielding Bessel's inequality as $n \to \infty$. □

The next result shows that the sequence of partial sums of the Fourier series of f converges to f in the weaker sense of *integral mean squares*. Here "weaker sense" means that uniform convergence in $C_{2\pi}$ implies "convergence of integral mean squares".

Theorem 3. (Formulas of Parseval and of Plancherel) *If* $f \in \mathcal{S}_{2\pi}$ *then*

$$\lim_{n\to\infty} \|s_n(f) - f\|_2^2 \equiv \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big| \sum_{k=-n}^{n} \widehat{f}_k\, e^{ikx} - f(x)\Big|^2 dx = 0$$

und

$$\sum_{k=-\infty}^{\infty} |\widehat{f}_k|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \|f\|_2^2\,.$$

Proof. The triangle inequality (w.r.t. the integral norm $\|\cdot\|_2$) yields

$$\text{(4)} \qquad \|f - s_n(f)\|_2 \le \|f - \sigma_n(f)\|_2 + \|\sigma_n(f) - s_n(f)\|_2 =: I_1 + I_2\,.$$

Let $\varepsilon > 0$ be given. By Cor. 1.7 and 1.8, Chap. V, f is bounded, say $\|f\|_\infty \le M_f$, and by Lemma 1 and 3 $\|\sigma_n(f)\|_\infty \le M_f$. Further, by Cor. 1.8, Chap. V, f has at most countably many points of discontinuity. Let the first such point be the center of an open interval with length $(\varepsilon/M_f)^2 2^{-1}$, the j-th be the center of an open interval with length $(\varepsilon/M_f)^2 2^{-j}$, $j \in \mathbb{N}$. Denote the union of all these intervals by E. Then, by the triangle inequality (w.r.t. $\|\cdot\|_2$-norm) and the notation $E^c := [-\pi,\pi] \setminus E$,

$$\text{(5)} \qquad I_1 \le 2M_f \Big(\sum_{j=1}^{\infty} (\varepsilon/M_f)^2 2^{-j}\Big)^{1/2} + \Big(\int_{E^c} \sup_{x\in E^c} |\sigma_n(f,x) - f(x)|^2 dx\Big)^{1/2}$$

By Part (b) of Fejér's Theorem, the last integrand becomes less than $\varepsilon/\sqrt{2\pi}$ for all $n \ge N_0$, hence $I_1 \le 2\varepsilon + \varepsilon,\ n \ge N_0$.
Since $\sigma_n(f)$ und $s_n(f)$ are trigonometric polynomials of degree less or equal n, we obtain by Theorem 2 (a)

$$\text{(6)} \qquad \|\sigma_n(f) - s_n(f)\|_2^2 = \Big\| \sum_{k=-n}^{n} \frac{|k|}{n+1}\, \widehat{f}_k\, e^{ikx} \Big\|_2^2 = \sum_{k=-n}^{n} \Big(\frac{|k|}{n+1}\Big)^2 |\widehat{f}_k|^2$$

for all $n \in \mathbb{N}$. Since $\sum_{k=-\infty}^{\infty} |\widehat{f}_k|^2$ converges by Bessel's inequality, choose $N_1 \in \mathbb{N}$ so large that

$$\sum_{N<|k|\le n} \Big(\frac{|k|}{n+1}\Big)^2 |\widehat{f}_k|^2 \le \sum_{|k|>N} |\widehat{f}_k|^2 < \varepsilon/2 \qquad \text{for all } n > N_1.$$

So $N_1 \in \mathbb{N}$ is fixed. Now choose $N_2 \in \mathbb{N}$, $N_2 > N_1$, so big that

$$\frac{|k|^2}{(n+1)^2} < \frac{\varepsilon}{2(\langle f, f\rangle + 1)} \qquad \text{for all } k,\ |k| \le N_1,\ \text{all } n \ge N_2\,.$$

In view of (6) the last two estimates and Bessel's inequality show that for all $n \geq \max(N_0, N_2)$ we have

$$I_2 \leq \frac{\varepsilon}{2(\langle f, f\rangle + 1)} \sum_{k=-N}^{N} |\widehat{f}_k|^2 + \sum_{N<|k|\leq n} \Big(\frac{|k|}{n+1}\Big)^2 |\widehat{f}_k|^2 < \varepsilon\,, \quad n \geq \max(N_0, N_2),$$

and, therefore, on account of (4), $\lim_{n\to\infty} \| f - s_n(f) \|_2 = 0$.

The final statement, i.e., $\sum_{k=-\infty}^{\infty} |\widehat{f}_k|^2 = \| f \|_2^2$, is an easy consequence of the first part and of Theorem 2 when $n \to \infty$. □

Application: $\displaystyle\sum_{j=0}^{\infty}(2j+1)^{-2} = \frac{\pi^2}{8}$; in particular $\displaystyle\sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6}$.

