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Introduction to Mathematical Logic

SS 2010, Exercise Sheet #12

EXERCISE 40:

a) Specify a consistent[†] first-order theory which is not complete.

b) Clearify the relation between completeness and the claim made by the completeness theorem.

EXERCISE 41:

a) Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ denote a finite set of first-order sentences of the theory of fields.
Prove: If $\mathcal{A}$ is valid in all fields of characteristic zero,
then there exists an integer P such that $\mathcal{A}$ is valid in all fields of characteristic $p \geq P$.

b) Prove: The theory of fields of characteristic 0 is not finitely axiomatizable.

EXERCISE 42:

Let T denote a first-order theory whose language has binary relation symbol "$<$" as only nonlogical symbol.

a) Suppose that *every* well-ordered set is a model of T. Consider the extended theory T' with additional constant symbols $c_1, c_2, \ldots, c_n, \ldots$ and axioms "$c_{n+1} < c_n$" for every integer n.
Show that each finitely axiomatized part of T' has a model.

b) Prove that T cannot have *precisely* the well-ordered sets as models.

EXERCISE 43:

a) Let φ be a sentence in the language of fields. Prove that the following are equivalent:

   i) φ is valid in the field $\mathbb{C}$ of complex numbers.
   ii) φ is valid in all algebraically closed fields of characteristic 0.
   iii) φ is valid in some algebraically closed field of characteristic 0.
   iv) There is some integer P such that, for all primes $p \geq P$,
   φ is valid in some algebraically closed field of characteristic p.
   v) There is some $P \in \mathbb{N}$ such that, for all primes $p \geq P$,
   φ is valid in all algebraically closed fields of characteristic p.

b*) Informally describe a computer algorithm which, given a system of multivariate polynomial equations "$p_j(z_1, \ldots, z_n) = 0$" and inequalities "$q_\ell(z_1, \ldots, z_n) \neq 0$" with integer coefficients, reports whether this system has some solution $(z_1, \ldots, z_n) \in \mathbb{C}^n$ or not.

[†] not necessarily provably
[*] Bonus exercise