Fachbereich Mathematik
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13th Exercise Sheet Analysis II (engl.)
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(G13.1) (The Cantor Set)
We repeat the definition of the Cantor set in $[0,1]$. Set

$$\begin{aligned} C_0 &:= [0,1] \\ C_{n+1} &:= (\frac{1}{3}C_n) \cup (\frac{1}{3}C_n + \frac{2}{3}). \end{aligned}$$

for $n \geq 1$; where $\frac{1}{3}C_n := \{\frac{1}{3}x : x \in C_n\}$ and $\frac{1}{3}C_n + \frac{2}{3} := \{\frac{1}{3}x + \frac{2}{3} : x \in C_n\}$. The Cantor set is defined as

$$C := \bigcap_{n=0}^{\infty} C_n.$$

Decide whether this set is a Lebesgue null set or a Jordan null set, i.e. a set of Lebesgue measure zero or a set of Jordan measure zero.

(G13.2) (Fubini's Theorem)

(a) Let $f : [a,b] \to \mathbb{R}$, $g : [c,d] \to \mathbb{R}$ be continuous functions, and $R = [a,b] \times [c,d]$. Show that

$$\int_R [f(x)g(y)]\, d(x,y) = \left(\int_a^b f(x)dx\right) \cdot \left(\int_c^d g(y)dy\right).$$

(b) Compute the following integrals:

$I_1 = \displaystyle\int_R (x \sin y - ye^x)\, d(x,y)$, where $R = [-1,1] \times [0, \pi/2]$;

$I_2 = \displaystyle\int_R \frac{x}{1+xy}\, d(x,y)$, where $R = [0,1] \times [0,1]$;

$I_3 = \displaystyle\int_R \frac{x^2 z^3}{1+y^2}\, d(x,y,z)$, where $R = [0,1] \times [0,1] \times [0,1]$.

(G13.3)

Define the function $f : [0,1] \to \mathbb{R}$ as follows: $f(x) = \frac{1}{n}$ if $\frac{1}{n+1} < x \leq \frac{1}{n}$, $(n = 1, 2, \ldots)$ and $f(0) = 0$. Prove that the function f is Riemann integrable on $[0,1]$ and compute the integral $\int_0^1 f(x)dx$.

Hint. Use Main Theorem 7.8 of the Chapter "The Riemann Integral on Rectangles". For the computation of the integral one may define $I_n := \int_{\frac{1}{n+1}}^1 f(x)dx$ for all $n \in \mathbb{N}$ and prove that $|I_n - \int_0^1 f(x)dx| \overset{n\to\infty}{\longrightarrow} 0$.