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Issued 2010-05-17
To be submitted on 2010-05-25

Introduction to Mathematical Logic

SS 2010, Exercise Sheet #6

EXERCISE 24:

a) Construct a subset of $\mathbb{R}$ which is not a countable union of closed sets, i.e. not in $\mathbf{\Sigma}_2$.

b)* Prove your claim from a).

c) Determine the 'ground level' $\mathbf{\Sigma}_1 \cap \mathbf{\Pi}_1$ of the Borel Hierarchy of subsets of $\mathbb{R}^n$.

d) Describe the Borel Hierarchy of subsets of $\mathbb{Q}^n$.
Start by determining $\mathbf{\Sigma}_2$.
What happens to Corollary 3.16?

EXERCISE 25:
Fix some sufficiently large ordinal λ and consider the set of all open-closed intervals

$$(\alpha, \beta] \quad := \quad \{\gamma \text{ ordinal}, \alpha < \gamma \leq \beta\}, \qquad \alpha, \beta \leq \lambda$$

a) Prove that the intersection of two open-closed intervals is again an open-closed interval.
How about the intersection of denumerably many open-closed intervals?

b) Let I be a set and, for each $i \in I$, U_i some open-closed interval. Suppose that $\bigcup_{i \in I} U_i = (0, \omega]$. Prove that there exists a finite subset $J \subseteq I$ such that $\bigcup_{i \in J} U_i = (0, \omega]$.

In topological terms, this means that $(0, \omega]$ is compact (with respect to the order topology).

c) Prove that $(0, \lambda]$ is compact for every ordinal λ.
Hint: transfinite induction.

d) Is $(0, \lambda)$ always compact as well (and if yes, why not)?

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*Bonus exercise