Martin Ziegler

Issued 2010-04-26
To be submitted on 2010-05-03

Introduction to Mathematical Logic

SS 2010, Exercise Sheet #3

EXERCISE 16:

The lecture defined ordinal addition and multiplication by constructing new well-orders from given ones. Here, we shall pursue an alternative approach similar to Exercise 7a+d+e) based on *transfinite recursion*: For ordinals α,β, let

$$\begin{aligned} A(\alpha,0) &:= \alpha \\ A(\alpha,\beta^+) &:= A(\alpha,\beta)^+ \\ A(\alpha,\beta) &:= \sup_{\gamma<\beta} A(\alpha,\gamma) \quad \text{in case } \beta \text{ is a limit ordinal} \end{aligned} \tag{1}$$

a) Prove that this defines an ordinal $A(\alpha,\beta)$ for every pair (α,β) of ordinals.

b) Prove that A is associative.

c) One can show that, for every fixed ordinal α, the mapping $\beta \mapsto \alpha+\beta$ on ordinals is *continuous* in the sense that it holds $\alpha+\beta = \sup_{\gamma<\beta} \alpha+\gamma$.
Conclude $A(\alpha,\beta) = \alpha+\beta$ for all ordinals α,β.

d) Prove or disprove: The mapping $\alpha \mapsto \alpha+\beta$ is continuous for every ordinal β.
Hint: Exercise 15a).

e) Use ordinal addition and multiplication to define ordinal exponentiation $E(\alpha,\beta)$ such that, for natural numbers n and m, $E(n,m)$ coincides with n^m.

f) Show that $E(2,\omega) = \omega < \omega\cdot\omega = E(\omega,2)$.

EXERCISE 17:

a) Show that $\mathbb{N} := \{0,1,2,\ldots\}$ is equipotent to $N := \{1,2,\ldots\}$
as well as to the sets $2\mathbb{N}$ of even numbers
and $\mathbb{P} = \{2,3,5,7,11,13,17,\ldots\}$ of prime numbers.

b) Prove that N is equipotent to $N \times N$.
Hint: $(x,y) \mapsto 2^{x-1}\cdot(2y-1)$.

c) Let X,Y denote sets. Prove that $(X\cup Y) \uplus (X\cap Y)$ is equipotent to $X \uplus Y$.

d) Prove that the powerset $\mathcal{P}(X) = \{Y : Y \subseteq X\}$ is equipotent to $2^X = \{ f : X \to \{0,1\}\}$.

e) Prove that X is *not* equipotent to 2^X.
Hint: For $F : X \to 2^X$, consider $g(x) := 1 - F(x)(x)$.