Fachbereich Mathematik
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10.1.2010

TECHNISCHE
UNIVERSITÄT
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6. Problems for Manifolds

Problem 24 – Flows and Lie brackets:

Consider $X(u,v) := (0,u)$ on $\mathbb{R}^2$.

a) Plot $X(u,v)$.

b) Find a chart $(x,U)\colon U \to \mathbb{R}^2$ around the point $(1,0)$ such that $X = e_1$, as in Lemma 26. Formulate this first as a condition on the differential $dx\colon \mathbb{R}^2 \to \mathbb{R}^2$. What is the maximal choice of U?

c) If you like: Discuss all choices for x. Remember to verify that x is a diffeomorphism.

d) Moreover, let $Y(u,v) := (1,0)$, see the example in class. Verify Lemma 27 at the point $(1,0)$.

Problem 25 – Cylindrical coordinates:

On $\Omega := \mathbb{R}^3 \setminus \{(0,0,w) : w \in \mathbb{R}\}$ consider the vector fields $X(u,v) := \frac{1}{\sqrt{u^2+v^2}}(u,v,0)$ and $Y(u,v) := J(u,v) := (-v,u,0)$.

a) Plot X and Y. Can you see what $[X,Y]$ is?

b) Verify they span an involutive distribution Δ.

c) Pick a point in Ω and determine a chart (x,U) as in the Frobenius Theorem 31.

Problem 26 – Skew symmetric bilinear forms:

The purpose of this problem is to prepare Thursday's class.

a) Give an example of a skew-symmetric bilinear form, $b\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, that is, $b(v,w) = -b(w,v)$ for all $v,w \in \mathbb{R}^n$.

b) Show that $b(v,v) = 0$ for all $v \in \mathbb{R}^n$ is equivalent to b skew-symmetric.

c) What is the dimension of the space of skew-symmetric forms $B(n)$? Exhibit a basis for $B(n)$, for instance in terms of the basis $e^i = \langle ., e_i\rangle$ of the dual space.

d) Can you find a projection which maps an arbitrary bilinear form to the skew-symmetric forms? What are the reasonable properties to ask for?

Problem 27 – Non-integrable distribution:

Check explicitely that $X(p) = e_1$ and $Y(p) = e_2 + p^1 e_3$ is non-integrable.