Fachbereich Mathematik
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9th Exercise Sheet Analysis II (engl.)
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(G9.1)
Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) \quad = \quad \begin{cases} \dfrac{xy^3}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

(a) Prove that the function $I : \mathbb{R} \to \mathbb{R}$, $I(y) = \displaystyle\int_0^1 f(x,y)\,dx$ is differentiable and calculate I'.

(b) Define $I^* : \mathbb{R} \to \mathbb{R}$, $I^*(y) = \displaystyle\int_0^1 \frac{\partial f}{\partial y}(x,y)\,dx$ and calculate $I^*(0)$.

(c) Remark that $I'(0) \neq I^*(0)$. Compare this with Theorem VII.6.1, and give an explanation.

(G9.2)

Let $U \subseteq \mathbb{R}$ be open, let $J \subseteq \mathbb{R}$ be a compact interval, and let $f : J \times U \to \mathbb{R}$ be continuous and continuously partial differentiable with respect to the second variable. Let $\alpha, \beta : U \to \mathbb{R}$ be differentiable on U and suppose that $\alpha(U), \beta(U) \subseteq J$. Let

$$I : U \to \mathbb{R}, \quad I(y) = \int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx.$$

Prove that I is differentiable on U and for all $y \in U$,

$$I'(y) = \int_{\alpha(y)}^{\beta(y)} \partial_2 f(x,y)\,dx + \beta'(y) f(\beta(y),y) - \alpha'(y) f(\alpha(y),y).$$

Hint. Consider the function $\Phi(u, v, y) = \displaystyle\int_u^v f(x, y)\, dx$ with $u, v \in J$ and $y \in U$, remark that $I(y) = \Phi(\alpha(y), \beta(y), y)$ and differentiate I.

(G9.3)

We say that two open sets $U, V \subseteq \mathbb{R}^n$ are *diffeomorphic* if there exists a diffeomorphism $f : U \to V$; (see Definition 1.1 Chap. VIII).

1. Prove that every two non-empty open intervals I and J of $\mathbb{R}$ are diffeomorphic.
2. Prove that $\mathbb{R}$ is differomorphic with $(0, 1)$.
3. Is $\mathbb{R}$ diffeomorphic with $(0, 1) \cup (1, 2)$?
4. Define $F : \mathbb{R}^2 \to \mathbb{R} : F(x, y) = y^2 + y + 3x + 1$. In which points $x \in \mathbb{R}$ can we solve the equation $F(x, y) = 0$ with respect to y as a differentiable function of x using only the definition?