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10th Exercise Sheet Analysis II (engl.)

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(G10.1)

Consider the set $A := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 1 = 0\}$ and $(x_0, y_0) \in A$. Give conditions for the pair (x_0, y_0) under which there is an open set W with $x_0 \in W$ and a differentiable function $\varphi : W \to \mathbb{R}$ such that $\varphi(x_0) = y_0$ and $(x, \varphi(x)) \in A$ for all $x \in W$. Show that for any suitable pair (x_0, y_0) the corresponding open set W can be chosen to be the interval $(-1, 1)$ but it cannot be chosen so that $(-1, 1) \subsetneqq W$ (i.e., the segment $(-1, 1)$ is a proper subset of W). Also show that for different values of y_0 we have different functions $\varphi : (-1, 1) \to \mathbb{R}$.

(G10.2)

Define the function $F(x, y, z) = x^3 z^2 - z^3 yx$ for $(x, y, z) \in \mathbb{R}^3$.

1. Prove that there is an open set $W \subseteq \mathbb{R}^2$ such that $(1, 1) \in W$ and a differentiable function $f : W \to \mathbb{R}$ such that $f(1, 1) = 1$ and $F(x, y, f(x, y)) = 0$ for all $(x, y) \in W$. Compute the partial derivatives $\dfrac{\partial f}{\partial x}(1, 1)$ and $\dfrac{\partial f}{\partial y}(1, 1)$.

2. Prove that there is a differentiable function $g : \mathbb{R} \to \mathbb{R}^2 : g = (g_1, g_2)$ such that $F(x, g_1(x), g_2(x)) = 0$ for all $x \in \mathbb{R}$. Is this function g unique?

(G10.3)

(Simple Zeros of Polynomials)

Suppose that $a = (a_0, \ldots, a_n)$, $p : \mathbb{R} \to \mathbb{R}$, $\quad p(x) = \sum_{k=0}^{n} a_k\, x^k$ is a polynomial of degree $n \geq 1$ and $x_0 \in \mathbb{R}$ is a simple zero of p.
Given $b = (b_0, b_1, \ldots, b_n) \in \mathbb{R}^{n+1}$ we define

$$p_b : \mathbb{R} \to \mathbb{R}, \qquad p_b(x) := \sum_{k=0}^{n} b_k\, x^k.$$

1. Show that for b sufficiently near a the polynomial p_b has a unique simple zero $\varphi(b)$ near the simple zero x_0 of p_a. Show that the function φ defined in this way is continuously differentiable near a.

2. Prove the following: If p possesses exactly n different zeros, then the polynomials p_b with b sufficiently near a also have n different zeros.