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Issued 2010-05-03
To be submitted on 2010-05-10

Introduction to Mathematical Logic

SS 2010, Exercise Sheet #4

EXERCISE 18:
Let X, Y, Z denote sets. Prove:

a) $X \uplus Y$ and $Y \uplus X$ are equipotent.

b) $(X \uplus Y) \uplus Z$ and $X \uplus (Y \uplus Z)$ are equipotent.

c) $X \times Y$ and $Y \times X$ are equipotent.

d) $(X \times Y) \times Z$ and $X \times (Y \times Z)$ are equipotent.

e) $X \times (Y \uplus Z)$ and $(X \times Y) \uplus (X \times Z)$ are equipotent.

f) $Y^X \times Z^X$ and $(Y \times Z)^X$ are equipotent.

g) $Z^{X \uplus Y}$ and $Z^X \times Z^Y$ are equipotent.

h) $(Z^Y)^X$ and $Z^{Y \times X}$ are equipotent.

EXERCISE 19:
Continuing Exercise 17b),

a) show that ω is equipotent to ω^2, to ω^3, to ω^4, and so on.

b) and even equipotent to $\bigcup_{n \in \omega} \omega^n$.

c) Specify explicitly an injective mapping f from $\mathbb{Q}_> := \mathbb{Q} \cap (0, \infty)$ to $N := \omega \setminus \{0\}$.

d) Together with the injection $\mathrm{id} : N \to \mathbb{Q}_>$, Cantor-Schröder-Bernstein implies that $X := N$ is equipotent to $Y := \mathbb{Q}_>$. For this particular application, describe (as explicit as possible) the sets X_X, X_Y, X_∞ occurring in the proof of this theorem.

e) If $h : X \to Y$ is surjective, does this imply that Y is subpotent to X?
If Y is subpotent to X, does there exist a surjective $h : X \to Y$?

EXERCISE 20:
Prove:

a) The union of two finite sets is a finite set.

b) The (Cartesian) product of two finite sets is a finite set.

c) Let I and X_i be a finite sets for each $i \in I$. Then $\bigcup_{i \in I} X_i$ and $\prod_{i \in I} X_i$ are finite sets.

d) If X, Y are finite sets, then so is X^Y.