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Introduction to Mathematical Logic

SS 2010, Exercise Sheet #10

EXERCISE 34:
Let L denote a language and $\mathcal{M}$ a structure of L. Prove the following:

a) A sentence "$A \wedge B$" is valid in $\mathcal{M}$ iff both "A" and "B" are valid in $\mathcal{M}$.

b) A sentence "$\forall v A[v]$" is valid in $\mathcal{M}$ iff for each $a \in M$, the sentence "$A[i_a]$" of $L_{\mathcal{M}}$ is valid in $\mathcal{M}$.

c) A sentence "$A \Leftrightarrow B$" is valid in $\mathcal{M}$ iff either both "A" and "B" are valid in $\mathcal{M}$ or none is valid in $\mathcal{M}$.

d) Let $A[v_1, \ldots, v_n]$ be a formula and $t_1, \ldots, t_n$ variable-free terms of L.
Prove that the following two formulas are valid in $\mathcal{M}$:

$$(\forall v_1 \forall v_2 \cdots \forall v_n : A) \Rightarrow A[t_1, \ldots, t_n] \qquad A[t_1, \ldots, t_n] \Rightarrow \exists v_1 \exists v_2 \cdots \exists v_n : A$$

EXERCISE 35:
A field $\mathbb{K}$ is algebraically closed if every nonconstant polynomial over $\mathbb{K}$ has a root in $\mathbb{K}$.
The characteristic of a field $\mathbb{K}$ is the least integer $p \geq 2$ such that $1 + 1 + \cdots + 1 = 0$ (p-fold).
If such p exists, it must be a prime; otherwise $\mathbb{K}$ is said to be of characteristic 0.

a) Prove *Freshman Exponentiation*: $(x + y)^p = x^p + y^p$ in fields of characteristic $p \geq 2$.

b) Prove that every algebraically closed field is infinite, as is every field of characteristic 0.

c) Define a first-order theory whose models are precisely the fields.

d) Fix p. Define a first-order theory $\mathsf{F}(p)$ whose models are precisely the fields of characteristic p.

e) Define a first-order theory $\mathsf{F}(0)$ whose models are precisely the fields of characteristic 0.

f) Define a first-order theory ACF whose models are precisely the algebraically closed fields.

g) Show that there exists an uncountable and a countable model of ACF. (Hint: Löwenheim-Skolem)

EXERCISE 36:
An ordered field is a field $\mathbb{K}$ equipped with an additional order relation "$<$" that complies with the field operations and constants.
A complete ordered is an ordered field $\mathbb{K}$ such that every nonempty subset of $\mathbb{K}$ which has an lower bound also has a greatest lower bound; recall Section 1.1 from the lecture.

a) Define a first-order theory whose models are precisely the ordered fields.
Give an example of a complete ordered field.

b)† Let $\mathbb{K}$ denote a complete ordered field. Prove that there exists an injective homomorphism of ordered fields $h : \mathbb{R} \to \mathbb{K}$, i.e. such that $h(0) = 0$, $h(1) = 1$, $h(x + y) = h(x) + h(y)$, $h(x \cdot y) = h(x) \cdot h(y)$, and $h(x) < h(y)$ whenever $x < y$.

c) Can you define a first-order theory whose models are precisely the complete ordered fields? Prove!

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*Recall that the lecture and exercise session intended for June 21 and 22 have moved.

†Bonus exercise