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Introduction to Mathematical Logic

SS 2010, Exercise Sheet #13

Recall from the lecture the following lemma of Vaught:

> *Let K be an infinite set and T a consistent theory subpotent to K such that all models of T are infinite and any two models of T equipotent to K are isomorphic. Then T is complete.*

EXERCISE 44:
A linear order "$<$" on a set X is dense if, to any $x < y$, there are u, v, w with $u < x < v < y < w$.
One can show that any two countable dense linear orders are isomorphic.

a) Define the theory DLO of dense linear orders. Are your axioms recursively enumerable? Explain!

b) Given an example of a model of DLO. Are there countable models of DLO? Finite models? Why?

c) Conclude that DLO is complete!

Recall Gödel's Incompleteness Theorem. In 1948, Julia Robinson presented a formula $\varphi[X]$ in the language of fields such that $\mathbb{Q} \models \varphi[i_x]$ for every $x \in \mathbb{N}$ and $\mathbb{Q} \models \neg\varphi[i_x]$ for every $x \in \mathbb{Q} \setminus \mathbb{N}$.

d) Prove or disprove: Th($\mathbb{Q}$) is complete.

e)* Prove or disprove: $\mathbb{Q}$ admits a recursively enumerable axiomatization in the language of fields.

EXERCISE 45:

a) Suppose that every set can be well-ordered; i.e. to every X, there is a set $R \subseteq X \times X$ which constitutes a well-ordering relation. Conclude that this implies the Axiom of Choice.

b) Let X, Y denote metric spaces† and $f : X \to Y$ an arbitrary function.
Show that, for every $n \in \mathbb{N}$, the following is an open subset of X:

$$\big\{x \in X \;\big|\; \exists m \in \mathbb{N}\ \forall y, z \in X : \big(d(x, y) < 1/m \wedge d(x, z) < 1/m\big) \Rightarrow d\big(f(y), f(z)\big) < 1/n\big\} \ .$$

c) Conclude that the set $\{x \in X : f \text{ continuous at } x\}$ is in Borel class $\mathbf{\Pi}_2$, i.e. a G_δ subset of X.
Give an example of a function $f : X \to Y$ where this set is not open nor closed.

d) Recall the non-commutativity of ordinal arithmetic and simplify the following ordinal expressions: $1 + \omega + \omega^2 + \omega^3 + \ldots + \omega^n + 1$, $\quad 1 + \omega + \omega^2 + \omega^3 + \ldots + \omega^\omega + 1$. What could $\omega - 1$ mean?

EXERCISE 46:

a) For fixed n, determine the number (i.e. the cardinality of the set) of functions $f : \{0, 1\}^n \to \{0, 1\}$.

b) For $x_1, \ldots, x_n \in \{0, 1\}$, consider the formula $\varphi_{x_1,\ldots,x_n}[X_1, \ldots, X_n] \ := \ \bigwedge_{i=1}^{n} \begin{cases} X_i & : x_i = 1 \\ \neg X_i & : x_i = 0 \end{cases}$
Show that $\varphi_{x_1,\ldots,x_n}[X_1, \ldots, X_n]$ is a formula in the language of propositional logic with variables $X_1, \ldots, X_n$ such that, for every valuation $v : \{X_1, \ldots, X_n\} \to \{0, 1\}$, it holds:
$v \models \varphi_{x_1,\ldots,x_n}$ iff $v(X_i) = x_i$ for all $i = 1, \ldots, n$.

c) Prove that, to every function $f : \{0, 1\}^n \to \{0, 1\}$, $\psi := \bigvee_{\bar{x} \in f^{-1}[1]} \varphi_{\bar{x}}$ is a formula in the language of propositional logic such that, for every valuation v, it holds: $v \models \psi$ iff $f\big(v(X_1), \ldots, v(X_n)\big) = 1$.

d) Prove that the number of formulas of length ℓ with n variables in the language of propositional logic is at most $(n + c)^\ell$ for some constant c.

e)* Verify that ψ from c) has length at most C^n for some constant C.
Show that (for n sufficiently large) there are functions $f : \{0, 1\}^n \to \{0, 1\}$ which cannot be expressed as formulas of length polynomial in n; and that in fact 'most' functions require formulas of length at least c^n for some constant c.

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*Bonus exercise   †you may assume $X = Y = \mathbb{R}$ with metric $d(x, y) = |x - y|$ if you like