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Introduction to Mathematical Logic

SS 2010, Exercise Sheet #7

EXERCISE 26:
In topology, a G_δ-set (in X) is one of the form $\bigcap_{n\in\omega} Y_n$ with $Y_n \subseteq X$ open for every $n \in \omega$.

a) Which Borel class ($\boldsymbol{\Sigma}_\alpha$ or $\boldsymbol{\Pi}_\alpha$ and for which α) do G_δ-sets belong to?

b) Prove that the Borel class you specified in a) is in general the least possible to contain G_δ-sets.

c) Fix $k \in \mathbb{N}$.
Generalize a+b) to describe all sets in $\boldsymbol{\Sigma}_k$ (all sets in $\boldsymbol{\Pi}_k$) in terms of open or closed sets.

d)* Let α denote some *un*countable ordinal. What is $\boldsymbol{\Sigma}_\alpha$?

e) Prove that $\boldsymbol{\Sigma}_\alpha$ is closed under countable unions.
Which countable operation is $\boldsymbol{\Pi}_\alpha$ closed under?

EXERCISE 27:
Consider the following subset C of $\mathbb{R}$:

$$\Big\{ \sum_{n=1}^{\infty} a_n 3^{-n} : a_n \in \{0, 2\} \Big\} \ .$$

a) What is the cardinality of C ?

b) Which (least) Borel class does C belong to?

c)* Prove your claim from b).

EXERCISE 28:

b) For every x, y, we define the ordered pair $(x, y) := \big\{\{x\}, \{x, y\}\big\}$.
Prove that $(x, y) = (x', y') \Leftrightarrow x = x' \wedge y = y'$ holds.

c) Inductively extend the definition, claim, and proof in b) from pairs to n-tuples.

d) What does n range over in c)? Which axioms does your proof rely on?

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*Bonus exercise