Fachbereich Mathematik
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4th Exercise Sheet Analysis II (engl.)
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(G4.1) Consider the metric spaces $([0,\infty), d_1)$, where d_1 is the discrete metric, and $([0,\infty), d_2)$, where $d_2(x,y) = |y-x|$. Let $f : [0,\infty) \to [0,\infty)$ be the function $f(x) = x$. Prove that $[0,\infty)$ is bounded in $([0,\infty), d_1)$, that f is uniformly continuous when considered as a function between the metric spaces $([0,\infty), d_1)$ and $([0,\infty), d_2)$, and that $f([0,\infty))$ is unbounded in $([0,\infty), d_2)$. Conclude that in general uniformly continuous functions do not preserve boundedness.

Consider the inverse function $f^{-1} : [0,\infty) \to [0,\infty)$, $f^{-1}(x) = x$. Is f^{-1} continuous when considered as a function between the metric spaces $([0,\infty), d_2)$ and $([0,\infty), d_1)$?

(G4.2)

For $0 < p < 1$, define ℓ^p to be the set of sequences $x = (x_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} |x_n|^p$ converges.

1. Prove that for real numbers $a, b \geq 0$ and $0 < p < 1$, we have $(a+b)^p \leq a^p + b^p$.
2. Show that the function $d_p(x,y) = \sum_{n=1}^{\infty} |x_n - y_n|^p$ defines a metric on ℓ^p.
3. Define $\|x\|_p = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{1/p}$. Does this define a norm on ℓ^p?

(G4.3) Consider the vector space $(C[0,1], \|\cdot\|_\infty)$ of continuous functions $f : [0,1] \to \mathbb{R}$ equipped with the supremum norm. For $n \in \mathbb{N}$ let $P_n \subset C[0,1]$ be the finite dimensional linear subspace of polynomials of degree less than or equal to n. We then call $p_b \in P_n$ a polynomial of best approximation to $f \in C[0,1]$ if

$$\|f - p_b\|_\infty = \inf\{\|f - p\|_\infty : p \in P_n\}.$$

1. Prove that any polynomial $p_b \in P_n$ of best approximation to f satisfies $\|p_b\|_\infty \leq 2\|f\|_\infty$.
2. Prove that each $f \in C[0,1]$ possesses a polynomial $p_b \in P_n$ of best approximation.