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Issued 2010-06-25
To be submitted on 2010-07-05

Introduction to Mathematical Logic

SS 2010, Exercise Sheet #11

EXERCISE 37:

a) Prove that $A \Rightarrow (B \Rightarrow C)$ is a tautological consequence of $(A \Rightarrow B) \Rightarrow C$ but not vice versa.

b) Let $B, A_1, \ldots, A_n$ be formulas. Prove:
B is a tautological consequence of $\{A_1, \ldots, A_n\}$ iff "$\bigwedge_i A_i \Rightarrow B$" is a tautology.

c) Let A, B, C denote formulas and $\mathcal{A}$ a set of formulas. Prove:

i) If $\mathcal{A} \models A$, then $\mathcal{A} \models B \vee A$

ii) If $\mathcal{A} \models A \vee A$, then $\mathcal{A} \models A$

iii) If $\mathcal{A} \models A \vee (B \vee C)$, then $\mathcal{A} \models (A \vee B) \vee C$

iv) If $\mathcal{A} \models A \vee B$ and $\mathcal{A} \models \neg A \vee C$, then $\mathcal{A} \models B \vee C$

v) $\models \neg A \vee A$

vi) If $\mathcal{A} \models A$ and $\mathcal{A} \models A \Rightarrow B$, then $\mathcal{A} \models B$.

EXERCISE 38:
Let $G = (V, E)$ denote a graph on a (not necessarily finite) vertex set V and E a set of unordered edges $\{u, v\}$, $u, v \in V$. A subgraph of G is a graph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$. For any natural number k, we say that G is k-colorable if its vertices can be painted with k colours such that any two adjacent ones receive different colours:

$$\exists c : V \to \{1, 2, \ldots, k\} \qquad \text{such that} \qquad \{u, v\} \in E \Rightarrow c(u) \neq c(v) \quad .$$

a) Prove that the graph on the back side is 4-colourable but not 3-colourable.

b) Let $G = (V, E)$ denote a graph, k an integer, and consider the set of variables $\{x_{u,i} : u \in V, 1 \leq i \leq k\}$. Furthermore consider the set Φ consisting of the following formulas:

$$\bigvee_{i=1}^{k} x_{u,i} : u \in V; \quad \neg x_{u,i} \vee \neg x_{u,j} : u \in V, 1 \leq i < j \leq k; \quad \neg x_{u,i} \vee \neg x_{w,i} : \{u, w\} \in E, 1 \leq i \leq k$$

Show that G is k-colourable iff Φ is satisfiable.

c) Prove that G is k-colourable iff each of its finite subgraphs is k-colourable.

EXERCISE 39:
Let $\mathcal{A}$ denote a set of formulas and A, B, C formulas. Prove:

b) If $\mathcal{A} \vdash A$, then $\mathcal{A} \vdash \neg\neg A$. Conversely, if $\mathcal{A} \vdash \neg\neg A$ then $\mathcal{A} \vdash A$.

c) If $\mathcal{A} \vdash A$ and $\mathcal{A} \vdash A \Rightarrow B$, then $\mathcal{A} \vdash B$. (Modus ponens)

d) If $\mathcal{A} \vdash (A \vee B) \vee C$, then $\mathcal{A} \vdash A \vee (B \vee C)$.

e) If $\mathcal{A} \vdash \neg A \vee C$ and $\mathcal{A} \vdash \neg B \vee C$, then $\mathcal{A} \vdash \neg(A \vee B) \vee C$.