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# 10th Tutorial Analysis II (engl.)
## Summer Semester 2010

**(T10.1)**

1. Show that for $(x, y)$ near $(1, 1)$ the equation $x^3 + y^2 - 2xy = 0$ may be solved uniquely with respect to $x$ and that the obtained function $x = \varphi(y)$ is continuously differentiable near $y = 1$. Compute $\varphi'(1)$.
2. Show that $\varphi$ is two times continuously differentiable near $y = 1$ and calculate $\varphi''(1)$.
3. Is the equation uniquely solvable with respect to $y$ near $(1, 1)$?

**(T10.2)**

1. Let $M_n(\mathbb{R})$ be the set of all $n \times n$-matrices with real entries. We identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$ and we consider the usual Euclidean norm.

   Prove that the function $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous. (This exercise is used in the proof of Lemma 1.5 Chap. VIII).

2. Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ be a differentiable function and $a, x \in U$ with $\overline{ax} \subseteq U$. For all $x_0 \in \mathbb{R}^n$ we view the matrix $Df(x_0)$ as a linear function from $\mathbb{R}^n \to \mathbb{R}^m$; by $\|Df(x_0)\|$ we mean the *operator norm* of the linear function $Df(x_0)$ (see the Remark 2.7 Chap. VI). Assume that

   $$\sup_{t \in [0,1]} \|Df(a + t(x - a))\| := L < \infty$$

   Then $\|f(x) - f(a)\| \leq L\|x - a\|$. (This is Lemma 1.4 Chap. VIII).

   *Hint.* Use the Mean Value Theorem (Theorem 2.8 Chap. VII). Recall though that this theorem is true only for functions which take real values.

**(T10.3)**

Let $f : [0,1] \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$ and $g : [0,1] \to \mathbb{R}$ be two continuously differentiable functions and assume that $|f_k'(t)| \leq g'(t)$ for all $k = 1, \ldots, n$ and all $t \in [0,1]$. Prove that

$$\|f(1) - f(0)\|_\infty \leq |g(1) - g(0)|.$$