Fachbereich Mathematik
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5th Exercise Sheet Analysis II (engl.)

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(G5.1)

1. Prove that a set $M \subseteq \mathbb{R}$ with at least two elements is connected if and only if M is an interval; this is Example 4.2 Chap. VI. (You may use that every path-connected metric space is connected).

2. Prove without using the statement of Theorem 1.8 Chap. VII that if $f = (f_1, \dots, f_m) : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x_0 \in U$ then the linear map A from Definition 1.1 Chap. VII is uniquely defined. (This is Remark 1.2 Chap VII).

(G5.2)

1. Compute the derivative of the function $f : \mathbb{R}^3 \to \mathbb{R}^2 : f(x, y, z) = (x \cdot y \cdot z, e^{(y+z)})$.

2. Suppose that we are given a function $f : \mathbb{R}^2 \to \mathbb{R}$ which satisfies $f(x, y) = \log(\sqrt{x^2 + y^2})$ for $(x, y) \neq (0, 0)$. Compute the directional derivative of f at $(1, 0)$ in the direction $u = (2/\sqrt{5}, 1/\sqrt{5})$.

3. Define the function $f : \mathbb{R}^2 \to \mathbb{R} : f(x, y) = \dfrac{xy^2}{x^2 + y^4}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Prove that the directional derivative $D_u f(0, 0)$ exists for all $u \in \mathbb{R}^2$ with $\|u\|_2 = 1$ but f is not continuous at $(0, 0)$. (This is the exercise given in the Script before Theorem 1.11 Chap. VII).

(G5.3)

We are familiar with the mean value theorem in one variable:

Let $f : [a, b] \to \mathbb{R}$ be continuous and let f be differentiable on (a, b). Then there is a $c \in (a, b)$ with

$$f(b) - f(a) = f'(c)(b - a).$$

Decide whether this theorem continues to hold for functions $f : [a, b] \to \mathbb{R}^m$ for arbitrary $m \in \mathbb{N}$. Give a proof or a counterexample.