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22-04-2010

2nd Exercise Sheet Analysis II (engl.)
Summer Semester 2010

(G2.1)

Examine whether the following integrals exist.

1. $I = \displaystyle\int_0^\infty \frac{x}{\sqrt{1+x^3}}dx.$
2. $B(p,q) = \displaystyle\int_0^1 t^{p-1} \cdot (1-t)^{q-1} dt$; where $p, q \in \mathbb{C}$ with $Re(p), Re(q) > 0$ (see Example 4.10 Chap. 5).

(G2.2)

1. Suppose that we are given a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(x) = |x|$ for all $x \in [-\pi, \pi]$. Compute the Fourier series of f.
2. **(Fourier coefficients of a shifted function)**
   Let $f : \mathbb{R} \to \mathbb{C}$ be Riemann integrable on $[-\pi, \pi]$ and 2π-periodic. Define for any $h \in \mathbb{R}$ the *shifted function* $\tau_h f : \mathbb{R} \to \mathbb{C}, \quad \tau_h f(x) = f(x+h)$. Prove that the Fourier coefficients $\widehat{\tau_h f}_k$ of $\tau_h f$ are given by: $\widehat{\tau_h f}_k = e^{ikh} \hat{f}_k$ for every $k \in \mathbb{Z}$.

(G2.3)

We give the following definition. Suppose that A, X are sets such that $A \subseteq X$. We define the function $\chi_A : X \to \{0,1\} : \chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. The function χ_A is *the characteristic function of* A.

For all naturals $n \geq 2$ we define the interval $A_n := [\sum_{k=1}^{n-1} \frac{1}{k}, \sum_{k=1}^{n} \frac{1}{k})$ and the function $f : [1, \infty) \to \mathbb{R} : f(x) = \sum_{n=1}^{\infty} (-1)^n \cdot \chi_{A_n}(x)$. Prove that (a) for all $x > 1$ there is some n_0 such that $f(t) = \sum_{k=1}^{n_0} (-1)^k \cdot \chi_{A_k}(t)$ for all $t \in [1, x]$ and so the series $\sum_{n=1}^{\infty} (-1)^n \cdot \chi_{A_n}$ converges to f uniformly on each fixed interval $[1, x]$; (b) the function f is jump continuous and integrable on $[1, \infty)$ and (c) the function f is not absolutely integrable on $[1, \infty)$.

Below we give the graph of f.