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Introduction to Mathematical Logic

SS 2010, Exercise Sheet #1

EXERCISE 7:
Recall the Peano axioms for $(\mathbb{N}, S)$.

a) Define addition $A(n,m) = n+m$ recursively in terms of S.

b) Use induction to conclude that your definition satisfies commutativity and associativity.

c) Now forget a) and b) and define A axiomatically.

d) Do similarly for multiplication $M(n,m) = n \cdot m$

e) and for exponentiation $E(n,m) = n^m$.

f) Conclude from the above axioms (!) that $(x+1)^2 = x^2 + 2 \cdot x + 1$ holds for all $x \in \mathbb{N}$.

g) Prove (*not* restricting[1] to the above axioms) that the following holds for all $x, y \in \mathbb{N}$:

$$\Big((1+x)^y + (1+x+x^2)^y \Big)^x \cdot \Big((1+x^3)^x + (1+x^2+x^4)^x \Big)^y$$
$$= \Big((1+x)^x + (1+x+x^2)^x \Big)^y \cdot \Big((1+x^3)^y + (1+x^2+x^4)^y \Big)^x$$

Hint: Check that $x^2 - x + 1$ divides both $x^3 + 1$ and $x^4 + x^2 + 1$.

EXERCISE 8:
Count: 1, 2, 3, 4, ..., ∞.
How would you continue? And how then on?
And how after infinitely many further steps? And how then?

EXERCISE 9:

a) How does the picture on the back of the page in landscape indicate a partially ordered set?
Why is the indicated order not total?
Write down explicitly (a set and) the order relation that gives rise to the picture.

b) Draw a similar picture for the set $\mathbb{N}$ of natural numbers (with respect to order "$\leq$")

c) and for the 'numbers' counted in Exercise 8.

EXERCISE 10:

a) Is the set $\mathbb{N}$ of natural numbers[2] well-ordered with respect to "$<$"?

b) How about the set $\mathbb{R}$ of real numbers?

c) How about $\mathbb{N}$, but now with respect to "$>$"?

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[1] In fact Alex Wilkie proved in 1980 that this cannot be concluded from the above axioms alone.
[2] as you know them, i.e. non-axiomatized