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Introduction to Mathematical Logic

SS 2010, Exercise Sheet #5

EXERCISE 21:

a) Prove that 2^ω is equipotent with $2^\omega \setminus \big\{\{n\} : n \in \omega\big\}$.

b) Prove that $\mathbb{R}$ is equipotent with $(-1,1)$, with $[0,1]$, with $\mathbb{R}^2$, and with $\mathbb{R}^\omega$.

c)* Which of the bijections are continuous; which ones cannot be, and why?

d) How many (total) continuous real functions are there?

e) How many partial continuous real functions are there?

EXERCISE 22:
The lecture showed that at least one string $\bar{x} \in \{0,1\}^n$ cannot be compressed below length n.
Prove: To every $n \in \mathbb{N}$, at least half of the strings $\bar{x} \in \{0,1\}^n$ cannot be compressed below $n-1$.

EXERCISE 23:
We identify *predicates* with 0-1-valued functions; e.g.,
"$x \leq y$": $\mathbb{N}^2 \to \mathbb{N}$, $(x,y) \mapsto 1$ for $x \leq y$ and $(x,y) \mapsto 0$ else.
Prove that the following functions are recursive:

a) Addition $A : \mathbb{N}^2 \to \mathbb{N}$, $(x,y) \mapsto x+y$.

b) Positive subtraction $(x,y) \mapsto \max(0, x-y)$.

c) Multiplication.

d) Predicates "$x = 0$"; "$x \leq y$", "$x < y$", "$x \neq y$", and "$x = y$";

e) $\bar{x} \mapsto \begin{cases} f(\bar{x}) & : \quad P(\bar{x}) \\ g(\bar{x}) & : \neg P(\bar{x}) \end{cases}$ for recursive functions $f, g : \mathbb{N}^n \to \mathbb{N}$ and recursive predicate P.

f) $x \text{ rem } (y+1)$ (division with remainder),

g) $n \mapsto n$-th prime number

*Bonus exercise