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Introduction to Mathematical Logic

SS 2010, Exercise Sheet #9

EXERCISE 32:
Consider the following claim:

> *Every infinite (in the sense of having infinitely many nodes), finitely-branching (i.e. s.t. each node has only finitely many children) tree contains an infinite branch.*

a) Prove that claim!
Hint: Recall the proof of Bolzano-Weierstraß.
You might first try to prove the case of infinite *binary* trees.

b) Consider the axiom of countable choice:

> *The cartesian product of a sequence of non-empty sets is non-empty.*

Let X be a countable set and $R \subseteq X \times X$ such that $\forall x \exists y : (x, y) \in R$. Use the axiom of countable choice to prove that there is a mapping $f : \mathbb{N} \to X$ such that $\big(f(n), f(n+1)\big) \in R$.

c) Now re-prove a) using only Zermelo-Fraenkel and b).

d)* Let $q \in \mathbb{N}$ and X_n subpotent to $q - 1$ for every $n \in \omega$; now consider the set $C := \prod_{n\in\omega} X_n$. Prove that

$$d : C \times C \to \mathbb{R}, \qquad d(\bar{x}, \bar{y}) := 2^{-\min\{n : x_n \neq y_n\}}$$

is a metric on C and renders $C \ni \bar{x} \mapsto \sum_n x_n q^{-n} \in \mathbb{R}$ continuous.

e) Prove that every sequence in C has a convergent subsequence.

EXERCISE 33:

a) Show that the set of all expressions of a countable language is countable.

b) In the language for the theory of rings, consider the following term/formula and list all its subterms/subformulae:

$$x \cdot y \cdot z \ + \ x \cdot z \ + \ 1 \qquad\qquad y = z \ \vee \ \exists y \forall z : (x \cdot y \cdot z = z \wedge z \cdot x \cdot y = z) \ .$$

In the above formula, which variables occur freely, which boundedly?

c) Recall the language of set theory (Example 5.4a) and list all atomic formulas of this language.

d) Recall the Definition 1.1 of a total order.
Now specify the theory of totally ordered sets according to Definition 5.9.

e)* In Example 5.12 (linear mappings as interpretation of the language of rings) and in combination with Example 5.6a), identify (and describe the philosophical differences between) three distinct conceptions of an 'integer'.

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*Bonus exercise

†Please indicate your preference of possible dates for the written examination at `http://www.doodle.com/7f5dh84cuswyxqy9`