Fachbereich Mathematik
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7. Problems for Manifolds

Problem 28 – Quiz:

a) For given $f \in \mathcal{V}(\mathbb{R}^n)$ find a form ω on $\mathbb{R}^n$ such that $d\omega = \operatorname{div} f e^1 \wedge \ldots \wedge e^n$.
b) Determine $\{v \in \mathbb{R}^2 : e^1 \wedge e^2(v, e_2) = 0\}$
c) For $w \in \mathbb{R}^3$ given determine $V(w) := \{v \in \mathbb{R}^3 : e^1 \wedge e^2(v, w) = 0\}$
d) Determine $L := \{\omega \in \Lambda^2\mathbb{R}^n : \omega(e_1, e_2) = 0\}$.
e) Let $\omega \in \Lambda^k M$ and $X_1, \ldots, X_k \in \mathcal{V}(M)$. Which of the following statements is true?
• The value of $d\omega(X_1, \ldots, X_k)$ at $p \in M$ depends only on the values of the X_i's at p, but not on the way they extend to M,
• this value depends only on the value of ω at p but not of the way the form ω extends to M.

Problem 29 – n-dimensional Cube:

Denote the standard unit cube by $C := \{x \in \mathbb{R}^n : 0 \leq x^1, \ldots, x^n \leq 1\}$.

a) Write down the faces of the standard cube (how many are there?).
b) For $1 \leq i \leq n$ let

$$\omega_i \in \Lambda^{n-1}\mathbb{R}^n, \qquad \omega_i := e^1 \wedge \ldots \wedge \widehat{e^i} \wedge \ldots \wedge e^n.$$

Describe those faces of C such that the form ω_i vanishes on multivectors formed by tangent vectors to the faces.

Problem 30 – Geometric interpretation of a two-form:

Let $P(v, w)$ be the planar parallelogram in $\mathbb{R}^3$, spanned by $v, w \in \mathbb{R}^3$. Let $\pi \colon \mathbb{R}^3 \to \mathbb{R}^2$ be projection to the xy-plane and let $\eta = e^1 \wedge e^2 \in \Lambda^2\mathbb{R}^3$.

a) Give a formula for the signed area of $\pi(P(v, w))$.
b) Prove that $\eta(v, w)$ agrees with the signed area of $\pi(P(v, w))$.

Problem 31 – Decomposable and undecomposable forms:

a) Show that in $\mathbb{R}^3$ any two-forms $\omega := v \wedge w$ and $\eta := r \wedge s$ have a sum $\omega + \eta = a \wedge b$ for some covectors $a, b \in \mathbb{R}^{3*}$.
b) Prove that $e^1 \wedge e^2 + e^3 \wedge e^4 \in \Lambda^2\mathbb{R}^4$ cannot be written in the form $v \wedge w$ for $v, w \in \mathbb{R}^{4*}$.
c) Find $\omega \in \Lambda^2\mathbb{R}^4$ such that $\omega \wedge \omega \neq 0$.

Problem 32 – Hodge star:

Let V^n be a vector space with inner product. With resprect to an orthonormal basis $(e^1, \ldots, e^n)$, define an operator

$$* \colon \Lambda^k V \to \Lambda^{n-k} V, \qquad *(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}) = e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \ldots \wedge e_{i_n},$$

if $\{i_1, \cdots i_k, i_{k+1} \cdots i_n\}$ is an even permutation of $\{1, 2, \ldots, n\}$.

a) What is $*(e^1 \wedge e^2)$ in $\mathbb{R}^3$? What is $*1$ in $\mathbb{R}^n$?
b) Prove $** = (-1)^{k(n-k)}$.
c) Prove that on $\Lambda^k V$ we can define an inner product by

$$\langle v, w \rangle := *(w \wedge *v).$$