Proof. Consider

$$f(x) = \begin{cases} -1 & , -\pi \leq x < 0 \\ 1 & , 0 \leq x < \pi \end{cases} \qquad \Longrightarrow \qquad \widehat{f}_k = \frac{1}{\pi i k} \begin{cases} 0 & , k \text{ even} \\ 2 & , k \text{ odd}\,. \end{cases}$$

Theorem 3 yields

$$1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \sum_{k \text{ ungerade}} \frac{4}{\pi^2 k^2} = \sum_{j=0}^{\infty} \frac{8}{\pi^2 (2j+1)^2}$$

or equivalently $\sum_{j=0}^{\infty}(2j+1)^{-2} = \pi^2/8$. Defining

$$S := \sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} + \sum_{j=1}^{\infty} \frac{1}{(2j)^2} = \frac{\pi^2}{8} + \frac{1}{4} S\,,$$

this equation in S admits the unique solution $S = \pi^2/6$.

Lemma 4. *Let $f \in C_{2\pi}$ be piecewise continuously differentiable, i.e., f' has at most finitely many jump discontinuities. Then $\widehat{[f']}_k = ik\, \widehat{f}_k$ for all $k \in \mathbb{Z}$.*

Proof: First assume that f' has only one jump discontinuity, say at $x_0 \in [-\pi, \pi]$. Then, integration by parts yields

$$\begin{aligned} 2\pi\, \widehat{[f']}_k &= \Big(\int_{-\pi}^{x_0-} + \int_{x_0+}^{\pi} \Big) f'(x)\, e^{-ikx}\, dx \\ &= f(x) e^{-ikx}\Big|_{-\pi}^{x_0} + f(x) e^{-ikx}\Big|_{x_0}^{\pi} - \int_{-\pi}^{\pi} f(x)\, (-ik)\, e^{-ikx}\, dx\,. \end{aligned}$$

The boundary terms vanish since $f,\, e^{-ikx} \in C_{2\pi}$. This proves the assertion in the case of just one jump discontinuity of f'. If f' has finitely many jump discontinuities, the previous procedure may be repeated finitely many times. □

Theorem 5 *Let $f \in C_{2\pi}$ be piecewise continuously differentiable. Then the partial sums of the Fourier series of f converge in the supremum norm to f, i.e.,*

$$\lim_{n\to\infty} \|f - s_n(f)\,\|_\infty \equiv \lim_{n\to\infty} \sup_{x\in[-\pi,\pi]} \left| f(x) - \sum_{k=-n}^{n} \widehat{f}_k\, e^{ikx} \right| = 0\,.$$

Proof. Since f' is piecewise continuous, $\int_{-\pi}^{\pi} |f'(x)|^2\, dx < \infty$. Then by Theorem 3 and Lemma 4

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |\widehat{f}_k| &= |\widehat{f}_0| + \sum_{k\neq 0} \left| \frac{\widehat{[f']}_k}{ik} \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|\, dx + \Big(\sum_k |\,\widehat{[f']}_k|^2 \Big)^{1/2} \Big(\sum_{k\neq 0} k^{-2} \Big)^{1/2} < \infty; \end{aligned}$$

here we used the Cauchy-Schwarz inequality in the vector space

$$\ell^2(\mathbb{Z}) = \{(a_k)_k : \sum_{k=-\infty}^{\infty} |a_k|^2 < \infty\}$$

endowed with the scalar product

$$\langle (a_k)_k, (b_k)_k \rangle_{\ell^2} := \sum_{k=-\infty}^{\infty} a_k\, \overline{b_k}\,,$$

see the Appendix. Using the Weierstraß M-test, the sequence of partial sums $\Big(\sum_{|k|\leq n} \widehat{f}_k\, e^{ikx} \Big)_n$ is uniformly convergent to a continuous (!) function. Now the Uniqueness Theorem yields the assertion. □

D. Appendix

Lemma. *Let* X *be a vector space over the field* $\mathbb{C}$ *(or* $\mathbb{R}$*) with scalar product* $\langle\cdot,\cdot\rangle$*. Then the* **inequality of Cauchy-Schwarz**

$$|\langle f,g\rangle| \leq \langle f,f\rangle^{1/2}\langle g,g\rangle^{1/2}$$

holds for all $f,g\in X$*. Defining* $\|f\|_X := \langle f,f\rangle^{1/2}$*, the triangle inequality*

$$\|f+g\|_X \leq \|f\|_X + \|g\|_X$$

holds for all $f,g\in X$*; moreover,* $\|\cdot\|_X$ *is a* **norm** *on* X*.*

Proof. For $f=0$, the inequalities are obvious. Now let $f\neq 0$, set $\alpha=\langle f,g\rangle$ and choose $\lambda=-\overline{\alpha}/\langle f,f\rangle$. Then a simple calculation shows that

$$0 \leq \langle \lambda f+g,\lambda f+g\rangle = |\lambda|^2\|f\|_X^2 + \lambda\langle f,g\rangle + \overline{\lambda}\langle g,f\rangle + \|g\|_X^2 = \|g\|_X^2 - \frac{|\alpha|^2}{\|f\|_X^2}\,,$$

yielding the Cauchy-Schwarz inequality.
The triangle inequality may now be proved analogously to the triangle inequality on $\mathbb{C}$:

$$\begin{aligned} \|f+g\|_X^2 &= \langle f+g,f+g\rangle = \|f\|_X^2 + \langle f,g\rangle + \langle g,f\rangle + \|g\|_X^2 \\ &= \|f\|_X^2 + 2\,\mathsf{Re}\,(\langle f,g\rangle) + \|g\|_X^2 \leq \|f\|_X^2 + 2\|f\|_X\|g\|_X + \|g\|_X^2\,. \end{aligned}$$

□