Fachbereich Mathematik
Prof. Dr. W. Trebels
Dr. V. Gregoriades
Dr. A. Linshaw

TECHNISCHE
UNIVERSITÄT
DARMSTADT

21-05-2010

7th Exercise Sheet Analysis II (engl.)

Summer Semester 2010

(G7.1) Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = \cos x \sin y \exp(z)$. Compute the Taylor polynomial of order 3 of f at the point $(0, 0, 0)$ in two ways:

(i) using Taylor's Theorem VII.4.1;

(ii) multiplying the Taylor polynomials for $\exp(z), \cos x, \sin y$.

(G7.2) Let $P : \mathbb{R}^n \to \mathbb{R}$ be the following homogeneous polynomial of degree k:

$$P(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha,$$

where $c_\alpha \in \mathbb{R}$, $\alpha \in \mathbb{N}_0^n$. Prove the following:

(i) For any multi-index $\beta \in \mathbb{N}_0^n$ with $|\beta| = k$,

$$D^\beta P(x) = \beta!\, c_\beta.$$

(ii) If $P(x) = 0$ for all x in a neighborhood U of 0, then $c_\alpha = 0$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$.

(iii) $\lim\limits_{x\to 0} \dfrac{P(x)}{\|x\|^m} = 0$ for all $m < k$.

(iv) If $\lim\limits_{x\to 0} \dfrac{P(x)}{\|x\|^k} = 0$, then $P(x) = 0$ for all $x \in \mathbb{R}^n$.

(G7.3) Let U be an open subset of $\mathbb{R}^n$ and $C_b^k(U)$ be the set of all $f \in C^k(U, \mathbb{R})$ with the property that $D^\alpha f$ is bounded on U for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$. For every $f \in C_b^k(U)$, we define

$$\|f\|_k := \sum_{|\alpha|\le k} \frac{1}{\alpha!} \sup\{|D^\alpha f(x)| : x \in U\}.$$

Prove the following:

(i.) The mapping

$$\| \cdot \|_k : C_b^k(U) \to \mathbb{R}, \quad f \mapsto \|f\|_k$$

is a norm on the real vector space $C_b^k(U)$.

(ii.) For any $f, g \in C_b^k(U)$,

$$\|fg\|_k \leq \|f\|_k \|g\|_k.$$

Hint: for (ii) you will need to use the Leibniz Formula from (T7.3).