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4th Tutorial Analysis II (engl.)

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(T4.1)

1. Let (X, d) be a metric space, and let $x \in X$ and $\epsilon > 0$. Recall that

$$U_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}.$$

For the purposes of this exercise we define

$$K_\epsilon(x) := \{y \in X : d(x, y) \leq \epsilon\}.$$

Give an example where $\overline{U_\epsilon(x)} \neq K_\epsilon(x)$.

Hint: Consider $X = (\mathbb{R} \setminus [\frac{1}{2}, \frac{3}{2}]) \cup \{1\}$ with the metric inherited from the natural metric on $\mathbb{R}$.

2. Let $n \in \mathbb{N}$ and let $(X_1, d_1), \ldots, (X_n, d_n)$ be nonempty metric spaces.

   i. Consider the product space $X = \Pi_{k=1}^n X_k$ with metric $d : X \times X \to \mathbb{R}$ defined by

   $$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{1 \leq k \leq n} d_k(x_k, y_k).$$

   Prove that this is really a metric on X.

   ii. For $1 \leq k \leq n$ we let the kth coordinate projection $\pi_k : X \to X_k$ be defined by $\pi_k(x_1, \ldots, x_n) = x_k$. Prove that coordinate projections are continuous.

(T4.2)

1. Let X be a metric space, $Y \subset X$ a subset regarded as a metric space with the induced metric. Show that a subset $V \subset Y$ is open relative to Y if and only if $V = U \cap Y$ for some open set $U \subset X$.

2. Let X be a metric space, and suppose that K and Y are subspaces of X with $K \subset Y \subset X$. Prove that K is compact relative to X if and only if K is compact relative to Y. Conclude that the property that K is compact is independent of the space in which K is embedded, and is an *intrinsic* property of K.

(T4.3) Let $a < b \in \mathbb{R}$ and consider the set $C^1[a,b]$ of continuously differentiable functions $f : [a,b] \to \mathbb{R}$. Prove that $C^1[-1,1]$ equipped with the supremum norm $\|\cdot\|_\infty$ is not a Banach space. (Hint: Consider the sequence $(f_n)_{n\in\mathbb{N}}$ given by $f_n(x) = \sqrt{x^2 + 1/n}$. Show that this sequence does not converge in $C^1[a,b]$.)