Fachbereich Mathematik
Prof. Dr. W. Trebels
Dr. V. Gregoriades
Dr. A. Linshaw

TECHNISCHE UNIVERSITÄT DARMSTADT

01-07-2010

12th Exercise Sheet Analysis II (engl.)
Summer Semester 2010

(G12.1)

1. Let $A \subset \mathbb{R}^2$ be the set of points on the ellipse $x^2 + 4y^2 = 1$ for which $x \geq 0$ or $y \geq 0$. Describe A as the image of a path $\gamma : I \to \mathbb{R}^2$, where I is an appropriate interval.

2. Compute the arc length of the path

$$\gamma : [1, 2] \to \mathbb{R}^2, \gamma(t) = \left(t^3, \frac{3}{2} t^2 \right)$$

3. Let $x, y \in \mathbb{R}^n$ and

$$\gamma : [0, 1] \to \mathbb{R}^n, \quad \gamma(t) = (1 - t)x + ty$$

be the *affine path joining x and y.* Prove that γ is rectifiable and that $L_\gamma = \|x - y\|_2$.

(G12.2) (Functions of bounded variation)

Let $a < b \in \mathbb{R}$ and define $BV[a, b]$ to be the set of all functions $f : [a, b] \to \mathbb{R}$ of bounded variation. Prove the following:

1. If $f \in BV[a, b]$, then f is bounded and

$$|f(a) - f(b)| \leq V_a^b(f).$$

2. If $f, g \in BV[a, b]$ and $\lambda, \mu \in \mathbb{R}$, then $\lambda f + \mu g, f \cdot g \in BV[a, b]$ and

$$\begin{aligned} V_a^b(\lambda f + \mu g) &\leq |\lambda| V_a^b(f) + |\mu| V_a^b(g), \\ V_a^b(f \cdot g) &\leq \|f\|_\infty V_a^b(g) + \|g\|_\infty V_a^b(f). \end{aligned}$$

3. If f is monotone, then f is of bounded variation and

$$V_a^b(f) = |f(b) - f(a)|.$$

4. Any continuously differentiable function $f : [a, b] \to \mathbb{R}$ is of bounded variation and $V_a^b(f) = \int_a^b |f'(t)|\, dt$.

(G12.3) (Arc length in polar coordinates)

Consider the following path

$$\gamma : [\alpha, \beta] \to \mathbb{R}^2,\ \ \gamma(\varphi) = r(\varphi) \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix},$$

where $r : [\alpha, \beta] \to [0, \infty)$ is continuously differentiable.

1. Prove that for any r, γ is rectifiable and its arc length is

$$L_\gamma = \int_\alpha^\beta \sqrt{(r(\varphi))^2 + (r'(\varphi))^2}\, d\varphi.$$

2. Let $r : [-\pi, \pi] \to [0, \infty),\ \ r(\varphi) = 1 + \cos\varphi$. For this choice of r, prove that γ is closed (that is, $\gamma(-\pi) = \gamma(\pi)$) and that γ is injective on $[-\pi, \pi)$. (This means that γ is a *Jordan curve*).

3. Let γ be as above, and compute the arc length of γ.